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Compactness of Hardy Operators Involving Suprema

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Abstract. – *We study compactness properties of Hardy operators involving suprema on weighted Banach function spaces. We first characterize the compactness of abstract operators assumed to have their range in the class of non-negative monotone functions. We then define a category of pairs of weighted Banach function spaces for which a suitable Muckenhoupt-type condition implies the boundedness of Hardy operators involving suprema, and prove a criterion for the compactness of these operators between such a couple of spaces. Finally, we characterize the compactness of these operators on weighted Lebesgue spaces including those which do not belong to the above-mentioned category.*

1. – Introduction

The *Hardy operator*

$$Hf(t) := \int_0^t f(s)\,ds,$$

together with its many various modifications, where $t \in (0, \infty)$ and f is a non-negative locally-integrable function on $(0, \infty)$, plays a central role in several branches of analysis and its applications. It becomes of a particular interest when functional-analytic methods are applied to finding solutions of partial differential equations. Important intrinsic properties of this operator, such as boundedness and compactness, on various function spaces, have been intensively studied over almost a century. Certain special attention has been paid to weighted spaces.

In particular, many authors studied the question, for which non-negative measurable functions on $(0, \infty)$ w and v and for which parameters $p, q \in (0, \infty]$ there exists a positive constant, C, possibly dependent on w, v, p, q, but not on f, such that

$$\left(\int_0^\infty \left(\int_0^t f(s)\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}} \le C \left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}}, \tag{1.1}$$

with the usual modification when $p = \infty$ and/or $q = \infty$. Inequalities of this type have been called the *weighted Hardy inequalities*. It turns out that in the case when $1<p\le q<\infty$, (1.1) holds if and only if (see for example [2])

$$\sup_{0<t<\infty}\left(\int_t^\infty w(s)\,ds\right)^{1/q}\left(\int_0^t v(s)^{1-p'}\,ds\right)^{1/p'}<\infty, \tag{1.2}$$

where $p'=\dfrac{p}{p-1}$, while, in the case when $1<q<p<\infty$, (1.1) holds if and only if (see [12, pp. 72-76] and [13, pp. 45-48])

$$\left(\int_0^\infty\left(\int_x^\infty w(t)\,dt\right)^{r/q}\left(\int_0^x v(t)^{1-p'}\,dt\right)^{r/q'}v(x)^{1-p'}\,dx\right)^{1/r}<\infty,$$

where $\dfrac{1}{r}=\dfrac{1}{p}-\dfrac{1}{q}$. The estimate (1.2) is called the *Muckenhoupt condition*, since it was first obtained (for the case $p=q$) in [14]. It is of interest to notice that the Muckenhoupt condition is *necessary* for (1.1) for *any* values of $p,q\in(1,\infty)$, but it is *not sufficient* when $q<p$.

Aside from the boundedness, the most important property of the Hardy operator from the point of view of applications, is its *compactness* on various function spaces, which, again, has been widely studied. For example, when $1<p\le q<\infty$, v and w are non-negative measurable functions on $(0,\infty)$, and $L^p(v)$ denotes the class of measurable functions f on $(0,\infty)$ such that $\int_0^\infty |f|^p v<\infty$ (analogously for $L^q(w)$), then the operator H is compact from $L^p(v)$ into $L^q(w)$ on $(0,\infty)$ if and only if

$$\lim_{x\to0+}\left(\int_x^\infty w(s)\,ds\right)^{1/q}\left(\int_0^x v(s)^{1-p'}\,ds\right)^{1/p'}=0$$

and

$$\lim_{x\to\infty}\left(\int_x^\infty w(s)\,ds\right)^{1/q}\left(\int_0^x v(s)^{1-p'}\,ds\right)^{1/p'}=0,$$

while, for $1<q<p<\infty$, it is compact from $L^p(v)$ into $L^q(w)$ on $(0,\infty)$ if and only if it is bounded.

When the action of the Hardy operator is considered in a more general setting of the so-called *weighted Banach function spaces* (whose precise definition will be given in Section 2), the Muckenhoupt condition, appropriate for these spaces, is still necessary for the boundedness of the operator. Indeed, suppose that the Hardy operator H is bounded from one weighted Banach function space,

$X = X(v)$, into another one, $Y = Y(w)$, that is, there exists a constant C such that $\|Hf\|_Y \le C\|f\|_X$ for every $f \ge 0$ and $f \in X$. Then, necessarily,

$$\sup_{0<x<\infty} \|\chi_{(x,\infty)}\|_Y \left\|\frac{\chi_{(0,x)}}{v}\right\|_{X'} < \infty, \tag{1.3}$$

where $X' = X'(v)$ is the associate space to X (see Section 2). Thus, naturally, as first observed in [6], the collection of all pairs of weighted Banach function spaces (X, Y) can be divided into two subclasses; one containing those pairs for which (1.3) is sufficient for the boundedness of the Hardy operator, and another one containing those pairs for which it is not. Following [6], we say that a pair of weighted Banach function spaces (X, Y) belongs to the *Muckenhoupt category* if the condition (1.3) implies the boundedness of H from X to Y. In [6], the compactness of the Hardy operator from X into Y was characterized for pairs of weighted Banach function spaces belonging to the Muckenhoupt category.

One of our main goals in this paper is to characterize compactness of the so-called *Hardy operators involving suprema*. These are operators which, aside from integration, involve the operation of taking a pointwise supremum. More precisely, we shall work with the operator $T_{u,h}$, which is defined for a pair of given weights u, h on $(0,\infty)$ (that is, measurable, positive and finite a.e. locally integrable functions on $(0,\infty)$) at a measurable function f on $(0,\infty)$ by

$$(T_{u,h}f)(t) = \sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds, \quad t \in (0,\infty),$$

where $H(t) := \int_0^t h(s)\,ds$.

Operators of this type have proved to be useful in several applications. For example, it was shown in [3] that a sharp estimate of the non-increasing rearrangement of the fractional maximal operator of a given function can be given in terms of such operator. This information can be in turn used to study the action of the fractional maximal operator on classical Lorentz spaces. Further, Hardy-type operators involving suprema have been found useful in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds ([10]). They also constitute a handy tool for characterizing the associate norm of an operator-induced norm (see, for example, [15] or [16]). An important role of the Hardy operators involving suprema in limiting interpolation theory can be observed for example in [5], [7], [4] or [17].

However, while the boundedness of such operators on various weighted spaces has been thoroughly investigated and comprehensive results were obtained (see e.g. [3] or [8]), almost no effort has been spent in order to characterize their compactness. In this paper we intend to focus on this problem.

We shall present three principal results. The first one, Theorem 3.1, is a characterization of compactness on weighted Banach function spaces of general operators whose range lies in the class of monotone functions. This approach is relatively new and it is motivated by the fact that the Hardy operators involving suprema constitute an example of a class of such operators. The characterizing condition is given in terms of the absolute continuity of norm. This is done in Section 3.

The second main result is Theorem 4.6 in which we establish a necessary and sufficient condition on a pair of weights in order that Hardy operators involving suprema are compact between the corresponding pair of weighted Banach function spaces as long as this pair belongs to a Muckenhoupt-type category suitable for such operators.

Finally, in Section 5, we prove a characterization of the compactness of Hardy operators involving suprema on a pair of weighted Lebesgue spaces. The innovative part of this result consists of the case which is not covered by Theorem 4.6. We use weighted inequalities obtained in [8] and the techniques of discretization and antidiscretization from [9].

2. – Preliminaries

Throughout the paper, we use the symbol λ to denote the one-dimensional Lebesgue measure on $\mathbb{R}$. Let (Ω, μ) be a totally σ-finite measure space, $\mathcal{M}(\Omega, \mu)$ the collection of all μ-measurable functions on Ω whose values lie in $[-\infty, \infty]$ and $\mathcal{M}^+(\Omega, \mu)$ the cone of all functions from $\mathcal{M}(\Omega, \mu)$ with their values in $[0, \infty]$. The characteristic function of a μ-measurable set E is denoted by χ_E. By a simple function we understand a finite sum of functions, each of which is defined as a finite real multiple of a characteristic function of a set having finite measure.

We shall summarize some background material from the theory of Banach function spaces. The standard general reference is [1].

A mapping $\rho : \mathcal{M}^+(\Omega, \mu) \to [0, \infty]$ is called a *Banach function norm* if, for all f, g, f_n, $(n = 1, 2, 3, \dots)$, in $\mathcal{M}^+(\Omega, \mu)$, for all constants $a \geq 0$, and for all μ-measurable subsets E of Ω, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.;
(P2) $\rho(af) = a\rho(f)$;
(P3) $\rho(f + g) \leq \rho(f) + \rho(g)$;
(P4) $g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;
(P5) $f_n \uparrow f$ μ-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;
(P6) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;
(P7) $\mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq C_E \rho(f)$,
for some constant $C_E \in (0, \infty)$ depending on E and ρ but independent of f.

For a Banach function norm $\rho : \mathcal{M}^+(\Omega, \mu) \to [0, \infty]$, we call a *Banach function space* the collection of all functions (as usual, any two functions coinciding μ-a.e. are identified) f in $\mathcal{M}(\Omega, \mu)$ for which $\rho(|f|) < \infty$. We denote it by (X, ρ), or shortly X. For each $f \in X$, we define $\|f\|_X = \rho(|f|)$.

A function f belonging to a Banach function space X has *absolutely continuous norm* in X if $\lim_{n\to\infty} \|f\chi_{E_n}\|_X = 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ of μ-measurable subsets of Ω such that $\chi_{E_n} \to \chi_{\emptyset}$ μ-a.e. on Ω. The set of all functions in X with absolutely continuous norm is denoted by X_a. Provided X_a coincides with X, the space X itself is said to have absolutely continuous norm.

In a Banach function space X a subset Y of X_a is of *uniformly absolutely continuous norm* if, for every sequence $\{E_n\}_{n=1}^{\infty}$ of μ-measurable subsets of Ω, such that $\chi_{E_n} \to \chi_{\emptyset}$ μ-a.e. on Ω, and each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ satisfying

$$f \in Y,\ n \geq n_0 \ \Rightarrow \ \|f\chi_{E_n}\|_X < \varepsilon.$$

If $f \in X$ has absolutely continuous norm, then to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that for every μ-measurable set $E \subset \Omega$ with $\mu(E) < \delta$ we have $\|f\chi_E\|_X < \varepsilon$. We further define X_b to be the closure of the set of simple functions in X in the topology given by the norm $\|\cdot\|_X$. Then, one always has $X_a \subset X_b$, and X_a and X_b coincide if and only if for every set E of finite measure, the characteristic function χ_E has absolutely continuous norm.

If ρ is a Banach function norm, we define its *associate norm* ρ' at $g \in \mathcal{M}^+(\Omega, \mu)$ by

$$\rho'(g) = \sup\left\{ \int_{\Omega} fg\, d\mu;\ f \in \mathcal{M}^+(\Omega, \mu), \rho(f) \leq 1 \right\}.$$

Then ρ' is a Banach function norm as well, and the Banach function space $X' = (X', \rho')$ determined by ρ' is called the *associate space* of X.

The definitions of "associate notions" imply that for a function g belonging to the associate space X',

$$\|g\|_{X'} = \sup\left\{ \int_{\Omega} |fg| d\mu;\ f \in X, \|f\|_X \leq 1 \right\},$$

where $\|g\|_{X'} = \rho'(|g|)$ by definition.

For any Banach function space X and every $f \in X$ and $g \in X'$, the *Hölder inequality* asserts that the function fg is integrable and

$$\int_{\Omega} |fg| d\mu \leq \|f\|_X \|g\|_{X'}.$$

We say that a function v is a *weight* if it is measurable, positive and finite λ-a.e. and locally integrable on $(0, \infty)$.

Let v be a weight. In a special case when the underlying measure space is the interval $(0,\infty)$ endowed with a measure ν given by $\nu(E)=\int_E v(t)dt$ for every Lebesgue-measurable subset E of $(0,\infty)$, we denote a Banach function space X built upon this setting by $X(v)$ and call it a *weighted Banach function space*. Note that from the definition of a weight it follows that compact sets have finite measure and hence their characteristic functions are elements of $X(v)$. In what follows, we shall work solely with weighted Banach function spaces.

For $p\in[1,\infty]$ and a weight v we define the *weighted Lebesgue space* $L^p(v)$ as the set of all Lebesgue-measurable functions f on $(0,\infty)$, for which the inequality $\|f\|_{p,v}<\infty$ holds, where

$$\|f\|_{p,v}=\begin{cases}\left(\int_0^\infty |f(t)|^p v(t)dt\right)^{\frac{1}{p}} & \text{when } 1\le p<\infty,\\ \operatorname*{ess\,sup}_{0<t<\infty}|f(t)| & \text{when } p=\infty.\end{cases}$$

It is a routine matter to verify that every weighted Lebesgue space $L^p(v)$ is a weighted Banach function space, whose associate space is $L^{p'}(v)$, where the conjugate number p' is given by

$$p'=\begin{cases}\dfrac{p}{p-1} & \text{when } 1<p<\infty,\\ \infty & \text{when } p=1,\\ 1 & \text{when } p=\infty,\end{cases}$$

and that it has absolutely continuous norm whenever $1\le p<\infty$.

Each function $f\in L^p(v)$ is p-mean continuous, which means that for every $\varepsilon>0$ there exists a $\delta>0$ such that for each $h\in\mathbb{R}$ with $|h|<\delta$ we have

$$\int_0^\infty |f(t+h)-f(t)|^p v(t)dt<\varepsilon^p,$$

where f is defined by 0 outside the interval $(0,\infty)$.

The study of compactness of operators on Banach spaces goes hand in hand with the theory of compact sets in corresponding Banach spaces. The well-known Kolmogorov theorem asserts that, for $1\le p<\infty$, a compact set $A\subset L^p(v)$ is p-mean equicontinuous, i.e.

$$\forall\varepsilon>0\ \exists\delta>0\ \forall f\in A:\ |h|<\delta\Rightarrow\int_0^\infty |f(t+h)-f(t)|^p v(t)dt<\varepsilon^p,$$

where f is extended by 0 outside the interval $(0,\infty)$.

3. – Compactness of operators having range in non-negative monotone functions on weighted Banach function spaces

In this section we shall establish our first main result in which we characterize compact bounded operators whose range lies in the family of non-negative monotone functions on weighted Banach function spaces in terms of uniform absolute continuity of norm. This class contains many important operators including those involving suprema. An analogous approach was presented by Luxemburg and Zaanen in [11] for integral kernel operators.

THEOREM 3.1. – *Let v, w be weights on $(0,\infty)$. Let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Assume that $Y_a = Y_b$. For a bounded operator R from X to Y, such that Rf is a non-negative monotone function for each $f \in X$ and $\{Rf; f \in X, \|f\|_X \le 1\} \subset Y_a$, the following two statements are equivalent:*

(i) *the operator R is compact from X to Y;*
(ii) *the set $\{Rf; f \in X, \|f\|_X \le 1\}$ is of uniformly absolutely continuous norm in Y.*

PROOF. – Assume first that R is compact. Consider $\varepsilon > 0$ and a sequence $\{E_n\}$ of λ-measurable subsets of $(0,\infty)$, such that $\chi_{E_n} \to \chi_\emptyset$ λ-a.e. on $(0,\infty)$ (this is in fact equivalent to the pointwise convergence on a set $A \subset (0,\infty)$ for which $\int_{(0,\infty)\setminus A} w(t)dt = 0$). Since $\overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ is compact, there exist $k \in \mathbb{N}$ and a set $\{g_1, \dots, g_k\} \subset \overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ with the following property:

$$\forall g \in \overline{\{Rf; f \in X, \|f\|_X \le 1\}} \ \exists i \in \{1, \dots, k\} : \ \|g - g_i\|_Y < \frac{\varepsilon}{2}.$$

According to the assumption, all functions in $\overline{\{Rf; f \in X, \|f\|_X \le 1\}}$ have absolutely continuous norms. Therefore, there is an $n_0 \in \mathbb{N}$ satisfying that whenever $n \ge n_0$, the inequality $\|g_i\chi_{E_n}\|_Y < \frac{\varepsilon}{2}$ holds for every $i = 1, \dots, k$. Thus,

$$\|(Rf)\chi_{E_n}\|_Y \le \|(Rf - g_i)\chi_{E_n}\|_Y + \|g_i\chi_{E_n}\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where $i \in \{1, \dots, k\}$ is chosen to satisfy $\|Rf - g_i\|_Y < \frac{\varepsilon}{2}$. Hence the set $\{Rf; f \in X, \|f\|_X \le 1\}$ is of uniformly absolutely continuous norm in Y.

Conversely, suppose that the set $\{Rf; f \in X, \|f\|_X \le 1\}$ is of uniformly absolutely continuous norm in Y. Then for any $\eta > 0$, there exist $0 < a < b < \infty$, such that $\|(Rf)\chi_{(0,a)}\|_Y < \frac{\eta}{2}$ and $\|(Rf)\chi_{(b,\infty)}\|_Y < \frac{\eta}{2}$ for each $f \in X$ with $\|f\|_X \le 1$. Hence

$$\sup_{\|f\|_X \le 1} \|(Rf)\chi_{(0,a)}\|_Y + \sup_{\|f\|_X \le 1} \|(Rf)\chi_{(b,\infty)}\|_Y \le \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Set $(Tf)(t) = (Rf)(t)\chi_{[a,b]}(t)$ for any $f \in X$ and $t \in (0,\infty)$. It is enough to show that T is a compact operator in order to obtain the compactness of R. Take an arbitrary $\varepsilon > 0$. Due to the absolute continuity of the Lebesgue integral we can find a $\delta > 0$ for which $\|\chi_{[c,d]}\|_Y < \varepsilon$ for every $a \le c < d \le b$ such that $d - c < \delta$. Consider a partition $a = \alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha_n = b$ of the interval $[a,b]$ with $\alpha_i - \alpha_{i-1} < \delta$ for all $i \in \{1,\ldots,n\}$. Denote $I_i = [\alpha_{i-1},\alpha_i)$ for $i \in \{1,\ldots,n-1\}$ and $I_n = [\alpha_{n-1},\alpha_n]$. Define the mapping S by

$$(Sf)(t) = \sum_{i=1}^{n} \min\{(Rf)(\alpha_{i-1}),(Rf)(\alpha_i)\}\chi_{I_i}(t), \quad f \in X,\ t \in (0,\infty).$$

By virtue of the boundedness of R, there exists a constant $c(R)$ satisfying $\|Rf\|_Y \le c(R)\|f\|_X$ for all $f \in X$. Since Rf is non-negative and non-increasing or non-decreasing function for each $f \in X$, for arbitrary $x \in (0,\infty)$, $y \in (0,x)$, $z \in (x,\infty)$ and $f \in X$ we have

$$\begin{aligned}(Rf)(x) &= \|\chi_{(y,x]}\|_Y^{-1}\|(Rf)(x)\chi_{(y,x]}\|_Y\\ &\le \|\chi_{(y,x]}\|_Y^{-1}\|Rf\chi_{(y,x]}\|_Y\\ &\le \|\chi_{(y,x]}\|_Y^{-1}\|Rf\|_Y\\ &\le \|\chi_{(y,x]}\|_Y^{-1}c(R)\|f\|_X,\end{aligned}$$

or

$$\begin{aligned}(Rf)(x) &= \|\chi_{[x,z)}\|_Y^{-1}\|(Rf)(x)\chi_{[x,z)}\|_Y\\ &\le \|\chi_{[x,z)}\|_Y^{-1}\|Rf\chi_{[x,z)}\|_Y\\ &\le \|\chi_{[x,z)}\|_Y^{-1}\|Rf\|_Y\\ &\le \|\chi_{[x,z)}\|_Y^{-1}c(R)\|f\|_X,\end{aligned}$$

respectively. For both cases of monotonicity we can thus carry out a common estimate

$$(Rf)(x) \le \left(\min\left\{\|\chi_{(y,x]}\|_Y, \|\chi_{[x,z)}\|_Y\right\}\right)^{-1}c(R)\|f\|_X.$$

Application to $x = \alpha_i$ for any $i \in \{0,\ldots,n\}$ gives

$$\begin{aligned}(Rf)(\alpha_i) &\le \left(\min\left\{\left\|\chi_{(\frac{a}{2},\alpha_i]}\right\|_Y, \|\chi_{[\alpha_i,2b)}\|_Y\right\}\right)^{-1}c(R)\|f\|_X\\ &\le \left(\min\left\{\left\|\chi_{(\frac{a}{2},a]}\right\|_Y, \|\chi_{[b,2b)}\|_Y\right\}\right)^{-1}c(R)\|f\|_X.\end{aligned}$$

Moreover, $\chi_{I_i} \in Y$ because $I_i \subset [a,b]$. It follows from the preceding estimates that S is a bounded finite-rank, consequently compact, operator from X to Y. For

$f \in X$ we get

$$
\begin{aligned}
\|Tf - Sf\|_Y &= \left\| Rf\chi_{[a,b]} - \sum_{i=1}^{n} \min\{(Rf)(\alpha_{i-1}), (Rf)(\alpha_i)\}\chi_{I_i} \right\|_Y \\
&= \left\| \sum_{i=1}^{n} [Rf - \min\{(Rf)(\alpha_{i-1}), (Rf)(\alpha_i)\}]\chi_{I_i} \right\|_Y \\
&\leq \sum_{i=1}^{n} \left\| [Rf - \min\{(Rf)(\alpha_{i-1}), (Rf)(\alpha_i)\}]\chi_{I_i} \right\|_Y \\
&\leq \sum_{i=1}^{n} \left\| [\max\{(Rf)(\alpha_{i-1}), (Rf)(\alpha_i)\} - \min\{(Rf)(\alpha_{i-1}), (Rf)(\alpha_i)\}]\chi_{I_i} \right\|_Y \qquad (3.1) \\
&= \sum_{i=1}^{n} |(Rf)(\alpha_i) - (Rf)(\alpha_{i-1})| \left\| \chi_{I_i} \right\|_Y \\
&< \varepsilon \sum_{i=1}^{n} |(Rf)(\alpha_i) - (Rf)(\alpha_{i-1})| \\
&= \varepsilon |(Rf)(a) - (Rf)(b)| \\
&\leq \varepsilon((Rf)(a) + (Rf)(b)) \leq \varepsilon 2 \Big(\min\Big\{ \|\chi_{(\frac{a}{2},a]}\|_Y, \left\|\chi_{[b,2b)}\right\|_Y \Big\} \Big)^{-1} c(R)\|f\|_X,
\end{aligned}
$$

by using the fact that Rf is monotone and the estimate for $(Rf)(a)$ and $(Rf)(b)$ derived above. This yields

$$\sup_{\|f\|_X \leq 1} \|Tf - Sf\|_Y < c(a, b, R)\varepsilon,$$

where $c(a, b, R)$ is a constant depending only on a, b and $c(R)$. Hence we arrive at the compactness of the operator T and so finally at the compactness of the operator R. □

REMARK 3.2. – Let us present an example showing that without the assumption that R maps all functions from X to the class of monotone functions the statement of Theorem 3.1 is no longer true in general.

Consider weights v, w, such that $v(t) = 1$ for all $t \in (0, \infty)$ and $\int_0^\infty w(t)dt < \infty$. The spaces $L^\infty(v)$ and $L^1(w)$ are thus Banach function spaces. In addition, $L^1(w)$ has absolutely continuous norm by virtue of the Lebesgue dominated convergence theorem. For any $f \in L^\infty(v)$ we have

$$\int_0^\infty |f(t)|w(t)dt \leq \int_0^\infty w(t)dt\|f\|_{\infty,v}.$$

Set

$$Rf = |f|, \quad f \in L^\infty(v).$$

Then R is a well-defined bounded operator from $L^\infty(v)$ to $L^1(w)$, which assigns a non-negative but not necessarily monotone function from $L^1(w)$ to each function from $L^\infty(v)$. We claim that $\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}$ is of uniformly absolutely continuous norm, however R is not compact. Indeed, take a sequence $\{E_n\}$ of λ-measurable subsets of $(0,\infty)$, such that $\chi_{E_n} \to \chi_\emptyset$ λ-a.e. on $(0,\infty)$ and $\varepsilon > 0$. Using the Lebesgue dominated convergence theorem, we find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $f \in L^\infty(v)$ with $\|f\|_{\infty,v} \le 1$,

$$\|\chi_{E_n} Rf\|_{1,w} = \int_0^\infty \chi_{E_n}(t)|f(t)|w(t)dt \le \int_0^\infty \chi_{E_n}(t)w(t)dt < \varepsilon.$$

Hence, $\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}$ is of uniformly absolutely continuous norm. By the Kolmogorov theorem, if $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ was a compact set in $L^1(w)$, then it would be mean equicontinuous, which means that it would satisfy

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall g \in \overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}} :$$
$$|h| < \delta \Rightarrow \int_0^\infty |g(t+h) - g(t)|w(t)dt < \varepsilon,$$

considering g defined by 0 outside the interval $(0,\infty)$. To show that the set $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ is not mean equicontinuous, take $\varepsilon = \frac{1}{2}\int_0^\infty w(t)dt$ and for each $\delta > 0$ put

$$f_\delta(t) = \chi_{\bigcup\limits_{k \in \mathbb{N}\cup\{0\}} \left((2k+1)\frac{\delta}{2}, (2k+2)\frac{\delta}{2}\right]}(t), \quad t \in (0,\infty).$$

Then $f_\delta \in \overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$, and for $h = \frac{\delta}{2}$ we get

$$\int_0^\infty |f_\delta(t+h) - f_\delta(t)|w(t)dt = \int_0^\infty w(t)dt > \varepsilon.$$

Therefore, the set $\overline{\{Rf; f \in L^\infty(v), \|f\|_{\infty,v} \le 1\}}$ and consequently the operator R are not compact.

4. – Compactness of operators involving suprema on weighted Banach function spaces

From now on, we shall focus on the operators involving suprema. In this section we shall present a result in the spirit of [6], where Hardy-type integral operators were treated. Given a mapping involving supremum, we will determine a class of pairs of spaces, for which we can prove a general necessary and sufficient condition for this mapping to be compact. Similarly to [6], the class of couples of spaces is related to the boundedness of the operator under consideration and the characterization of the compactness of the operator is expressed in terms of the norms of weights that occur in the definitions of the spaces and the operator.

Before we will come to the principal theorem, we shall fix some notation, recall certain definitions and state and prove some auxiliary assertions.

NOTATION 4.1. – In keeping with notation of Section 2, $\mathcal{M}((0,\infty),\lambda)$ denotes the set of all Lebesgue-measurable functions on $(0,\infty)$.

For a weight h we put

$$H(t)=\int_0^t h(s)ds, \quad t\in(0,\infty). \tag{4.1}$$

Let u, h be weights, let H be given by (4.1) and let $I\subset(0,\infty)$ be an interval. We define

$$\bar{u}_I(t)=H(t)\sup_{t\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}, \quad t\in(0,\infty).$$

It is obvious that $\bar{u}_I(t)\ge u(t)\chi_I(t)$ for every $t\in(0,\infty)$ and that the function $\dfrac{\bar{u}_I}{H}$ is non-increasing. We abbreviate

$$\bar{u}(t)=H(t)\sup_{t\le\tau<\infty}\frac{u(\tau)\chi_{(0,\infty)}(\tau)}{H(\tau)}, \quad t\in(0,\infty).$$

We use the symbol T^I to denote the mapping given at a function f by $\chi_I Tf$, where T is some mapping defined at f and $I\subset(0,\infty)$ is an interval.

DEFINITION 4.2. – *Let h be a weight, let H be given by* (4.1), *and assume that $H(t)<\infty$ for every $t\in(0,\infty)$. Given another weight u, we define the mapping $T_{u,h}$ at $f\in\mathcal{M}((0,\infty),\lambda)$ by*

$$(T_{u,h}f)(t)=\sup_{t\le\tau<\infty}\frac{u(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)ds, \quad t\in(0,\infty). \tag{4.2}$$

Let $I\subset(0,\infty)$ be an interval. For $f\in\mathcal{M}((0,\infty),\lambda)$ we set

$$(T_{u,h,I}f)(t)=\sup_{t\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)\chi_I(s)ds, \quad t\in(0,\infty). \tag{4.3}$$

One can easily see that $T_{u,h}f$ is a non-negative non-increasing function for each $f \in \mathcal{M}((0,\infty),\lambda)$.

Let's have a look at a consequence of the boundedness of an operator $T^I_{u,h,I} : X(v) \to Y(w)$.

LEMMA 4.3. – *Given a pair of weights v, w, let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. For an interval $I \subset (0,\infty)$ and another pair of weights u, h such that the function H from* (4.1) *satisfies $H(t) < \infty$ for every $t \in (0,\infty)$, we define the operator $T_{u,h,I}$ by* (4.3). *If the operator $T^I_{u,h,I} : X \to Y$ is bounded, then*

$$\sup_{x\in I}\left\|\chi_{(0,x)}\chi_I\frac{\bar{u}_I(x)}{H(x)} + \chi_{[x,\infty)}\chi_I\frac{\bar{u}_I}{H}\right\|_Y \left\|\chi_{(0,x)}\chi_I\frac{h}{v}\right\|_{X'} < \infty. \tag{4.4}$$

PROOF. – Since $T^I_{u,h,I}$ is bounded, there exists a constant $c(T^I_{u,h,I}) > 0$ such that

$$\left\| T^I_{u,h,I}f\right\|_Y \le c(T^I_{u,h,I})\|f\|_X, \quad f \in X.$$

Take $f \in X$ with $\|f\|_X \le 1$ and $x \in I$. Then for $t \in (0,x) \cap I$ we have

$$\begin{aligned}
(T_{u,h,I}f)(t) &= \sup_{t\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{x\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{x\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^x |f(s)|h(s)\chi_I(s)ds \\
&= \frac{\bar{u}_I(x)}{H(x)}\int_0^x |f(s)|h(s)\chi_I(s)ds,
\end{aligned}$$

while for $t \in [x,\infty) \cap I$ we have

$$\begin{aligned}
(T_{u,h,I}f)(t) &= \sup_{t\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^\tau |f(s)|h(s)\chi_I(s)ds \\
&\ge \sup_{t\le\tau<\infty}\frac{u(\tau)\chi_I(\tau)}{H(\tau)}\int_0^x |f(s)|h(s)\chi_I(s)ds \\
&= \frac{\bar{u}_I(t)}{H(t)}\int_0^x |f(s)|h(s)\chi_I(s)ds.
\end{aligned}$$

Hence,

$$\begin{aligned} c(T^I_{u,h,I}) &\geq c(T^I_{u,h,I})\|f\|_X \geq \left\| T^I_{u,h,I}f\right\|_Y = \left\| \chi_{(0,x)}T^I_{u,h,I}f + \chi_{[x,\infty)}T^I_{u,h,I}f\right\|_Y \\ &= \left\|\chi_{(0,x)}\chi_I T_{u,h,I}f + \chi_{[x,\infty)}\chi_I T_{u,h,I}f\right\|_Y \\ &\geq \left\|\chi_{(0,x)}\chi_I \frac{\bar{u}_I(x)}{H(x)} + \chi_{[x,\infty)}\chi_I \frac{\bar{u}_I}{H}\right\|_Y \int_0^x |f(s)|\chi_I(s)\frac{h(s)}{v(s)}v(s)ds. \end{aligned}$$

By the definition of the associate norm, passing to the supremum over all $f \in X$ with $\|f\|_X \leq 1$ gives

$$\left\|\chi_{(0,x)}\chi_I \frac{\bar{u}_I(x)}{H(x)} + \chi_{[x,\infty)}\chi_I \frac{\bar{u}_I}{H}\right\|_Y \left\|\chi_{(0,x)}\chi_I \frac{h}{v}\right\|_{X'} \leq c(T^I_{u,h,I}).$$

In conclusion, we take the supremum over all $x \in I$ to obtain (4.4). □

Lemma 4.3 shows that (4.4) is always necessary for the boundedness of $T^I_{u,h,I} : X(v) \to Y(w)$. It turns out that for some spaces it is also sufficient, while for the other spaces it is not. This justifies our following definition.

DEFINITION 4.4. – *Let v, w, u, h be weights such that the function H from* (4.1) *satisfies $H(t) < \infty$ for every $t \in (0,\infty)$, and let the operators $T_{u,h}$ and $T_{u,h,I}$ be defined by* (4.2) *and* (4.3), *respectively. We say that a pair of weighted Banach function spaces $(X, Y) = (X(v), Y(w))$ belongs to the category $\mathbb{M}(T_{u,h})$ and write $(X, Y) \in \mathbb{M}(T_{u,h})$, if for each interval $I \subset (0,\infty)$ the condition* (4.4) *implies that the mapping $T^I_{u,h,I}$ is a bounded operator from X to Y and*

$$\begin{aligned} &\sup_{x\in I}\left\|\chi_{(0,x)}\chi_I \frac{\bar{u}_I(x)}{H(x)} + \chi_{[x,\infty)}\chi_I \frac{\bar{u}_I}{H}\right\|_Y \left\|\chi_{(0,x)}\chi_I \frac{h}{v}\right\|_{X'} \\ &\quad\leq \sup\{\|T^I_{u,h,I}f\|_Y; f \in X, \|f\|_X \leq 1\} \\ &\quad\leq K \sup_{x\in I}\left\|\chi_{(0,x)}\chi_I \frac{\bar{u}_I(x)}{H(x)} + \chi_{[x,\infty)}\chi_I \frac{\bar{u}_I}{H}\right\|_Y \left\|\chi_{(0,x)}\chi_I \frac{h}{v}\right\|_{X'}, \end{aligned} \tag{4.5}$$

where $K \geq 1$ is a constant independent of v, w, u, h and I.

Next, we shall need an auxiliary lemma that will turn out to be of a crucial importance in the proofs of the main results.

LEMMA 4.5. – *Let v, w, u, h be weights such that the function H from* (4.1) *satisfies $H(t) < \infty$ for every $t \in (0,\infty)$ and let $X = X(v)$, $Y = Y(w)$ be weighted Banach function spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.*

On X', the associate space of X, consider the norm $\|\cdot\|_{X'}$. Suppose that

$$\lim_{a\to 0_+} \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} = 0 \tag{4.6}$$

and

$$\lim_{b\to \infty} \sup_{b<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(b,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} = 0. \tag{4.7}$$

Then

$$\sup_{0<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} < \infty. \tag{4.8}$$

PROOF. – Conditions (4.6) and (4.7) give existence of $a \in (0,\infty)$ and $b \in (0,\infty)$, such that $a<b$ and

$$\sup_{0<x\le a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \le 1 \tag{4.9}$$

and

$$\sup_{b\le x<\infty} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(b,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \le 1. \tag{4.10}$$

Denote

$$\Psi(x) = \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'}.$$

We note that

$$\sup_{0<x<\infty} \Psi(x) = \max\left\{ \sup_{0<x<a} \Psi(x),\ \sup_{a\le x\le b} \Psi(x),\ \sup_{b<x<\infty} \Psi(x) \right\},$$

and we shall estimate the supremum of the function Ψ over each of the intervals $(0,a)$, $[a,b]$ and (b,∞) separately.

For $a>0$ we have

$$\begin{aligned}
\sup_{0<x<a} \Psi(x) &= \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} + \frac{\bar{u}}{H} \chi_{[a,b]} + \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\le \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\quad + \sup_{0<x<a} \left\| \frac{\bar{u}}{H} \chi_{[a,b]} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} + \sup_{0<x<a} \left\| \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'}
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\quad + \left\| \frac{\bar{u}}{H} \chi_{[a,b]} \right\|_Y \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'} + \left\| \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'} \\
&\le \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,x)} + \frac{\bar{u}}{H} \chi_{[x,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\quad + \left\| \frac{\bar{u}}{H} \chi_{[a,b]} \right\|_Y \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'} + \left\| \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,b)} \right\|_{X'} .
\end{aligned}
$$

Due to (4.9), the first summand is less than or equal to one, and due to (4.10) evaluated at $x = b$, also the last summand is less than or equal to one. As $\frac{\bar{u}}{H}$ is non-increasing, the middle term can be treated as

$$
\left\| \frac{\bar{u}}{H} \chi_{[a,b]} \right\|_Y \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'} \le \frac{\bar{u}(a)}{H(a)} \|\chi_{[a,b]}\|_Y \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'} \le \frac{\|\chi_{[a,b]}\|_Y}{\|\chi_{(0,a)}\|_Y} < \infty,
$$

where the last but one estimate follows from (4.9) evaluated at $x = a$, while the last one then stems from the properties of w, namely that w is locally integrable and positive λ-a.e. on $(0,\infty)$. Thus, $\sup_{0<x<a} \Psi(x)$ is finite.

As for the interval $[a,b]$, we write

$$
\begin{aligned}
\sup_{a\le x\le b} \Psi(x) &= \sup_{a\le x\le b} \left\| \frac{\bar{u}(x)}{H(x)} \chi_{(0,a)} + \frac{\bar{u}(x)}{H(x)} \chi_{[a,x)} + \frac{\bar{u}}{H} \chi_{[x,b]} + \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\le \sup_{a\le x\le b} \left\| \frac{\bar{u}(a)}{H(a)} \chi_{(0,a)} + \frac{\bar{u}}{H} \chi_{[a,b]} + \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\le \sup_{a\le x\le b} \left\| \frac{\bar{u}(a)}{H(a)} \chi_{(0,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} + \sup_{a\le x\le b} \left\| \frac{\bar{u}}{H} \chi_{[a,b]} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\quad + \sup_{a\le x\le b} \left\| \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,x)} \right\|_{X'} \\
&\le \left\| \frac{\bar{u}(a)}{H(a)} \chi_{(0,a)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,b)} \right\|_{X'} + \frac{\bar{u}(a)}{H(a)} \|\chi_{[a,b]}\|_Y \left\| \frac{h}{v} \chi_{(0,b)} \right\|_{X'} \\
&\quad + \left\| \frac{\bar{u}}{H} \chi_{(b,\infty)} \right\|_Y \left\| \frac{h}{v} \chi_{(0,b)} \right\|_{X'},
\end{aligned}
$$

where we again used that $\dfrac{\bar{u}}{H}$ is non-increasing. Thanks to (4.10), the last term is less than or equal to one. Inequalities (4.9) and (4.10), respectively, yield

$$
\frac{\bar{u}(a)}{H(a)} \le \|\chi_{(0,a)}\|_Y^{-1} \left\| \frac{h}{v} \chi_{(0,a)} \right\|_{X'}^{-1}
$$

and

$$\left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'}\leq\left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_{Y}^{-1}.$$

Hence,

$$\sup_{a\leq x\leq b}\Psi(x)$$

$$\leq\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1}\left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_{Y}^{-1}+\left\|\chi_{(0,a)}\right\|_{Y}^{-1}\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1}\left\|\chi_{[a,b]}\right\|_{Y}\left\|\frac{\bar{u}}{H}\chi_{[b,\infty)}\right\|_{Y}^{-1}+1.$$

Because all the weights are positive λ-a.e. on $(0,\infty)$ and locally integrable, the expression on the right hand side of the above inequality is finite.

Concerning the interval (b,∞), we proceed as follows:

$$\begin{aligned}\sup_{b<x<\infty}\Psi(x)&=\sup_{b<x<\infty}\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,b]}+\frac{\bar{u}(x)}{H(x)}\chi_{(b,x)}+\frac{\bar{u}}{H}\chi_{[x,\infty)}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\&\leq\sup_{b<x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\&\quad+\sup_{b<x<\infty}\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(b,x)}+\frac{\bar{u}}{H}\chi_{[x,\infty)}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}.\end{aligned}$$

The expression $\|\chi_{(0,b]}\|_Y$ makes sense because $\chi_{(0,b]}=\chi_{(0,a)}+\chi_{[a,b]}$ and $\chi_{(0,a)}\in Y$ according to (4.9) and $\chi_{[a,b]}\in Y$ as $[a,b]$ is compact. The latter term of the above estimate is exactly the formula from the left hand side of (4.10), therefore it is less than or equal to one. To deal with the first summand, pick an arbitrary $c\in(b,\infty)$. Since w is positive λ-a.e. on $(0,\infty)$ and locally integrable, there is a constant $0<L<\infty$, such that

$$\|\chi_{(0,b]}\|_Y\leq L\|\chi_{(b,c)}\|_Y.$$

Using this fact, we arrive at

$$\begin{aligned}&\sup_{b<x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\\&=\max\left\{\sup_{b<x<c}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'},\sup_{c\leq x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\&\leq\max\left\{\frac{\bar{u}(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},L\sup_{c\leq x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(b,c)}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\&\leq\max\left\{\frac{\bar{u}(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},L\sup_{c\leq x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(b,x)}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\&\leq\max\left\{\frac{\bar{u}(b)}{H(b)}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,c)}\right\|_{X'},L\sup_{b\leq x<\infty}\frac{\bar{u}(x)}{H(x)}\left\|\chi_{(b,x)}\right\|_{Y}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{X'}\right\}\\&\leq\max\left\{\left\|\chi_{(0,a)}\right\|_{Y}^{-1}\left\|\frac{h}{v}\chi_{(0,a)}\right\|_{X'}^{-1}\left\|\chi_{(0,b]}\right\|_{Y}\left\|\frac{\bar{u}(c)}{H(c)}\chi_{(b,c)}+\frac{\bar{u}}{H}\chi_{[c,\infty)}\right\|_{Y}^{-1},L\right\},\end{aligned}$$

where the last inequality is derived from (4.10), the monotonicity of $\frac{\bar{u}}{H}$ and the estimate for $\frac{\bar{u}(a)}{H(a)}$ carried out above. Again, as we suppose that weights are positive λ-a.e. on $(0,\infty)$ and $\chi_{(0,b]} \in Y$, the maximum, which we focus on, is finite. Thus also $\sup\limits_{b<x<\infty} \Psi(x) < \infty$. This, finally, shows (4.8), as desired. □

Now we are in a position to present a characterization of the compactness of $T_{u,h}$ from $X = X(v)$ to $Y = Y(w)$ for any couple (X, Y) in the category $\mathbb{M}(T_{u,h})$.

THEOREM 4.6. – *Let v, w, u, h be weights such that the function H from* (4.1) *satisfies $H(t) < \infty$ for every $t \in (0,\infty)$, and let the operator $T_{u,h}$ be defined by* (4.2)*. Let $X = X(v)$ and $Y = Y(w)$ be weighted Banach function spaces, such that $(X, Y) \in \mathbb{M}(T_{u,h})$ and $Y = Y_a$, and let them be equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Then $T_{u,h}$ is a compact operator from X into Y if and only if both of the following conditions are satisfied:*

$$\lim_{a\to 0_+} \sup_{0<x<a} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,a)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} = 0 \tag{4.11}$$

and

$$\lim_{b\to \infty} \sup_{b<x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} = 0. \tag{4.12}$$

PROOF. – *Necessity:* For contradiction, suppose that $T_{u,h}$ is a compact operator from X to Y, but (4.11) is not true. Then there exist $\varepsilon > 0$, a decreasing sequence $\{a_n\} \subset (0,\infty)$ with $\lim\limits_{n\to\infty} a_n = 0$ and points $x_n \in (0, a_n)$, such that

$$\left\| \frac{\bar{u}(x_n)}{H(x_n)}\chi_{(0,x_n)} + \frac{\bar{u}}{H}\chi_{[x_n,a_n)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x_n)} \right\|_{X'} > \varepsilon.$$

From the definition of the associate norm and by the absolute continuity of the Lebesgue integral, there exist a sequence $\{f_n\} \subset X$ with $\|f_n\|_X \le 1$ and numbers $\beta_n \in (0, x_n)$ satisfying

$$\int_{\beta_n}^{x_n} |f_n(s)|h(s)ds > \frac{1}{2}\int_0^{x_n} |f_n(s)|h(s)ds > \frac{1}{4}\left\| \frac{h}{v}\chi_{(0,x_n)} \right\|_{X'}.$$

Define the functions $F_n = f_n\chi_{(\beta_n,x_n)}$. Clearly, the lattice property of X gives that these functions lie in the closed unit ball of X. Since $\{a_n\}$ is decreasing, for every $n \in \mathbb{N}$ we can find $m_0 \in \mathbb{N}$, such that for every $m \ge m_0$ the inequalities $a_m < \beta_n$ and $\|\chi_{(0,a_m)}T_{u,h}F_n\|_Y < \frac{1}{8}\varepsilon$ hold. The latter inequality is guaranteed by the absolute continuity of the norm of the function $T_{u,h}F_n$, which follows from the assumptions that $T_{u,h} : X \to Y$ and $Y = Y_a$. Now, for $m \geq m_0$ and $t \geq x_m$, we get

$$\begin{aligned}
(T_{u,h}F_m)(t) &= \sup_{t\leq\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f_m(s)|\chi_{(\beta_m,x_m)}(s)h(s)ds \\
&= \sup_{t\leq\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_{\beta_m}^{x_m} |f_m(s)|h(s)ds \\
&= \frac{\bar{u}(t)}{H(t)} \int_{\beta_m}^{x_m} |f_m(s)|h(s)ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|T_{u,h}F_m - T_{u,h}F_n\|_Y \\
&\quad \geq \left\|\chi_{(0,a_m)}(T_{u,h}F_m - T_{u,h}F_n)\right\|_Y \\
&\quad \geq \left\|\chi_{(0,a_m)}T_{u,h}F_m\right\|_Y - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad \geq \left\|\chi_{(0,x_m)}T_{u,h}F_m(x_m) + \chi_{[x_m,a_m)}T_{u,h}F_m\right\|_Y \\
&\qquad - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad = \left\|\chi_{(0,x_m)}\frac{\bar{u}(x_m)}{H(x_m)}\int_{\beta_m}^{x_m} |f_m(s)|h(s)ds + \chi_{[x_m,a_m)}\frac{\bar{u}}{H}\int_{\beta_m}^{x_m} |f_m(s)|h(s)ds\right\|_Y \\
&\qquad - \left\|\chi_{(0,a_m)}T_{u,h}F_n\right\|_Y \\
&\quad \geq \frac{1}{4}\left\|\frac{\bar{u}(x_m)}{H(x_m)}\chi_{(0,x_m)} + \frac{\bar{u}}{H}\chi_{[x_m,a_m)}\right\|_Y \left\|\frac{h}{v}\chi_{(0,x_m)}\right\|_{X'} - \frac{1}{8}\varepsilon \\
&\quad \geq \frac{1}{8}\varepsilon > 0.
\end{aligned}$$

We have used the fact that $T_{u,h}$ maps X into the class of non-increasing functions and the definition of $\{F_n\}$, a_m and x_m. So, we have found the sequence $\{F_n\} \subset \{f \in X; \|f\|_X \leq 1\}$ such that none of the subsequences of $\{T_{u,h}F_n\}$ can be Cauchy, thus neither convergent in Y. This is a contradiction with the compactness of $T_{u,h}$.

The proof of the necessity of (4.12) is analogous.

Sufficiency: Given an interval $I \subset (0,\infty)$, set

$$(T_{u_I,h}f)(t) = \sup_{t\leq\tau<\infty} \frac{u(\tau)\chi_I(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds, \quad f \in X, t \in (0,\infty).$$

Observe that for $0<a<b<\infty, f \in X$ with $\|f\|_X \le 1$ and $t \in (0,\infty)$ we have

$$(T^{(0,a)}_{u_{[a,\infty)},h}f)(t) + (T^{[a,b]}_{u,h}f)(t) \le (T_{u,h}f)(t) \le (T^{(0,a)}_{u_{(0,a)},h}f)(t) + (T^{(0,a)}_{u_{[a,\infty)},h}f)(t) + (T^{[a,b]}_{u,h}f)(t) + (T^{(b,\infty)}_{u,h}f)(t).$$

So,

$$0 \le T_{u,h}f - T^{(0,a)}_{u_{[a,\infty)},h}f - T^{[a,b]}_{u,h}f \le T^{(0,a)}_{u_{(0,a)},h}f + T^{(b,\infty)}_{u,h}f$$

(a pointwise inequality). In order to establish the compactness of the operator $T_{u,h}$, we shall prove that for an appropriate choice of a and b, the function on the right hand side of the inequality lies in Y and has small norm and that the mapping $T^{(0,a)}_{u_{[a,\infty)},h} + T^{[a,b]}_{u,h}$ is, under our assumptions, a compact operator from X to Y.

Condition (4.11) guarantees for each $\varepsilon > 0$ the existence of $a \in (0,\infty)$ such that

$$\sup_{0<x\le a} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}}{H}\chi_{[x,a)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} < \varepsilon. \tag{4.13}$$

Hence,

$$\sup_{0<x\le a} \left\| \frac{\bar{u}_{(0,a)}(x)}{H(x)}\chi_{(0,x)} + \frac{\bar{u}_{(0,a)}}{H}\chi_{[x,a)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} < \varepsilon,$$

since the function in the norm of Y is at each point less than or equal to the function standing ibidem in (4.13). Because the pair $(X(v), Y(w))$ belongs to the category $\mathbb{M}(T_{u,h})$, the operator $T^{(0,a)}_{u_{(0,a)},h} = T^{(0,a)}_{u,h,(0,a)} : X \to Y$ is bounded and $\sup\{\|T^{(0,a)}_{u,h,(0,a)}f\|_Y; f \in X, \|f\|_X \le 1\} \le K\varepsilon$, where $K \ge 1$ is a constant independent of v, w, u, h and a.

By (4.12), for a given $\varepsilon > 0$ we find $b \in (a,\infty)$ such that

$$\sup_{b\le x<\infty} \left\| \frac{\bar{u}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(0,x)} \right\|_{X'} < \varepsilon, \tag{4.14}$$

where a is from the previous paragraph and corresponds to ε. Then also

$$\sup_{b\le x<\infty} \left\| \frac{\bar{u}_{(b,\infty)}(x)}{H(x)}\chi_{(b,x)} + \frac{\bar{u}_{(b,\infty)}}{H}\chi_{[x,\infty)} \right\|_Y \left\| \frac{h}{v}\chi_{(b,x)} \right\|_{X'} < \varepsilon. \tag{4.15}$$

Therefore $T^{(b,\infty)}_{u,h,(b,\infty)} : X \to Y$ is a bounded operator from X to Y and

$$\sup\{\|T^{(b,\infty)}_{u,h,(b,\infty)}f\|_Y; f \in X, \|f\|_X \le 1\} \le K\varepsilon,$$

where $K \ge 1$ is a constant independent of v, w, u, h and b. For $f \in X$ with

$\|f\|_X \le 1$ and $t \in (0,\infty)$, we estimate

$$(T_{u,h}^{(b,\infty)} f)(t) \le \chi_{(b,\infty)}(t)\frac{\bar{u}(t)}{H(t)}\int_0^b |f(s)|h(s)ds + (T_{u,h,(b,\infty)}^{(b,\infty)} f)(t)$$

$$\le \chi_{(b,\infty)}(t)\frac{\bar{u}(t)}{H(t)}\left\|\frac{h}{v}\chi_{(0,b)}\right\|_{X'} + (T_{u,h,(b,\infty)}^{(b,\infty)} f)(t).$$

With reference to (4.14) and (4.15), the function given at $t \in (0,\infty)$ by the expression on the right hand side is an element of Y. In agreement with the lattice property of Y, so is the function defined on the left and

$$\left\| T_{u,h}^{(b,\infty)} f\right\|_Y \le (1+K)\varepsilon.$$

To summarize our achievements so far, we have found $a \in (0,\infty)$ and $b \in (0,\infty)$ corresponding to a given $\varepsilon > 0$, such that $a < b$ and, for any $f \in X$ with $\|f\|_X \le 1$, the function

$$T_{u,h}f - T_{u_{[a,\infty)},h}^{(0,a)} f - T_{u,h}^{[a,b]} f$$

falls into Y and

$$\left\| T_{u,h}f - T_{u_{[a,\infty)},h}^{(0,a)} f - T_{u,h}^{[a,b]} f\right\|_Y < C\varepsilon,$$

where $C > 0$ is a constant independent of v, w, u, h, a and b.

Now, we are left with the proof of the statement that the mapping

$$T_{u_{[a,\infty)},h}^{(0,a)} + T_{u,h}^{[a,b]}$$

is a compact operator from X to Y.

The function $\chi_{(0,a)}$ is in Y, because, according to (4.13), $\frac{\bar{u}(a)}{H(a)}\chi_{(0,a)}$ is in Y. For any $f \in X$ and $t \in (0,a)$, we can write

$$(T_{u_{[a,\infty)},h} f)(t) = (T_{u,h}f)(a) \le \|\chi_{(0,a)}\|_Y^{-1} \|\chi_{(0,a)}(T_{u,h}f)(a)\|_Y$$

$$\le \|\chi_{(0,a)}\|_Y^{-1} \|T_{u,h}f\|_Y \le \|\chi_{(0,a)}\|_Y^{-1} c(T_{u,h})\|f\|_X.$$

Here, $c(T_{u,h}) > 0$ is a constant satisfying $\|T_{u,h}f\|_Y \le c(T_{u,h})\|f\|_X$ for every $f \in X$. This conclusion is based on the monotonicity of the function $T_{u,h}f$ and the boundedness of the operator $T_{u,h} : X \to Y$ following from the assumption that $(X,Y) \in \mathbb{M}(T_{u,h})$ and from Lemma 4.5 in combination with (4.11) and (4.12). Clearly, the expression standing before $\|f\|_X$ at the end of the formula is, due to the fact that w is positive λ-a.e., a positive and finite constant independent of f. We obtained that the mapping $T_{u_{[a,\infty)},h}^{(0,a)}$ is a bounded finite rank, hence compact, operator from X to Y.

Since the interval $[a, b]$ is compact, the function $\chi_{[a,b]}$ belongs to Y. Further, we use the boundedness of the operator $T_{u,h} : X \to Y$ again and the lattice property of Y to arrive at the observation that $T_{u,h}^{[a,b]}$ is a bounded operator from X to Y. Obviously, the image of each function f is non-negative on $(0, \infty)$ and non-increasing on $[a, b]$. Since the features of the operator $T_{u,h}^{[a,b]}$ meet the requirements imposed on operators in formulation of Theorem 3.1, to show the compactness of this operator, we can apply the method which we used in the proof of Theorem 3.1 after we had restricted the problem to an interval $[a, b]$. To be concrete, thanks to the assumption that $Y = Y_a$, for an arbitrary $\eta > 0$ we find a decomposition $a = \alpha_0 < \alpha_1 \ldots < \alpha_n = b$ such that $\|\chi_{[\alpha_{i-1},\alpha_i]}\|_Y < \eta$ for each $i \in \{1, \ldots, n\}$. Set $I_i = [\alpha_{i-1}, \alpha_i)$ for $i \in \{1, \ldots, n-1\}$ and $I_n = [\alpha_{n-1}, \alpha_n]$. Define

$$(Sf)(t) = \sum_{i=1}^{n} (T_{u,h}f)(\alpha_i)\chi_{I_i}(t), \quad f \in X,\, t \in (0, \infty).$$

Then $S : X \to Y$ is a compact operator (for more details see the proof of Theorem 3.1) and via the same process as in (3.1), used for appropriate operators, we obtain

$$\sup_{\|f\|_X \le 1} \left\| T_{u,h}^{[a,b]} f - Sf \right\|_Y \le \eta (T_{u,h}f)(a) \le \eta \left\|\chi_{(0,a)}\right\|_Y^{-1} c(T_{u,h}),$$

where the constant $c(T_{u,h}) > 0$ satisfies $\| T_{u,h}f \|_Y \le c(T_{u,h})\|f\|_X$ for every $f \in X$. So, the operator $T_{u,h}^{[a,b]}$ is compact from X to Y.

To conclude, note that we have shown that the mapping $T^{(0,a)}_{u_{[a,\infty)},h} + T_{u,h}^{[a,b]}$ is a compact operator from X to Y, roughly speaking, close to the operator $T_{u,h}$. This gives the compactness of the operator $T_{u,h}$, as desired. □

REMARK 4.7. – The statement of Theorem 4.6 remains true if we replace the condition $Y = Y_a$ with either one of the following assumptions:

(a) $T_{u,h}(X) \subset Y_a$;
(b) the weight w satisfies $\int_0^x w(s)ds < \infty$ for each $x \in (0, \infty)$, $Y_a = Y_b$ and $\lim_{x \to \infty} \| \frac{\bar{u}}{H} \chi_{[x,\infty)} \|_Y = 0$.

The proof can be carried out along the same lines as that of Theorem 4.6, therefore it is omitted.

5. – Compactness of operators involving suprema on weighted Lebesgue spaces

In this final section, we will consider weights u, h on $(0, \infty)$ such that $H(t) < \infty$ for every $t \in (0, \infty)$ and u is continuous on $(0, \infty)$, and the operators $T_{u,h}$ and

$T_{u,h,I}$ defined by (4.2) and (4.3), respectively. We shall characterize the weights v and w for which $T_{u,h}$ is a compact operator from $X(v)$ to $Y(w)$ in the special case when X and Y are Lebesgue spaces. We shall assume that $\int_0^x v(t)dt<\infty$ and $\int_0^x w(t)dt<\infty$ for every $x\in(0,\infty)$. Under such circumstances, the boundedness of $T_{u,h}$ from $L^p(v)$, $1\le p<\infty$, into $L^q(w)$, $0<q<\infty$, was studied in [8]. In [8, Theorem 4.2], it was showed that for $p\le q$, a mapping $T_{u,h}$ is a bounded operator from $L^p(v)$ to $L^q(w)$ if and only if

$$\sup_{0<x<\infty}\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,x)}+\frac{\bar{u}}{H}\chi_{[x,\infty)}\right\|_{q,w}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{p',v}<\infty,$$

and that

$$\begin{aligned}\sup_{0<x<\infty}&\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,x)}+\frac{\bar{u}}{H}\chi_{[x,\infty)}\right\|_{q,w}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{p',v}\\&\le\sup\{\|T_{u,h}f\|_{q,w};f\in L^p(v),\|f\|_{p,v}\le 1\}\\&\le c(p,q)\sup_{0<x<\infty}\left\|\frac{\bar{u}(x)}{H(x)}\chi_{(0,x)}+\frac{\bar{u}}{H}\chi_{[x,\infty)}\right\|_{q,w}\left\|\frac{h}{v}\chi_{(0,x)}\right\|_{p',v}.\end{aligned}$$

The method of the proof however works equally well for the mapping $T^I_{u,h,I}$, where $I\subset(0,\infty)$ is any open interval. This is so since we have no requirements on the integrability of weights over $(0,\infty)$ and that a weight is surely positive and finite λ-a.e. on I and integrable at the left endpoint of I. If $I\subset(0,\infty)$ is such that one or both of its endpoints belong to I, both the equivalence (4.4) to the boundedness of $T^I_{u,h,I}:L^p(v)\to L^q(w)$ and the inequality (4.5) follow from the continuity of the Lebesgue integral and the continuity of u. So, we have $(L^p(v),L^q(w))\in\mathbb{M}(T_{u,h})$ for $1\le p\le q\le\infty$. Hence, the case $p\le q$ in the main theorem of this section is covered by Theorem 4.6. Nevertheless, Theorem 4.6 does not answer the question for $q<p$. We shall bring a complete characterization of the compactness of $T_{u,h}$ from $L^p(v)$ to $L^q(w)$ for any $1\le p,q<\infty$. The point of departure is the result about boundedness introduced in [8].

DEFINITION 5.1. – *Let $I\in\mathbb{Z}\cup\{-\infty\}$ and $J\in\mathbb{Z}\cup\{\infty\}$, $J\ge I$. An increasing sequence $\{x_k\}_{k=I}^{k=J}\subset[0,\infty]$ is called a* covering sequence *if $\lim_{k\to-\infty}x_k=0$ for $I=-\infty$, $x_I=0$ for $I\in\mathbb{Z}$, $\lim_{k\to\infty}x_k=\infty$ for $J=\infty$ and $x_J=\infty$ for $J\in\mathbb{Z}$.*

Consider $a\in(0,\infty)$, $I\in\mathbb{N}\cup\{0\}$ and $J\in\mathbb{N}\cup\{\infty\}$, $J\ge I$. We say that $\{x_k\}_{k=I}^{k=J}\subset[0,a]$ is a *sequence convenient for the interval* $[0,a]$ if it is a decreasing sequence satisfying $x_I=a$, $x_J=0$ for $J\in\mathbb{N}$ and $\lim_{k\to\infty}x_k=0$ for $J=\infty$.

NOTATION 5.2. – For $1 \leq p < \infty$, $0 \leq \alpha < \beta \leq \infty$ and weights h, v, we denote

$$\sigma_{p,h}(\alpha,\beta) = \begin{cases} \left(\int\limits_{\alpha}^{\beta} [v(s)]^{1-p'} [h(s)]^{p'} ds \right)^{\frac{1}{p'}} & \text{when } 1 < p < \infty, \\ \operatorname*{ess\,sup}\limits_{\alpha < s < \beta} \dfrac{h(s)}{v(s)} & \text{when } p = 1. \end{cases}$$

The symbol $\sigma_{p,h}(\alpha,\beta)$ does not reflect its dependence on v, but we shall use it only in context with a fixed v, where no confusion should occur.

If not otherwise stated, we stick to definitions and notation from Section 4.

LEMMA 5.3. – *Let $1 \leq p, q < \infty$, $q < p$, and let u, h, v, w be weights, such that u is continuous on $(0,\infty)$ and $\int\limits_0^x h(t)dt < \infty$, $\int\limits_0^x v(t)dt < \infty$, $\int\limits_0^x w(t)dt < \infty$ for every $x \in (0,\infty)$. Define r by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Suppose*

$$\sup_{\{x_k\}} \left(\sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k-1},x_k)\right]^r \right)^{\frac{1}{r}} < \infty, \tag{5.1}$$

where the supremum is taken over all covering sequences $\{x_k\}$. Then for every $\varepsilon > 0$ there exist $a \in (0,\infty)$ and $b \in (0,\infty)$, such that $a < b$ and

$$\sup_{\{x_k\}} \left(\sum_k \left(\int\limits_{x_{k+1}}^{x_{k-1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1},x_k)\right]^r \right)^{\frac{1}{r}} < \varepsilon, \tag{5.2}$$

where the supremum is taken over all sequences $\{x_k\}$ convenient for the interval $[0,a]$, and

$$\sup_{\{x_k\}} \left(\sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k-1},x_k)\right]^r \right)^{\frac{1}{r}} < \varepsilon, \tag{5.3}$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k=I}^{k=J}$ with $x_I = b$ for $I \in \mathbb{Z}$ or $\lim\limits_{k\to-\infty} x_k = b$ for $I = -\infty$ and $x_J = \infty$ for $J \in \mathbb{N}$ or $\lim\limits_{k\to\infty} x_k = \infty$ for $J = \infty$.

PROOF. – If (5.2) was not true, there would exist an $\tilde{\varepsilon} > 0$ such that, for each $0 < x < \infty$,

$$\sup_{\{x_k\}} \left(\sum_k \left(\int\limits_{x_{k+1}}^{x_{k-1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1},x_k)\right]^r \right)^{\frac{1}{r}} \geq \tilde{\varepsilon}, \tag{5.4}$$

where the supremum would be taken over all sequences $\{x_k\}$ convenient for the interval $[0,x]$. But then we would find a covering sequence $\{y_k\}$ for which

$$\sum_k \left(\int_{y_{k-1}}^{y_{k+1}} \min\left\{ \frac{\bar{u}(y_k)}{H(y_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(y_{k-1}, y_k)\right]^r = \infty,$$

and that would lead to a contradiction with (5.1). Indeed, here we perform the construction of such a covering sequence $\{y_k\}$. Set $x_1 = 1$. According to (5.4), there is a sequence $\{x_k^1\}_{k=1}^{k=J_1}$ convenient for the interval $[0, x_1]$ with the property

$$\sum_k \left(\int_{x_{k+1}^1}^{x_{k-1}^1} \min\left\{ \frac{\bar{u}(x_k^1)}{H(x_k^1)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1}^1, x_k^1)\right]^r \geq \left(\frac{\tilde{\varepsilon}}{2}\right)^r.$$

Take the smallest possible $K_1 \in \mathbb{N}$ such that $x_{K_1}^1 < \dfrac{1}{2}$ and

$$\sum_{k=2}^{K_1} \left(\int_{x_{k+1}^1}^{x_{k-1}^1} \min\left\{ \frac{\bar{u}(x_k^1)}{H(x_k^1)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1}^1, x_k^1)\right]^r > \frac{\tilde{\varepsilon}^r}{2^{r+1}}.$$

If $K_1 < J_1$, put $x_2 = x_{K_1}^1$. In the situation when $K_1 = J_1$, thus $x_{K_1}^1 = 0$, there must be some $x_2 \in \left(x_{K_1}^1, \min\left\{\frac{1}{2}, x_{K_1-1}^1\right\}\right)$, for which the inequality

$$\sum_{k=2}^{K_1-2} \left(\int_{x_{k+1}^1}^{x_{k-1}^1} \min\left\{ \frac{\bar{u}(x_k^1)}{H(x_k^1)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k+1}^1, x_k^1)\right]^r$$

$$+ \left(\int_{x_2}^{x_{K_1-2}^1} \min\left\{ \frac{\bar{u}(x_{K_1-1}^1)}{H(x_{K_1-1}^1)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_2, x_{K_1-1}^1)\right]^r > \frac{\tilde{\varepsilon}^r}{2^{r+1}}$$

holds. Define $y_0 = \infty$ and $y_k = x_{-k}^1$ for $k = -K_1+1, \ldots, -1$. Assume that we have already built a sequence $\{y_k\}_{k=n-\sum_{j=1}^n K_j}^0$ and that we know a point $x_{n+1} \in \left(0, \min\left\{\frac{1}{n+1}, x_{K_n-1}^n\right\}\right)$. Like in the case of $n=1$, we find a sequence $\{x_k^{n+1}\}_{k=1}^{k=J_{n+1}}$ convenient for the interval $[0, x_{n+1}]$, a natural number K_{n+1} and a point $x_{n+2} \in \left(0, \min\left\{\frac{1}{n+2}, x_{K_{n+1}-1}^{n+1}\right\}\right)$ satisfying

$$\sum_{k=2}^{K_{n+1}-2}\left(\int_{x_{k+1}^{n+1}}^{x_{k-1}^{n+1}}\min\left\{\frac{\bar u(x_k^{n+1})}{H(x_k^{n+1})},\frac{\bar u(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}\left[\sigma_{p,h}(x_{k+1}^{n+1},x_k^{n+1})\right]^r$$

$$+\left(\int_{x_{n+2}}^{x_{K_{n+1}-2}^{n+1}}\min\left\{\frac{\bar u(x_{K_{n+1}-1}^{n+1})}{H(x_{K_{n+1}-1}^{n+1})},\frac{\bar u(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}$$

$$\times\left[\sigma_{p,h}(x_{n+2},x_{K_{n+1}-1}^{n+1})\right]^r>\frac{\tilde\varepsilon^r}{2^{r+1}}.$$

We continue our construction of the required covering sequence by setting

$$y_{k+n-\sum_{j=1}^n K_j}=x_{-k}^{n+1}\text{ for }k=-K_{n+1}+1,\dots,-1.$$

This way we obtain a sequence $\{y_k\}_{k=-\infty}^0\subset(0,\infty]$, which is increasing with $y_0=\infty$ and $\lim_{k\to-\infty}y_k=0$. Furthermore,

$$\sum_k\left(\int_{y_{k-1}}^{y_{k+1}}\min\left\{\frac{\bar u(y_k)}{H(y_k)},\frac{\bar u(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}\left[\sigma_{p,h}(y_{k-1},y_k)\right]^r=\infty.$$

So, the described sequence $\{y_k\}$ is a covering sequence implementing a contradiction with (5.1). An analogous reasoning leads to (5.3).

THEOREM 5.4. – *Let $1\le p,q<\infty$ and assume that u, h, v, w are weights such that u is continuous on $(0,\infty)$ and $\int_0^x h(t)dt<\infty$, $\int_0^x v(t)dt<\infty$, $\int_0^x w(t)dt<\infty$ for every $x\in(0,\infty)$. We define the operator $T_{u,h}$ by (4.2).*

(i) *Let $p\le q$. Then $T_{u,h}$ is a compact operator from $L^p(v)$ to $L^q(w)$ if and only if both of the following conditions are satisfied:*

$$\lim_{a\to0_+}\sup_{0<x<a}\left(\left(\frac{\bar u(x)}{H(x)}\right)^q\int_0^x w(t)dt+\int_x^a\left(\frac{\bar u(t)}{H(t)}\right)^q w(t)dt\right)^{\frac1q}\sigma_{p,h}(0,x)=0\tag{5.5}$$

and

$$\lim_{b\to\infty}\sup_{b<x<\infty}\left(\left(\frac{\bar u(x)}{H(x)}\right)^q\int_b^x w(t)dt+\int_x^\infty\left(\frac{\bar u(t)}{H(t)}\right)^q w(t)dt\right)^{\frac1q}\sigma_{p,h}(0,x)=0.\tag{5.6}$$

(ii) *Let* $q<p$. *Define* r *by* $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$. *Then* $T_{u,h}$ *is a compact operator from* $L^p(v)$ *to* $L^q(w)$ *if and only if*

$$\sup_{\{x_k\}}\left(\sum_k\left(\int_{x_{k-1}}^{x_{k+1}}\min\left\{\frac{\bar{u}(x_k)}{H(x_k)},\frac{\bar{u}(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}\left[\sigma_{p,h}(x_{k-1},x_k)\right]^r\right)^{\frac{1}{r}}<\infty, \tag{5.7}$$

where the supremum is taken over all covering sequences $\{x_k\}$.

PROOF. – Part (i) is a direct consequence of an application of Theorem 4.6 to the couple $(L^p(v),L^q(w))$, as explained at the beginning of this section.

We shall show (ii). So, in the remainder consider only $q<p$.

Necessity: Since $T_{u,h}$, being a compact operator, is also bounded and formula (5.7) coincides with the condition equivalent to the boundedness of the operator $T_{u,h}$ (see [8, Theorem 4.2]), the necessity of condition (5.7) for the compactness of the operator $T_{u,h}$ is obvious.

Sufficiency: We divide the proof into three steps. In the first one we consider a slightly modified operator and show, roughly speaking, that all functions from the image of the unit ball of $L^p(v)$ are small near 0 in the sense of norm in $L^q(w)$. The second step is devoted to an analogous situation in which 0 is replaced by ∞. In the third step we derive the conclusion using Theorem 3.1.

STEP 1. – Let $\varepsilon>0$. By virtue of condition (5.7) and Lemma 5.3, there must be an $a\in(0,\infty)$ satisfying

$$\sup_{\{x_k\}}\left(\sum_k\left(\int_{x_{k+1}}^{x_{k-1}}\min\left\{\frac{\bar{u}(x_k)}{H(x_k)},\frac{\bar{u}(t)}{H(t)}\right\}^q w(t)dt\right)^{\frac{r}{q}}\left[\sigma_{p,h}(x_{k+1},x_k)\right]^r\right)^{\frac{1}{r}}<\varepsilon, \tag{5.8}$$

where the supremum is taken over all sequences $\{x_k\}$ convenient for the interval $[0,a]$. Because $\int_0^a w(t)dt<\infty$ and the Lebesgue integral is continuous, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ convenient for the interval $[0,a]$ such that $\int_{x_{k+1}}^{x_k} w(t)dt=2^{-k}\int_0^a w(t)dt$ for each $k\in\mathbb{N}$.

Taking $f \in L^p(v)$, we have

$$
\begin{aligned}
\left\| T^{(0,a)}_{u,h,(0,a)} f \right\|^q_{q,w} &= \int_0^a \left[\sup_{t \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&= \sum_{k=1}^\infty \int_{x_{k+1}}^{x_k} \left[\sup_{t \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&\le \sum_{k=1}^\infty \int_{x_{k+1}}^{x_k} \left[\sup_{x_{k+1} \le \tau < a} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q w(t) dt \\
&= \sum_{k=1}^\infty \max_{2 \le i \le k+1} \left\{ \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \right\} \\
&\qquad \times 2^{-k} \int_0^a w(t) dt \\
&\le \sum_{k=1}^\infty \sum_{i=2}^{k+1} \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q 2^{-k} \int_0^a w(t) dt \\
&= \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \sum_{k=i-1}^\infty 2^{-k} \int_0^a w(t) dt \\
&= \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q 2^{-i+2} \int_0^a w(t) dt \\
&= 4 \sum_{i=2}^\infty \left[\sup_{x_i \le \tau < x_{i-1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)| h(s) ds \right]^q \int_{x_{i+1}}^{x_i} w(t) dt \\
&\le 8 \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&\le c(q) \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_{z_{i+2}}^{z_i} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&\quad + c(q) \sum_{i=2}^\infty \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_{i+2}} |f(s)| h(s) ds \right]^q \int_{z_{i+2}}^{z_i} w(t) dt \\
&=: S_1^a + S_2^a,
\end{aligned}
$$

where $z_i \in [x_i, x_{i-1})$ and

$$\frac{u(z_i)}{H(z_i)}\int_0^{z_i} |f(s)|h(s)ds > \frac{1}{2}\sup_{x_i\le\tau<x_{i-1}} \frac{u(\tau)}{H(\tau)}\int_0^{\tau} |f(s)|h(s)ds,$$

for each $i \geq 2$.

Hölder's inequality yields

$$\int_{z_{i+2}}^{z_i} |f(s)|h(s)ds \le \sigma_{p,h}(z_{i+2}, z_i)\left(\int_{z_{i+2}}^{z_i} |f(s)|^p v(s)ds\right)^{\frac{1}{p}}.$$

In view of this, we have

$$S_1^a \le c(q)\sum_{i=2}^{\infty}\left[\frac{u(z_i)}{H(z_i)}\sigma_{p,h}(z_{i+2}, z_i)\left(\int_{z_{i+2}}^{z_i} |f(s)|^p v(s)ds\right)^{\frac{1}{p}}\right]^q \int_{z_{i+2}}^{z_i} w(t)dt.$$

Applying Hölder's inequality for sums with the exponents $\frac{p}{q}$ and $\frac{r}{q}$, we obtain

$$\begin{aligned}
S_1^a &\le c(q)\left(\sum_{i=2}^{\infty}\left(\frac{u(z_i)}{H(z_i)}\right)^r\left(\int_{z_{i+2}}^{z_i} w(t)dt\right)^{\frac{r}{q}}\left(\sigma_{p,h}(z_{i+2}, z_i)\right)^r\right)^{\frac{q}{r}} \\
&\quad \times \left(\sum_{i=2}^{\infty}\int_{z_{i+2}}^{z_i} |f(s)|^p v(s)ds\right)^{\frac{q}{p}} \\
&= c(q)\left(\sum_{i=2}^{\infty}\left(\int_{z_{i+2}}^{z_i}\left(\frac{u(z_i)}{H(z_i)}\right)^q w(t)dt\right)^{\frac{r}{q}}\left(\sigma_{p,h}(z_{i+2}, z_i)\right)^r\right)^{\frac{q}{r}} \\
&\quad \times \left(\sum_{i=2}^{\infty}\int_{z_{i+2}}^{z_i} |f(s)|^p v(s)ds\right)^{\frac{q}{p}}.
\end{aligned} \tag{5.9}$$

We can rewrite the first sum on the right hand side as follows:

$$\begin{aligned}
&\sum_{i=2}^{\infty}\left(\int_{z_{i+2}}^{z_i}\left(\frac{u(z_i)}{H(z_i)}\right)^q w(t)dt\right)^{\frac{r}{q}}\left(\sigma_{p,h}(z_{i+2}, z_i)\right)^r \\
&\qquad = \sum_{i=1}^{\infty}\left(\int_{z_{2i+2}}^{z_{2i}}\left(\frac{u(z_{2i})}{H(z_{2i})}\right)^q w(t)dt\right)^{\frac{r}{q}}\left(\sigma_{p,h}(z_{2i+2}, z_{2i})\right)^r \\
&\qquad\quad + \sum_{i=1}^{\infty}\left(\int_{z_{2i+3}}^{z_{2i+1}}\left(\frac{u(z_{2i+1})}{H(z_{2i+1})}\right)^q w(t)dt\right)^{\frac{r}{q}}\left(\sigma_{p,h}(z_{2i+3}, z_{2i+1})\right)^r.
\end{aligned}$$

We set $z_0 = a$ and $z_1 = a$. Then both the sequences $\{z_{2i}\}_{i=0}^{\infty}$ and $\{z_{2i+1}\}_{i=0}^{\infty}$ become convenient for the interval $[0, a]$. In addition to this,

$$\sum_{i=1}^{\infty} \left(\int_{z_{2i+2}}^{z_{2i}} \left(\frac{u(z_{2i})}{H(z_{2i})} \right)^q w(t)dt \right)^{\frac{r}{q}} \big(\sigma_{p,h}(z_{2i+2}, z_{2i})\big)^r$$

$$\leq \sum_{i=1}^{\infty} \left(\int_{z_{2i+2}}^{z_{2i-2}} \min\left\{ \frac{\bar{u}(z_{2i})}{H(z_{2i})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \big(\sigma_{p,h}(z_{2i+2}, z_{2i})\big)^r$$

and

$$\sum_{i=1}^{\infty} \left(\int_{z_{2i+3}}^{z_{2i+1}} \left(\frac{u(z_{2i+1})}{H(z_{2i+1})} \right)^q w(t)dt \right)^{\frac{r}{q}} \big(\sigma_{p,h}(z_{2i+3}, z_{2i+1})\big)^r$$

$$\leq \sum_{i=1}^{\infty} \left(\int_{z_{2i+3}}^{z_{2i-1}} \min\left\{ \frac{\bar{u}(z_{2i+1})}{H(z_{2i+1})}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \big(\sigma_{p,h}(z_{2i+3}, z_{2i+1})\big)^r,$$

where the inequalities are implied by the relation $u(t) \leq \bar{u}(t)$ for all $t \in (0, \infty)$ combined with the fact that function $\frac{\bar{u}}{H}$ is non-increasing. In view of the above, we return to (5.9) and use (5.8). We arrive at

$$S_1^a \leq c(p,q)\varepsilon^q \|f\|_{p,v}^q.$$

As for S_2^a, observe that

$$S_2^a \leq c(q) \sum_{i=2}^{\infty} \left[\frac{\bar{u}(z_i)}{H(z_i)} \int_0^{z_{i+2}} |f(s)|h(s)ds \right]^q \int_{z_{i+2}}^{z_i} w(t)dt$$

$$= c(q) \sum_{i=2}^{\infty} \int_{z_{i+2}}^{z_i} \left[\frac{\bar{u}(z_i)}{H(z_i)} \int_0^{z_{i+2}} |f(s)|h(s)ds \right]^q w(t)dt$$

$$\leq c(q) \sum_{i=2}^{\infty} \int_{z_{i+2}}^{z_i} \left[\frac{\bar{u}(t)}{H(t)} \int_0^{t} |f(s)|h(s)ds \right]^q w(t)dt$$

$$\leq c(q) \int_0^{a} \left[\frac{\bar{u}(t)}{H(t)} \int_0^{t} |f(s)|h(s)ds \right]^q w(t)dt.$$

Using [18, Theorem 3], (5.8) implies that there is a constant $c(p,q)$ such that

$$S_2^a \leq c(p,q)\varepsilon^q \left(\int_0^a |f(t)|^p v(t)dt \right)^{\frac{q}{p}} \leq c(p,q)\varepsilon^q \|f\|_{p,v}^q.$$

When we combine the estimates for S_1^a and S_2^a, we arrive at

$$\left\| T_{u,h,(0,a)}^{(0,a)} f \right\|_{q,w} \leq c(p,q)\varepsilon \|f\|_{p,v}.$$

STEP 2. – Let's start the study of the situation near ∞ provided that the weight w satisfies $\int_0^\infty w(t)dt < \infty$. In this case, for an arbitrary $\varepsilon > 0$ we can find $b \in (a, \infty)$, where a is from Step 1 and corresponds to ε, such that $\int_b^\infty w(t)dt < \varepsilon^q$ and $\int_0^b w(t)dt \geq 2^{-1} \int_0^\infty w(t)dt$. For $f \in L^p(v)$, then,

$$\begin{aligned}
\left\| T_{u,h}^{(b,\infty)} f \right\|_{q,w} &= \left(\int_b^\infty \left[T_{u,h} f(t) \right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \left(\int_b^\infty \left[T_{u,h} f(b) \right]^q w(t)dt \right)^{\frac{1}{q}} \\
&= \left(\int_b^\infty w(t)dt \right)^{\frac{1}{q}} \left(\int_0^b w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^b \left[T_{u,h} f(b) \right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \varepsilon 2^{\frac{1}{q}} \left(\int_0^\infty w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^b \left[T_{u,h} f(t) \right]^q w(t)dt \right)^{\frac{1}{q}} \\
&\leq \varepsilon 2^{\frac{1}{q}} \left(\int_0^\infty w(t)dt \right)^{-\frac{1}{q}} \left\| T_{u,h} f \right\|_{q,w}.
\end{aligned}$$

Since the operator $T_{u,h}$ is bounded by (5.7), we have

$$\left\| T_{u,h}^{(b,\infty)} f \right\|_{q,w} \leq c(q, w, T_{u,h}) \varepsilon \|f\|_{p,v},$$

for some constant $c(q, w, T_{u,h})$, which depends only on q, the L^1-norm of w and $c(T_{u,h})$, the latter being from

$$\left\| T_{u,h} f \right\|_{q,w} \leq c(T_{u,h}) \|f\|_{p,v}, \quad f \in L^p(v).$$

Now, suppose that $\int_0^\infty w(t)dt = \infty$ for the weight w. Take an arbitrary $\varepsilon > 0$ and fix a corresponding a from Step 1. Due to (5.7) and Lemma 5.3 we can find $b \in (a, \infty)$ such that

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{ \frac{\bar{u}(x_k)}{H(x_k)}, \frac{\bar{u}(t)}{H(t)} \right\}^q w(t)dt \right)^{\frac{r}{q}} \left[\sigma_{p,h}(x_{k-1}, x_k) \right]^r \right)^{\frac{1}{r}} < \varepsilon, \tag{5.10}$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k=I}^{k=J}$ with $x_I = b$ for $I \in \mathbb{Z}$ or $\lim\limits_{k\to-\infty} x_k = b$ for $I = \infty$ and $x_J = \infty$ for $J \in \mathbb{N}$ or $\lim\limits_{k\to\infty} x_k = \infty$ for $J = \infty$. There exists an increasing sequence $\{x_k\}_{k=-\infty}^{\infty}$ lying in the interval (b,∞), such that $\lim\limits_{k\to\infty} x_k = \infty$, $\lim\limits_{k\to-\infty} x_k = b$ and $\int\limits_{x_k}^{x_{k+1}} w(t)dt = 2^k$ for every $k \in \mathbb{Z}$.

Similarly to the interval $(0,a)$, for $f \in L^p(v)$ we obtain

$$
\begin{aligned}
\left\| T_{u,h}^{(b,\infty)} f \right\|_{q,w}^q &= \int_b^\infty \left[\sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q w(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_{x_k}^{x_{k+1}} \left[\sup_{t\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q w(t)dt \\
&\le \sum_{k=-\infty}^{\infty} \int_{x_k}^{x_{k+1}} \left[\sup_{x_k\le\tau<\infty} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q w(t)dt \\
&= \sum_{k=-\infty}^{\infty} \sup_{k\le i<\infty} \left\{ \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q \right\} 2^k \\
&\le \sum_{k=-\infty}^{\infty} \sum_{i=k}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q 2^k \\
&= \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q \sum_{k=-\infty}^{i} 2^k \\
&= \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q 2^{i+1} \\
&= 4 \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds \right]^q \int_{x_{i-1}}^{x_i} w(t)dt.
\end{aligned}
$$

Again, take $\{z_i\}_{i=-\infty}^{\infty}$, for which $z_i \in [x_i, x_{i+1})$ and

$$
\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)|h(s)ds > \frac{1}{2} \sup_{x_i\le\tau<x_{i+1}} \frac{u(\tau)}{H(\tau)} \int_0^\tau |f(s)|h(s)ds.
$$

Then,

$$\begin{aligned}\left\| T^{(b,\infty)}_{u,h} f \right\|^q_{q,w} &\le 8 \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_i} |f(s)|h(s)ds \right]^q \int_{z_{i-2}}^{z_i} w(t)dt \\ &\le c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_{z_{i-2}}^{z_i} |f(s)|h(s)ds \right]^q \int_{z_{i-2}}^{z_i} w(t)dt \\ &\quad + c(q) \sum_{i=-\infty}^{\infty} \left[\frac{u(z_i)}{H(z_i)} \int_0^{z_{i-2}} |f(s)|h(s)ds \right]^q \int_{z_{i-2}}^{z_i} w(t)dt =: S^b_1 + S^b_2.\end{aligned}$$

The estimate of S^b_1 and S^b_2 is analogous to that in STEP 1.

STEP 3. – Finally, we are in the position to verify the compactness of the operator $T_{u,h}$. Actually, we shall show that the set $\{T_{u,h} f; f \in L^p(v), \|f\|_{p,v} \le 1\}$ is of uniformly absolutely continuous norm in $L^q(w)$ and apply Theorem 3.1. Note that the operator $T_{u,h}$ and the spaces $L^p(v)$ and $L^q(w)$ fall into the setting of Theorem 3.1, due to the properties of $T_{u,h}$, whose boundedness from $L^p(v)$ to $L^q(w)$ was shown in [8, Theorem 4.2].

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of λ-measurable subsets of $(0,\infty)$ such that $\chi_{E_n} \to \chi_{\emptyset}$ λ-a.e. on $(0,\infty)$. Consider an arbitrary $\varepsilon > 0$. By the previous two steps, we are able to find $0 < a < b < \infty$ satisfying

$$\left\| T^{(0,a)}_{u,h,(0,a)} f \right\|_{q,w} \le \frac{\varepsilon}{4} \|f\|_{p,v}$$

and

$$\left\| T^{(b,\infty)}_{u,h} f \right\|_{q,w} \le \frac{\varepsilon}{4} \|f\|_{p,v}.$$

As $\int_0^b w(t)dt < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have

$$\left(\int_0^b \chi_{E_n}(t) w(t) dt \right)^{\frac{1}{q}} < \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{4c(T_{u,h})} \left(\int_0^a w(t)dt \right)^{\frac{1}{q}} \right\},$$

where $c(T_{u,h})$ denotes a constant from the inequality describing the boundedness of $T_{u,h}$, i.e.

$$\| T_{u,h} f \|_{q,w} \le c(T_{u,h}) \|f\|_{p,v}, \quad f \in L^p(v).$$

For $n \ge n_0$ and $f \in L^p(v)$ with $\|f\|_{p,v} \le 1$ we can write

$$\begin{aligned}\| \chi_{E_n} T_{u,h} f \|_{q,w} &\le \left\| \chi_{E_n} T^{(0,a)}_{u,h} f \right\|_{q,w} + \left\| \chi_{E_n} T^{[a,b]}_{u,h} f \right\|_{q,w} + \left\| \chi_{E_n} T^{(b,\infty)}_{u,h} f \right\|_{q,w} \\ &=: N_1 + N_2 + N_3.\end{aligned}$$

Now,

$$\begin{aligned}
N_1 &\le \left\| T^{(0,a)}_{u,h,(0,a)} f \right\|_{q,w} + \| \chi_{E_n} \chi_{(0,a)} T_{u,h} f(a) \|_{q,w} \\
&\le \frac{\varepsilon}{4} + \left(\int_0^a [T_{u,h} f(a)]^q w(t)dt \right)^{\frac{1}{q}} \left(\int_0^a w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt \right)^{\frac{1}{q}} \\
&\le \frac{\varepsilon}{4} + \| T_{u,h} f \|_{q,w} \left(\int_0^a w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt \right)^{\frac{1}{q}} \\
&\le \frac{\varepsilon}{4} + c(T_{u,h}) \| f \|_{p,v} \left(\int_0^a w(t)dt \right)^{-\frac{1}{q}} \left(\int_0^a \chi_{E_n} w(t)dt \right)^{\frac{1}{q}} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Concerning N_2, we use the monotonicity of the function $T_{u,h} f$ and the same estimate for $T_{u,h} f(a)$ as we used while treating the second term of the previous calculation and arrive at

$$\begin{aligned}
N_2 &= \left(\int_a^b [T_{u,h} f(t)]^q \chi_{E_n}(t) w(t)dt \right)^{\frac{1}{q}} \\
&\le \left(\int_a^b [T_{u,h} f(a)]^q \chi_{E_n}(t) w(t)dt \right)^{\frac{1}{q}} \\
&\le c(T_{u,h}) \left(\int_0^a w(t)dt \right)^{-\frac{1}{q}} \left(\int_a^b \chi_{E_n} w(t)dt \right)^{\frac{1}{q}} \\
&< \frac{\varepsilon}{4}.
\end{aligned}$$

Obviously,

$$N_3 \le \| \chi_{(b,\infty)} T_{u,h} f \|_{q,w} \le \frac{\varepsilon}{4}.$$

Altogether,

$$\| \chi_{E_n} T_{u,h} f \|_{q,w} < \varepsilon.$$

This finishes the proof of the fact that the set $\{T_{u,h} f; f \in L^P(v), \|f\|_{p,v} \le 1\}$ is of uniformly absolutely continuous norm in $L^q(w)$ and thus completes the whole proof of Theorem 5.4.

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