Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **6** (2013), n.1, p. 1–27.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2013_9_6_1_1_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2013.

Bollettino U. M. I.
(9) **VI** (2013), 1-27

Weighted Inertia-Dissipation-Energy Functionals for Semilinear Equations

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In memory of Professor E. Magenes

Abstract. – *We address a global-in-time variational approach to semilinear PDEs of either parabolic or hyperbolic type by means of the so-called Weighted Inertia-Dissipation-Energy (WIDE) functional. In particular, minimizers of the WIDE functional are proved to converge, up to subsequences, to weak solutions of the limiting PDE. This entails the possibility of reformulating the limiting differential problem in terms of convex minimization. The WIDE formalism can be used in order to discuss parameters asymptotics via Γ-convergence and is extended to some time-discrete situation as well.*

1. – Introduction

This paper is concerned with the classical semilinear PDE

$$\rho u_{tt} + \nu u_t - \Delta u + f(u) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

posed in a bounded smooth domain $\Omega \subset \mathbb{R}^d$ up to some reference time $T > 0$. Here, the *density* ρ and *viscosity* ν are nonnegative parameters such that $\rho + \nu > 0$, $f = F'$ is of polynomial growth, and F is λ-convex (see (2.2)). In particular, in the following we explicitly consider the case $\rho > 0$ and $\nu > 0$. Let us however mention that the theory includes the limiting cases of the semilinear wave equation ($\nu = 0$) and the semilinear heat equation ($\rho = 0$) as well. We complement equation (1.1) with homogeneous Dirichlet boundary conditions (for simplicity) and initial conditions $u(x, 0) = u^0(x)$, $\rho u_t(x, 0) = \rho u^1(x)$.

The aim of this paper is to discuss a global-in-time variational approach to (1.1). In particular, for all $\varepsilon > 0$ we shall be concerned with the functional

$$W_\varepsilon(u) = \int_0^T \int_\Omega \mathrm{e}^{-t/\varepsilon} \left(\frac{\varepsilon^2 \rho}{2} |u_{tt}|^2 + \frac{\varepsilon \nu}{2} |u_t|^2 + \frac{1}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x \, \mathrm{d}t.$$

The latter is called *Weighted Inertia-Dissipation-Energy* (WIDE) functional as it features the weighted sum of the *inertial* term $\rho |u_{tt}|^2/2$, the *dissipative* term $\nu |u_t|^2/2$, and the *energetic* term $|\nabla u|^2/2 + F(u)$.

The main result of the paper consists in showing that the WIDE functional W_ε is uniformly convex for ε small, hence admitting a unique minimizer u^ε with $u(x,0) = u^0(x)$, $\rho u_t(x,0) = \rho u^1(x)$, and that, up to not relabeled subsequences,

$$\lim_{\varepsilon \to 0} u^\varepsilon = u \quad \text{where } u \text{ solves (1.1).} \tag{1.2}$$

The interest of this perspective resides in the possibility of connecting the *difficult* semilinear PDE problem (1.1) with a comparably *easier* problem: the constrained minimization of the uniformly convex functional W_ε. This possibility provides a novel variational insight to the differential problem by opening the way to the application of the tools of the calculus of variations to (1.1). For instance, we recall that the functional W_ε admits a unique minimizer whereas no uniqueness is known for (1.1) under general nonlinearities f. In this regard, the WIDE functional approach can be expected to possibly serve as a *variational selection criterion* in some nonuniqueness situation. This possibility has been already checked for a specific ODE case in [11].

One has to mention that the WIDE variational program has already been successfully developed in [23, 25] for the semilinear wave case $\nu = 0$ and in [17] for the semilinear heat case $\rho = 0$. We aim here at combining the two techniques in order to deal with the mixed case of equation (1.1), namely for $\rho > 0$ *and* $\nu > 0$.

For semilinear wave equations $\nu = 0$, the convergence (1.2) corresponds to a conjecture by De Giorgi [8] which is originally stated for $T = \infty$. This conjecture has been checked positively by Serra & Tilli [23] and by the second author [25] (for $T < \infty$) following completely different approaches. Moreover, both for T finite and infinite, we have considered some ODE analogue of the De Giorgi conjecture in [11].

As for the semilinear parabolic case $\rho = 0$ one has to mention that the pioneering paper by Lions [12] as well as the monograph by Lions & Magenes [13]. As for the nonlinear case, one has to recall the paper by Ilmanen [9] where the WIDE approach is used in order to prove the existence and partial regularity of the so-called Brakke mean curvature flow of varifolds. In this regard, the reader is also referred to [24] for an application to mean curvature flow of cartesian surfaces. The general case of abstract gradient flows of λ-convex functionals is discussed in [17] in the Hilbertian setting and then in [21, 22] for curves of maximal slope in metric spaces. Results and applications to rate-independent dissipative systems have been presented by Mielke & Ortiz [15] and then extended and coupled with time-discretization in [16]. Two relaxation and scaling examples in Mechanics are provided by Conti & Ortiz [6], and some application to crack propagation is given by Larsen, Ortiz, & Richardson [10]. Moreover, the extension of the WIDE principle to doubly nonlinear parabolic equations is discussed in [1, 2, 3]. Eventually, a similar functional approach (with ε fixed though) has been considered by Lucia, Muratov, & Novaga in connection with traveling waves in reaction-diffusion-advection problems [14, 19, 20].

The aim of this paper is the extension of the analysis of [25] in order to take into account dissipative effects $\nu > 0$ as well. The outcome of this extension is a theory which is indeed independent of the character of equation (1.1), provided either ρ or ν is positive. This is a quite remarkable feature of the WIDE formalism which in principle could make it of use in relation with a significant range of evolution problems. We exploit this fact in Subsection 4.3 where the limits $\rho \to 0$ and $\nu \to 0$ are discussed by means of a Γ-convergence analysis. In order to illustrate the independence of the theory from the character of the equation we resolved in keeping track of the parameters ρ and ν throughout the analysis.

After having introduced the relevant notion of weak solution, in Section 2 we state our main convergence result Theorem 2.1. Section 3 is focusing on the well-posedness of the minimization problem for the WIDE functional W_ε (Theorem 3.1) as well as on the related Euler-Lagrange equation (Lemma 3.2). Combined with initial and boundary conditions, the latter corresponds to some weak form of the following

$$\varepsilon^2 \rho u_{tttt} - 2\varepsilon\rho u_{ttt} + (\rho - \varepsilon\nu) u_{tt} + \nu u_t - \Delta u + f(u) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

$$u(\cdot,0) = u^0(\,\cdot\,) \quad \text{in } \Omega, \qquad \rho u_t(\cdot,0) = \rho u^1(\,\cdot\,) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.4}$$

$$\varepsilon^2 \rho u_{tt}(\cdot,T) = 0, \qquad \varepsilon^2 \rho u_{ttt}(\cdot,T) = \varepsilon\nu u_t(\cdot,T) \quad \text{in } \Omega. \tag{1.5}$$

Clearly equation (1.1) is nothing but the formal limit in (1.3) for $\varepsilon \to 0$. In particular, the minimization of W_ε corresponds to an *elliptic regularization* in time of the original problem. Note that, as the above problem is of fourth order in time, the two extra *final conditions* (1.5) arise and, at all levels $\varepsilon > 0$, causality is lost. Owing to this fact, the convergence (1.2) is generally referred to as the *causal limit* for it results in restoring causality.

The proof of our main result is presented in Section 4 and rests upon the validity of an a priori estimate on the minimizers of the WIDE functional (Lemma 4.1). The proof of this estimate requires the discussion of some time-discrete version of the WIDE principle which might be also of independent interest. We develop such a discretization in Section 5.

2. – Main result

We shall start by recalling some assumptions and introducing our weak solution notion for problem (1.3)-(1.5). Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open, and Lipschitz domain and $f = F' \in C(\mathbb{R})$ be of polynomial growth. In particular, we ask for some constant $C > 0$ such that, for all $v \in \mathbb{R}$,

$$\frac{1}{C}|v|^p \le F(v) + C \quad \text{and} \quad |f(v)|^{p'} \le C(1 + |v|^p) \tag{2.1}$$

where $p \geq 2$ and $1/p + 1/p' = 1$. Moreover, we assume that F is λ-convex for some given $\lambda \in \mathbb{R}$, i.e.,

$$v \mapsto F(u) - \frac{\lambda}{2}|v|^2 \qquad \text{is convex.} \tag{2.2}$$

Equivalently, F is λ-convex if and only if

$$F(\theta u + (1-\theta)v) \leq \theta F(u) + (1-\theta)F(v) - \frac{\lambda}{2}\theta(1-\theta)|u-v|^2 \qquad \forall\, u,\, v \in \mathbb{R},\ 0 \leq \theta \leq 1.$$

Note that the growth assumptions in (2.1) imply that F has at most p-growth.

We define $H = \mathrm{L}^2(\Omega)$, $X = \mathrm{L}^p(\Omega)$, and $V = \mathrm{H}^1_0(\Omega)$ so that $V \subset H$ compactly and $X \subset H$ continuously and assume u^0, $u^1 \in V \cap X$. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing both on $V' \times V$ and $X' \times X$ and by $(\cdot, \cdot)$ the usual scalar product on H. Moreover, $|\cdot|$ denotes the modulus as well as the norm on H and $\|\cdot\|_B$ stands for the norm of the normed space B. We define the (*energy*) functional $E : V \cap X \to \mathbb{R}$ as

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega F(u)\,\mathrm{d}x$$

and the operators $A : V \to V'$ and $B : X \to X'$ as

$$\langle Au, v\rangle := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad \langle B(u), v\rangle := \int_\Omega f(u) v\,\mathrm{d}x$$

so that $\langle Au, u\rangle = \|u\|_V^2$ and $B(u) = f(u)$ almost everywhere. We have that

$$\mathrm{D}E = A + B : V \cap X \to V' + X'$$

being bounded. Finally, we introduce the spaces

$$\mathcal{U} := \mathrm{H}^1(0,T;H) \cap \mathrm{L}^2(0,T;V) \cap \mathrm{L}^p(0,T;X), \quad \mathcal{V} := \{u \in \mathcal{U} \ : \ \rho u \in \mathrm{H}^2(0,T;H)\}$$

and assume that B is weakly continuous, i.e., we have that

$$u_k \rightharpoonup u \text{ in } \mathcal{U} \implies B(u_k) \rightharpoonup B(u) \text{ in } \mathcal{U}'. \tag{2.3}$$

A choice for the function f fulfilling these assumptions is $f(u) = |u|^{p-2}u + \ell(u)$ where $\ell \in \mathrm{C}^{0,1}(\mathbb{R})$.

The above assumptions will be tacitly assumed throughout the remainder of the paper. We are now in position to present the main result of this text which is proved in Subsection 4.2.

THEOREM 2.1 (WIDE principle). – *Let u^ε minimize the WIDE functional W_ε on the nonempty and convex set $\mathcal{K}(u^0, u^1) := \{u \in \mathcal{V} : u(0) = u^0,\, \rho u_t(0) = \rho u^1\}$.*

Then, for some not relabeled subsequence we have that $u^\varepsilon \rightharpoonup u$ *in* $\mathcal{U}$, *where* u *solves*

$$\rho u_{tt} + \nu u_t + \mathrm{D}E(u) = 0 \quad a.e \ in\ (0,T), \quad u(0) = u^0,\ \rho u_t(0) = \rho u^1. \tag{2.4}$$

Before moving on let us stress that the latter result can be adapted in order to encompass more general situations. In particular, we can consider unbounded domains (see [25]) as well as different boundary conditions or the presence of additional source terms with no particular intricacy. Moreover, the WIDE approach can be applied to other classes of dissipative equations including. For instance, one could recast the WIDE principle for the *strongly damped* wave equation

$$\rho u_{tt} - \nu \Delta u_t - \Delta u + f(u) = 0,$$

suitably combined with boundary and initial conditions.

3. – Well-posedness of the minimum problem

Let us start by checking that indeed W_ε admits a unique minimizer in the set $\mathcal{K}(u^0, u^1)$. In the convex case

$$\lambda^- := \max\{0, -\lambda\} = 0$$

the existence of a (unique) minimizer is a direct consequence of the Direct Method. As for the general nonconvex case $\lambda^- > 0$, existence and uniqueness of minimizers follow by letting ε be small enough.

THEOREM 3.1 (Well-posedness of minimum problem). – *For* ε *small the* WIDE *functional* W_ε *is uniformly convex with respect to the metric of* $\mathrm{H}^1(0,T;H) \cap \mathrm{L}^2(0,T;V)$. *In particular,* W_ε *admits a unique minimizer* $u^\varepsilon \in \mathcal{K}(u^0, u^1)$.

PROOF. – As already mentioned the convex case $\lambda^- = 0$ is quite straightforward so let us assume from the very beginning that $\lambda^- > 0$ and decompose W_ε into the sum of a quadratic part Q_ε and a convex remainder R_ε as follows

$$\begin{aligned} W_\varepsilon(u) &= \int_0^T \frac{\mathrm{e}^{-t/\varepsilon}}{2}\left(\rho\varepsilon^2|u_{tt}|^2 + \nu\varepsilon|u_t|^2 + \|u\|_V^2 - \lambda^-|u|^2\right)\mathrm{d}t + \int_0^T\int_\Omega \mathrm{e}^{-t/\varepsilon} G(u)\,\mathrm{d}t\,\mathrm{d}x \\ &=: Q_\varepsilon(u) + R_\varepsilon(u) \end{aligned}$$

with $G(u) = F(u) + \lambda^-|u|^2/2$ convex by (2.2). In order to handle the quadratic part Q_ε, we will exploit the auxiliary function $v(t) := \mathrm{e}^{-t/(2\varepsilon)}u(t)$ and readily check that

$$\mathrm{e}^{-t/(2\varepsilon)}u_t(t) = v_t(t) + \frac{1}{2\varepsilon}v(t), \quad \mathrm{e}^{-t/(2\varepsilon)}u_{tt}(t) = v_{tt}(t) + \frac{1}{\varepsilon}v_t(t) + \frac{1}{4\varepsilon^2}v(t).$$

Note that, by possibly letting ε be small, standard computations ensure that

$$\mathrm{e}^{-T/\varepsilon}\|u\|^2_{\mathrm{L}^2(0,T;V)} \le \|v\|^2_{\mathrm{L}^2(0,T;V)} \le \|u\|^2_{\mathrm{L}^2(0,T;V)}, \tag{3.1}$$

$$\varepsilon^4\mathrm{e}^{-T/\varepsilon}\|u\|^2_{\mathrm{H}^2(0,T;H)} \le \|v\|^2_{\mathrm{H}^2(0,T;H)} \le \varepsilon^{-4}\|u\|^2_{\mathrm{H}^2(0,T;H)}. \tag{3.2}$$

Moreover, $Q_\varepsilon(u)$ can be rewritten in terms of v as

$$\begin{aligned} Q_\varepsilon(u) = &\int_0^T \left(\frac{\rho\varepsilon^2}{2}|v_{tt}|^2 + \frac{\rho+\varepsilon\nu}{2}|v_t|^2 + \frac{\rho+4\varepsilon\nu-16\varepsilon^2\lambda^-}{32\varepsilon^2}|v|^2 + \frac{1}{2}\|v\|^2_V\right)\mathrm{d}t \\ &+ \int_0^T \left(\rho\varepsilon(v_{tt},v_t) + \frac{\rho}{4}(v_{tt},v) + \frac{\rho+2\varepsilon\nu}{4\varepsilon}(v_t,v)\right)\mathrm{d}t \\ = &\int_0^T \left(\frac{\rho\varepsilon^2}{2}|v_{tt}|^2 + \frac{\rho+2\varepsilon\nu}{4}|v_t|^2 + \frac{\rho+4\varepsilon\nu-16\varepsilon^2\lambda^-}{32\varepsilon^2}|v|^2 + \frac{1}{2}\|v\|^2_V\right)\mathrm{d}t \\ &+ \frac{\rho\varepsilon}{2}\left(|v_t(T)|^2 - |v_t(0)|^2\right) + \frac{\rho}{4}\Big(\big(v_t(T),v(T)\big) - \big(v_t(0),v(0)\big)\Big) \\ &+ \frac{\rho+2\varepsilon\nu}{8\varepsilon}\left(|v(T)|^2 - |v(0)|^2\right) =: V_\varepsilon(v) + S_\varepsilon(v) \end{aligned} \tag{3.3}$$

where V_ε is the integral contribution whereas S_ε collects all boundary terms. By letting

$$\varepsilon < \max\left\{\sqrt{\frac{\rho}{16\lambda^-}}, \frac{\nu}{4\lambda^-}\right\}$$

the quadratic form V_ε is convex. Moreover, an elementary yet tedious calculation shows that for $u \in \mathcal{K}(u^0,u^1)$ we have

$$\begin{aligned} S_\varepsilon(v) = &\frac{3\varepsilon\rho}{8}\mathrm{e}^{-T/\varepsilon}|u_t(T)|^2 + \frac{\varepsilon\rho}{8}\mathrm{e}^{-T/\varepsilon}|u_t(T) - u(T)/\varepsilon|^2 + \frac{\nu}{4}\mathrm{e}^{-T/\varepsilon}|u(T)|^2 \\ &- \frac{3\varepsilon\rho}{8}|u^1|^2 - \frac{\varepsilon\rho}{8}|u^1 - u^0/\varepsilon|^2 - \frac{\nu}{4}|u^0|^2, \end{aligned}$$

which is convex in $u(T)$, $u_t(T)$ as well. Let now $\theta \in [0,1]$ and $u, \tilde{u} \in \mathcal{K}(u^0,u^1)$ be given. Moreover, define $v(t) := \mathrm{e}^{-t/(2\varepsilon)}u(t)$ and $\tilde{v}(t) := \mathrm{e}^{-t/(2\varepsilon)}\tilde{u}(t)$. For all ε small enough one deduces

$$
\begin{aligned}
&Q_\varepsilon\big(\theta u + (1-\theta)\tilde{u}\big) = V_\varepsilon\big(\theta v + (1-\theta)\tilde{v}\big) + S_\varepsilon\big(\theta v + (1-\theta)\tilde{v})\big) \\
&\leq \theta V_\varepsilon(v) + (1-\theta)V_\varepsilon(\tilde{v}) + \theta S_\varepsilon(v) + (1-\theta)S_\varepsilon(\tilde{v}) \\
&\quad - \frac{\theta(1-\theta)}{2}\int_0^T \rho\varepsilon^2|v_{tt}-\tilde{v}_{tt}|^2 + \frac{\rho+2\varepsilon\nu}{2}|v_t-\tilde{v}_t|^2 + \frac{\rho+4\varepsilon\nu-16\varepsilon^2\lambda^-}{16\varepsilon^2}|v-\tilde{v}|^2 + \|v-\tilde{v}\|_V^2\,\mathrm{d}t \\
&= \theta Q_\varepsilon(u) + (1-\theta)Q_\varepsilon(\tilde{u}) \\
&\quad - \frac{\theta(1-\theta)}{2}\int_0^T \rho\varepsilon^2|v_{tt}-\tilde{v}_{tt}|^2 + \frac{\rho+2\varepsilon\nu}{2}|v_t-\tilde{v}_t|^2 + \frac{\rho+4\varepsilon\nu-16\varepsilon^2\lambda^-}{16\varepsilon^2}|v-\tilde{v}|^2 + \|v-\tilde{v}\|_V^2\,\mathrm{d}t.
\end{aligned}
$$

By exploiting the first estimates in (3.1)-(3.2), we have proved that Q_ε is uniformly convex in with respect to the metric of $\mathrm{H}^1(0,T;H)\cap \mathrm{L}^2(0,T;V)$ (or even $\mathrm{H}^2(0,T;H)$ if $\rho>0$). As $W_\varepsilon = Q_\varepsilon + R_\varepsilon$ and R_ε is convex, the uniform convexity of W_ε and the existence of a unique minimizer $u^\varepsilon \in \mathcal{K}(u^0,u^1)$ ensues. □

3.1 – *Euler-Lagrange equation*

Our analysis, in particular the derivation of a priori estimates, relies on the specific structure of the Euler-Lagrange equation for W_ε. Let u^ε minimize W_ε in $\mathcal{K}(u^0,u^1)$. By considering $r \mapsto W_\varepsilon(u^\varepsilon + rv)$ for $v \in \mathcal{K}(0,0)$ we obtain that

$$
(3.4)\qquad 0 = \int_0^T \mathrm{e}^{-t/\varepsilon}\big(\varepsilon^2\rho(u^\varepsilon_{tt}, v_{tt}) + \varepsilon\nu(u^\varepsilon_t, v_t) + \langle \mathrm{D}E(u^\varepsilon), v\rangle\big)\,\mathrm{d}t \quad \forall\, v \in \mathcal{K}(0,0).
$$

Hence, we have the following.

LEMMA 3.2 (Euler-Lagrange equation). – *Let u^ε be the unique minimizer of the* WIDE *functional W_ε in $\mathcal{K}(u^0,u^1)$. Then, u^ε solves*

$$
(3.5)\qquad \varepsilon^2\rho u^\varepsilon_{tttt} - 2\varepsilon\rho u^\varepsilon_{ttt} + (\rho-\varepsilon\nu)u^\varepsilon_{tt} + \mathrm{D}E(u^\varepsilon) = 0 \quad \textit{a.e. in } (0,T),
$$

$$
(3.6)\qquad u^\varepsilon(0) = u^0,\ \rho u^\varepsilon_t(0) = \rho u^1,
$$

$$
(3.7)\qquad \rho u^\varepsilon_{tt}(T) = 0,\ \varepsilon^2\rho u^\varepsilon_{ttt}(T) - \varepsilon\nu u^\varepsilon_t(T) = 0.
$$

4. – Proof of the main result

The key step in the proof of Theorem 2.1 is to establish an integral energy estimate on u^ε which is independent of ε. Henceforth, the symbol C stands for any constant depending on data and independent of ρ, ν, and ε (and, later, τ) and possibly changing from line to line. We shall prove the following lemma.

LEMMA 4.1 (Estimate). – *Let u^ε minimize the* WIDE *functional W_ε over $\mathcal{K}(u^0, u^1)$. Then, we have*

$$(\rho + \nu)\int_0^T |u_t^\varepsilon|^2 \,\mathrm{d}t + \int_0^T E(u^\varepsilon)\,\mathrm{d}t \le C. \tag{4.1}$$

Note that, owing to the growth conditions (2.1), the latter estimate entails in particular that minimizers of W_ε on $\mathcal{K}(u^0, u^1)$ are uniformly bounded in $\mathcal{U}$. This provides the necessary compactness in order to prove our main result Theorem 2.1.

4.1 – *A formal argument*

Let us however provide here a formal argument toward Lemma 4.1. In particular, let us *assume* that the minimizers u^ε of the WIDE functional W_ε on the convex set $\mathcal{K}(u^0, u^1)$ are actually *strong solutions* of the Euler-Lagrange equation (3.5)-(3.7). Note that this smoothness assumption is presently not justified. In particular, the argument below will be made rigorous by means of a time-discretization technique in Section 5. Although quite technical, we believe this procedure to bear some interest in itself for it provides the complete analysis of a time-discrete version of the WIDE principle.

Let us focus on the (more difficult) case $\rho > 0$ only. Test (3.5) on the function $t \mapsto v(t) := (1 + T - t)(u_t^\varepsilon(t) - u^1)$ and take the integral on $(0, T)$. By recalling that

$$\forall g \in \mathrm{L}^1(0,T), \quad \int_0^T (1 + T - t) g(t)\,\mathrm{d}t = \int_0^T g(t)\,\mathrm{d}t + \int_0^T \left(\int_0^t g(s)\,\mathrm{d}s \right) \mathrm{d}t.$$

we easily compute that

$$\begin{aligned}
\int_0^T \varepsilon^2 \rho (u_{tttt}^\varepsilon, v)\,\mathrm{d}t &= \int_0^T \varepsilon^2 \rho (u_{tttt}^\varepsilon, u_t^\varepsilon - u^1)\,\mathrm{d}t + \int_0^T \int_0^t \varepsilon^2 \rho (u_{tttt}^\varepsilon, u_t^\varepsilon - u^1)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{(1+T)\varepsilon^2 \rho}{2} |u_{tt}^\varepsilon(0)|^2 - \frac{\varepsilon^2 \rho}{2} |u_{tt}(T)|^2 + \varepsilon^2 \rho (u_{ttt}^\varepsilon(T), u_t^\varepsilon(T) - u^1) \\
&\quad + \varepsilon^2 \rho (u_{tt}^\varepsilon(T), u_t^\varepsilon(T) - u^1) - \frac{3\varepsilon^2 \rho}{2} \int_0^T |u_{tt}^\varepsilon|^2 \,\mathrm{d}t
\end{aligned} \tag{4.2}$$

where we have used integration by parts several times. Analogously, we obtain

$$-\int_0^T 2\varepsilon\rho(u^\varepsilon_{ttt},v)\,\mathrm{d}t = 2\varepsilon\rho\int_0^T |u^\varepsilon_{tt}|^2\,\mathrm{d}t + 2\varepsilon\rho\int_0^T\int_0^t |u^\varepsilon_{tt}|^2\,\mathrm{d}s\,\mathrm{d}t$$
$$- 2\varepsilon\rho(u^\varepsilon_{tt}(T),u^\varepsilon_t(T)-u^1) - \varepsilon\rho|u^\varepsilon_t(T)-u^1|^2,$$

$$\int_0^T (\rho-\varepsilon\nu)(u^\varepsilon_{tt},v)\,\mathrm{d}t = \frac{\rho-\varepsilon\nu}{2}|u^\varepsilon_t(T)-u^1|^2 + \frac{\rho-\varepsilon\nu}{2}\int_0^T |u^\varepsilon_t-u^1|^2\,\mathrm{d}t.$$

Finally, we have that

$$\int_0^T \nu(u^\varepsilon_t,v)\,\mathrm{d}t = \nu\int_0^T |u^\varepsilon_t|^2\,\mathrm{d}t + \nu\int_0^T\int_0^t |u^\varepsilon_t|^2\,\mathrm{d}s\;\mathrm{d}t - \nu\int_0^T (u^\varepsilon_t,u^1)\,\mathrm{d}t$$
$$- \nu\int_0^T\int_0^t (u^\varepsilon_t,u^1)\,\mathrm{d}s\;\mathrm{d}t, \tag{4.3}$$

$$\int_0^T \langle \mathrm{D}E(u^\varepsilon),v\rangle = E(u^\varepsilon(T)) - (1+T)E(u^0) + \int_0^T E(u^\varepsilon)\,\mathrm{d}t - \int_0^T \langle \mathrm{D}E(u^\varepsilon),u^1\rangle\,\mathrm{d}t$$
$$- \int_0^T\int_0^t \langle \mathrm{D}E(u^\varepsilon),u^1\rangle\,\mathrm{d}s\,\mathrm{d}t. \tag{4.4}$$

Therefore, by summing up equations (4.2)-(4.4) and using the final conditions in (3.7) we come to the following

$$\frac{(1+T)\varepsilon^2\rho}{2}|u^\varepsilon_{tt}(0)|^2 + \Big(\frac{\rho-\varepsilon\nu}{2} - \varepsilon\rho\Big)|u^\varepsilon_t(T)-u^1|^2 + \nu\int_0^T |u^\varepsilon_t|^2\,\mathrm{d}t + 2\varepsilon\rho\int_0^T\int_0^t |u^\varepsilon_{tt}|^2\,\mathrm{d}s\,\mathrm{d}t \tag{4.5}$$
$$+ E(u^\varepsilon(T)) + \int_0^T E(u^\varepsilon)\,\mathrm{d}t + \rho\Big(2\varepsilon - \frac{3\varepsilon^2}{2}\Big)\int_0^T |u^\varepsilon_{tt}|^2\,\mathrm{d}t + \nu\int_0^T\int_0^t |u^\varepsilon_t|^2\,\mathrm{d}s\;\mathrm{d}t + \frac{\rho-\varepsilon\nu}{2}\int_0^T |u^\varepsilon_t-u^1|^2\,\mathrm{d}t$$
$$= (1+T)E(u^0) + \int_0^T \langle \mathrm{D}E(u^\varepsilon),u^1\rangle\,\mathrm{d}t + \int_0^T\int_0^t \langle \mathrm{D}E(u^\varepsilon),u^1\rangle\,\mathrm{d}s\;\mathrm{d}t$$
$$+ \nu\int_0^T (u^\varepsilon_t,u^1)\,\mathrm{d}t + \nu\int_0^T\int_0^t (u^\varepsilon_t,u^1)\,\mathrm{d}s\,\mathrm{d}t.$$

Hence, by using the growth conditions (2.1) and Young's inequality we have shown that for small ε estimate (4.1) holds.

4.2 – *Proof of Theorem 2.1*

Let us now come to the proof of our main result. Let u^ε be the unique minimizer of W_ε. Owing to Lemma 4.1 we can extract a not relabeled subsequence u^ε such that $u^\varepsilon \rightharpoonup u$ in $\mathcal{U}$. In order to check that u solves (2.4) let $w \in \mathrm{C}_0^\infty([0,T);V\cap X)$ be given and define $v^\varepsilon(t) := \mathrm{e}^{t/\varepsilon}w(t) - t(w_t(0)+w(0)/\varepsilon) - w(0)$ such that $v^\varepsilon \in \mathcal{K}(0,0)$. We have that

$$v_t^\varepsilon(t) = \mathrm{e}^{t/\varepsilon}w_t(t) + \frac{1}{\varepsilon}\mathrm{e}^{t/\varepsilon}w(t) - w_t(0) - \frac{1}{\varepsilon}w(0) \text{ and}$$
$$v_{tt}^\varepsilon(t) = \mathrm{e}^{t/\varepsilon}w_{tt}(t) + \frac{2}{\varepsilon}\mathrm{e}^{t/\varepsilon}w_t(t) + \frac{1}{\varepsilon^2}\mathrm{e}^{t/\varepsilon}w(t).$$

Since $v^\varepsilon \in \mathrm{C}_0^\infty([0,T);V\cap X) \subset \mathcal{K}(u^0,u^1)$, from the variational equality (3.4) one obtains

$$\begin{aligned}
0 &= \int_0^T \mathrm{e}^{-t/\varepsilon}\big(\varepsilon^2\rho(u_{tt}^\varepsilon, v_{tt}^\varepsilon) + \varepsilon\nu(u_t^\varepsilon, v_t^\varepsilon) + \langle \mathrm{D}E(u^\varepsilon), v^\varepsilon\rangle\big)\,\mathrm{d}t \\
&= \int_0^T \Big(\rho(u_{tt}^\varepsilon, \varepsilon^2 w_{tt} + 2\varepsilon w_t + w) + \nu(u_t^\varepsilon, \varepsilon w_t + w) - \varepsilon\nu(u_t^\varepsilon, h^\varepsilon) \\
&\qquad + \langle \mathrm{D}E(u^\varepsilon), w - th_\varepsilon - \mathrm{e}^{-t/\varepsilon}w(0)\rangle\Big)\,\mathrm{d}t,
\end{aligned}$$

where we have used the short-hand notation $h^\varepsilon(t) = \mathrm{e}^{-t/\varepsilon}(w_t(0)+w(0)/\varepsilon)$. Hence, by integration by parts we obtain

$$\begin{aligned}
&\int_0^T \big(-\rho(u_t^\varepsilon, w_t) + \nu(u_t^\varepsilon, w) + \langle \mathrm{D}E(u^\varepsilon), w\rangle\big)\,\mathrm{d}t + \rho(u_t^\varepsilon(0), w(0)) \\
&= \int_0^T \big(u_t^\varepsilon, \varepsilon^2\rho w_{ttt} + 2\varepsilon\rho w_{tt} + \varepsilon\nu w + \varepsilon\nu h_\varepsilon\big)\,\mathrm{d}t + \int_0^T \langle \mathrm{D}E(u^\varepsilon), th_\varepsilon + \mathrm{e}^{-t/\varepsilon}w(0))\rangle\,\mathrm{d}t \\
&- \rho\big(u_t^\varepsilon(0), \varepsilon^2 w_{tt}(0) + 2\varepsilon w_t(0)\big).
\end{aligned}$$

We easily check that $t \mapsto th_\varepsilon(t) + \mathrm{e}^{-t/\varepsilon}w(0)$ converges strongly in $\mathrm{L}^q(0,T;V\cap X)$ to 0 for every $q\in[1,\infty)$. In particular, letting $\varepsilon\to 0$ we exploit the weak continuity (2.3) and $u_t^\varepsilon(0) = u^1$ to obtain

$$\int_0^T \big(-\rho(u_t, w_t) + \nu(u_t, w) + \langle \mathrm{D}E(u), w\rangle\big)\,\mathrm{d}t + \rho(u^1, w(0)) = 0.$$

Namely, u solves the equation in (2.4), where u_{tt} makes sense in $\mathrm{L}^2(0,T;V') + \mathrm{L}^p(0,T;X')$. The initial condition $u(0) = u^0$ follows from the precompactness of u^ε in $\mathcal{U}$ while $u_t(0) = u^1$ follows from the weak formulation of the limit equation.

4.3 – *Γ-convergence*

As already mentioned, a remarkable trait of the WIDE approach is its independence of the character of the equation (1.1) as long as $\rho + \nu > 0$. In particular, the WIDE formalism is well-suited in order to describe limiting behaviors in the parameters. First of all, by inspecting the proof of Theorem 2.1 it is apparent that minimizers of the WIDE functional pass to limits $\rho \to 0$ and $\nu \to 0$ as well as to joint limits $(\rho, \varepsilon) \to (0, 0)$ and $(\nu, \varepsilon) \to (0, 0)$. On the other hand, by keeping ε fixed we can argue from a variational viewpoint by addressing the limits $\rho \to 0$ and $\nu \to 0$ within the frame of Γ-convergence [7].

Let us momentarily modify the notation for the WIDE functionals W_ε, the function space $\mathcal{V}$, and the set $\mathcal{K}$ by highlighting the dependence on the parameters ρ and ν as $W_\varepsilon^{\rho\nu}$, $\mathcal{V}^\rho$, and $\mathcal{K}^\rho(u^0, u^1)$, respectively. Moreover, for the sake of notational simplicity we incorporate the constraint $u \in \mathcal{K}^\rho(u^0, u^1)$ directly in the functional by letting

$$\overline{W}_\varepsilon^{\rho\nu} = \mathrm{W}_\varepsilon^{\rho\nu} \text{ on } \mathcal{K}^\rho(u^0, u^1) \text{ and } \overline{W}_\varepsilon^{\rho\nu} = \infty \text{ elsewhere.}$$

We have the following.

LEMMA 4.2 (Γ-limit $\nu \to 0$). – $\overline{W}_\varepsilon^{\rho\nu} \xrightarrow{\Gamma} \overline{W}_\varepsilon^{\rho 0}$ *w.r.t. both the strong and weak topology of $\mathcal{V}^\rho$.*

PROOF. – The existence of a recovery sequence is immediate by pointwise convergence. The Γ-lim inf inequality follows from the fact that $\overline{W}_\varepsilon^{\rho\nu} \geq \overline{W}_\varepsilon^{\rho 0}$ pointwise and $\overline{W}_\varepsilon^{\rho 0}$ is lower semicontinuous with respect to the weak topology of $\mathcal{V}^\rho$. □

As for the purely viscous limit we have the following.

LEMMA 4.3 (Γ-limit $\rho \to 0$). – $\overline{W}_\varepsilon^{\rho\nu} \xrightarrow{\Gamma} \overline{W}_\varepsilon^{0\nu}$ *w.r.t. both the strong and weak topology of $\mathcal{U}$.*

PROOF. – The Γ-lim inf inequality is immediate as $\overline{W}_\varepsilon^{\rho\nu} \geq \overline{W}_\varepsilon^{0\nu}$ pointwise and the latter is lower semicontinuous with respect to the weak topology of $\mathcal{U}$. As for the recovery sequence, we shall resort here to some singular perturbation technique (in time). In particular, for any given $u \in \mathcal{K}(u^0, u^1)$ and almost every $x \in \Omega$ we can find $t \mapsto v^\rho(x, t) \in \mathrm{H}_0^1(0, T)$ solving weakly

$$v^\rho(x, \cdot) - \sqrt{\rho} v_{tt}^\rho(x, \cdot) = u_t(x, \cdot) - u^1(x).$$

Then, it is a standard matter to prove that $u^\rho(x, t) := u^0(x) + tu^1(x) + \int_0^t v^\rho(x, s)\,\mathrm{d}s \in \mathcal{K}(u^0, u^1)$ is such that $u^\rho \to u$ strongly in $\mathcal{U}$ and $\sqrt{\rho} u_{tt}^\rho \to 0$ strongly in $\mathrm{L}^2(0, T; H)$. We hence have that $\overline{W}_\varepsilon^{\rho\nu}(u^\rho) \to \overline{W}_\varepsilon^{0\nu}(u)$. □

Note that the above results entail Γ-convergence with respect to both strong and weak topology. This circumstance is usually referred to as *Mosco convergence* [18] and plays a prominent role in classical convex analysis [4].

Before closing this subsection let us stress that the above Γ-limits are taken for ε fixed and record that combined Γ-convergence analyses for both parameters and $\varepsilon \to 0$ are presently not available. Additional material on Γ-convergence for WIDE functionals in the parabolic case is however to be found in [2, 15, 16].

5. – Time-discretization

The above proof of Theorem 2.1 rests upon the possibility of proving the key estimate (4.1). We achieve this by investigating a time-discrete version of the WIDE principle and, in particular, of the argument of Subsection 4.1. We replace the functional W_ε by a time-discrete WIDE functional $W_{\varepsilon\tau}$. From here on we directly focus on the situation $\rho > 0$, the case $\rho = 0$ being covered in [17]. We start by recalling the notation for the constant time-step $\tau := T/n$ ($n \in \mathbb{N}$) and introduce the space

$$\mathcal{V}_\tau := \left\{(u_0, \dots, u_n) \in H^{n+1} \, : \, (u_2, \dots, u_{n-2}) \in (V \cap X)^{n-3}\right\}.$$

Moreover, we define the functional $W_{\varepsilon\tau} : \mathcal{V}_\tau \to \mathbb{R}$ by

$$W_{\varepsilon\tau}(u_0, \dots, u_n) = \frac{\varepsilon^2 \rho}{2} \sum_{j=2}^{n} \tau e_{\varepsilon\tau,j} |\delta^2 u_j|^2 + \frac{\varepsilon \nu}{2} \sum_{j=2}^{n-1} \tau e_{\varepsilon\tau,j+1} |\delta u_j|^2 + \sum_{j=2}^{n-2} \tau e_{\varepsilon\tau,j+2} E(u_j).$$

Given the vector $(w_0, \dots, w_n)$, in the latter we have used the notation δw for its *discrete derivative* $\delta w_j := (w_j - w_{j-1})/\tau$ for $j = 1, \dots, n$ and $\delta^2 w = \delta(\delta w)$, $\delta^3 w = \delta(\delta^2 w)$ and so on. Moreover we have used the *weights* $e_{\varepsilon\tau,1}, \dots, e_{\varepsilon\tau,n}$ given by

$$e_{\varepsilon\tau,i} = \left(\frac{\varepsilon}{\varepsilon + \tau}\right)^i \qquad \text{for } i = 1, \dots, n. \tag{5.1}$$

These weights are nothing but the discrete version of the exponentially decaying weight $t \mapsto \exp(-t/\varepsilon)$ for we have that $\delta e_{\varepsilon\tau,i} + e_{\varepsilon\tau,i}/\varepsilon = 0$. Namely, $e_{\varepsilon\tau,i}$ is the solution of the constant time-step implicit Euler discretization of the problem $e' + e/\varepsilon = 0$, with the initial condition $e(0) = 1$. Finally, we denote the discrete counterpart of $\mathcal{K}(u^0, u^1)$ by $\mathcal{K}_\tau(u^0, u^1)$, i.e.,

$$\mathcal{K}_\tau(u^0, u^1) = \{(u_0, \dots, u_n) \in \mathcal{V}_\tau \, : \, u_0 = u^0, \, \rho \delta u_1 = \rho u^1\}.$$

The discrete WIDE functional $W_{\varepsilon\tau}$ represents a discrete version of the original time-continuous WIDE functional W_ε. We shall drop the subscript $\varepsilon\tau$ from $e_{\varepsilon\tau,j}$ in the remainder of this section for the sake of notational simplicity.

5.1 – *Well-posedness of the discrete minimum problem*

Exactly as in the time-continuous situation, in case E is λ-convex the functional $W_{\varepsilon\tau}$ turns out to be uniformly convex for all sufficiently small ε. Note that for all $(u_0, \dots, u_n) \in \mathcal{K}_\tau(u^0, u^1)$ we have that

$$\sum_{k=2}^{n} \tau |u_k|^2 \leq C\left(|u^0|^2 + |u^1|^2 + \sum_{k=2}^{n} \tau |\delta^2 u_k|^2 \right) \tag{5.2}$$

where C depends on T. Hence, the functional $W_{\varepsilon\tau}$ is coercive on $\mathcal{K}_\tau(u^0, u^1)$. Indeed, the coercivity of $W_{\varepsilon\tau}$ in V^{n-3} with respect to $(u_2, \dots, u_{n-2})$ is immediate. As for the coercivity in H we see that, due to (5.2), the discrete WIDE functional $W_{\varepsilon\tau}$ controls the norm in H (up to constants depending on $T, \rho, \nu, \varepsilon$, and τ).

LEMMA 5.1 (Well-posedness of the discrete minimum problem). – *For ε and τ small and all $u^0, u^1 \in H$, the discrete* WIDE *functional $W_{\varepsilon\tau}$ admits a unique minimizer in $\mathcal{K}_\tau(u^0, u^1)$.*

PROOF. – This argument is the discrete analogue of the proof of Theorem 3.1. In particular, we start by decomposing $W_{\varepsilon\tau}$ into a quadratic part $Q_{\varepsilon\tau}$ and a convex remainder $R_{\varepsilon\tau}$ as

$$\begin{aligned}
&W_{\varepsilon\tau}(u_0, \dots, u_n) \\
&= \left(\varepsilon^2 \rho \sum_{j=2}^{n} \tau \frac{e_j}{2} |\delta^2 u_j|^2 + \varepsilon\nu \sum_{j=2}^{n-1} \tau \frac{e_{j+1}}{2} |\delta u_j|^2 - \lambda^- \sum_{j=2}^{n-2} \tau \frac{e_{j+2}}{2} |u_j|^2 + \sum_{j=2}^{n-2} \tau \frac{e_{j+2}}{2} \|u_j\|_V^2 \right) \\
&+ \sum_{j=2}^{n-2} \tau e_{j+2} \int_\Omega G(u_j)\, \mathrm{d}x =: Q_{\varepsilon\tau}(u_0, \dots, u_n) + R_{\varepsilon\tau}(u_0, \dots, u_n)
\end{aligned}$$

with $G(u) = F(u) + \lambda^- |u|^2/2$. The result follows by checking that, for small ε and τ, the functional $Q_{\varepsilon\tau}$ is uniformly convex. To this end, for $(u_0, \dots, u_n) \in \mathcal{K}_\tau(u^0, u^1)$ let $(v_0, \dots, v_n)$ be defined as $v_i = \sqrt{e_i} u_i$. Then, we compute that

$$\begin{aligned}
\sqrt{e_i}\delta u_i &= \ell_{\varepsilon\tau} \delta v_i + \sigma_{\varepsilon\tau} v_i \\
\sqrt{e_i}\delta^2 u_i &= \ell_{\varepsilon\tau} \delta^2 v_i + \ell_{\varepsilon\tau}\sigma_{\varepsilon\tau} \delta v_{i-1} + \sigma_{\varepsilon\tau} \delta v_i + \sigma_{\varepsilon\tau}^2 v_{i-1}
\end{aligned}$$

where $\ell_{\varepsilon\tau} = \sqrt{\varepsilon/(\varepsilon+\tau)}$ and $\sigma_{\varepsilon\tau} = (1-\ell_{\varepsilon\tau})/\tau$ such that $\ell_{\varepsilon\tau} \to 1$ and $\sigma_{\varepsilon\tau} \to 1/(2\varepsilon)$ for $\tau \to 0$. Moreover, we used that

$$\delta \frac{1}{\sqrt{e_i}} = \sigma_{\varepsilon\tau} \frac{1}{\sqrt{e_i}}.$$

Hence, we can rewrite $Q_{\varepsilon\tau}(u_0,\dots,u_n)$ as

$$
\begin{aligned}
Q_{\varepsilon\tau}(u_0,\dots,u_n) = \varepsilon^2\rho\sum_{j=2}^{n}\frac{\tau}{2}\Big(&\ell_{\varepsilon\tau}^2|\delta^2 v_j|^2+\sigma_{\varepsilon\tau}^2|\delta v_j|^2+\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}^2|\delta v_{j-1}|^2+\sigma_{\varepsilon\tau}^4|v_{j-1}|^2\Big)\\
&+\varepsilon\nu\sum_{j=2}^{n-1}\frac{\tau\ell_{\varepsilon\tau}^2}{2}\Big(\ell_{\varepsilon\tau}^2|\delta v_j|^2+\sigma_{\varepsilon\tau}^2|v_j|^2\Big)-\lambda^-\sum_{j=2}^{n-2}\frac{\tau\ell_{\varepsilon\tau}^4}{2}|v_j|^2\\
&+\varepsilon^2\rho\sum_{j=2}^{n}\tau\Big(\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}(\delta^2 v_j,\delta v_{j-1})+\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}(\delta^2 v_j,\delta v_j)+\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}^2(\delta^2 v_j,v_{j-1})\\
&\qquad+\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}^2(\delta v_j,\delta v_{j-1})+\sigma_{\varepsilon\tau}^3(\delta v_j,v_{j-1})+\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}^3(\delta v_{j-1},v_{j-1})\Big)\\
&+\varepsilon\nu\sum_{j=2}^{n-1}\tau\ell_{\varepsilon\tau}^3\sigma_{\varepsilon\tau}(\delta v_j,v_j)+\sum_{j=2}^{n-2}\frac{\tau\ell_{\varepsilon\tau}^4}{2}\|v_j\|_V^2.
\end{aligned}
$$

Next, we collect all terms involving $|v_j|$ and obtain

$$
\begin{aligned}
&\varepsilon^2\rho\sum_{j=2}^{n}\frac{\tau}{2}\sigma_{\varepsilon\tau}^4|v_{j-1}|^2+\varepsilon\nu\sum_{j=2}^{n-1}\frac{\tau}{2}\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}^2|v_j|^2-\lambda^-\sum_{j=2}^{n-2}\frac{\tau\ell_{\varepsilon\tau}^4}{2}|v_j|^2\\
&=\big[\varepsilon^2\rho\sigma_{\varepsilon\tau}^4+\varepsilon\nu\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}^2-\lambda^-\ell_{\varepsilon\tau}^4\big]\sum_{j=2}^{n-2}\frac{\tau}{2}|v_j|^2\\
&+\frac{\tau\varepsilon^2\rho\sigma_{\varepsilon\tau}^4}{2}(|v_1|^2+|v_{n-1}|^2)+\frac{\tau\varepsilon\nu\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}^2}{2}|v_{n-1}|^2.
\end{aligned}
$$

We check that the coefficient in square brackets in front of the sum in the above right-hand side is positive for sufficiently small ε and τ. For instance, one can choose $\tau=3\varepsilon$ and compute $\ell_{\varepsilon\tau}=1/2$ and $(1-\ell_{\varepsilon\tau})/\tau=1/(6\varepsilon)$. In particular, the coefficient reads

$$
\big[\varepsilon^2\rho\sigma_{\varepsilon\tau}^4+\varepsilon\nu\ell_{\varepsilon\tau}^2\sigma_{\varepsilon\tau}^2-\lambda^-\ell_{\varepsilon\tau}^4\big]=\frac{\rho}{6^4\varepsilon^2}+\frac{\nu}{4(6^2)\varepsilon}-\frac{\lambda^-}{16}
$$

which basically corresponds to the coefficient multiplying $|v|^2$ in the first line of (3.3). Next, we look at the mixed terms. Using summing by parts, we obtain

$$
\sum_{j=2}^{n}\tau(\delta^2 v_j,v_{j-1})=(\delta v_n,v_n)-(\delta v_1,v_1)-\sum_{j=2}^{n}\tau|\delta v_j|^2.
$$

Hence, we have that

$$
\begin{aligned}
&\varepsilon^2\rho\sum_{j=2}^{n}\tau\Big(\frac{1}{2}\sigma_{\varepsilon\tau}^2|\delta v_j|^2+\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}^2(\delta^2 v_j,v_{j-1})\Big)\\
&\qquad=\varepsilon^2\rho\sum_{j=2}^{n}\tau\sigma_{\varepsilon\tau}^2\Big(\frac{1}{2}-\ell_{\varepsilon\tau}\Big)|\delta v_j|^2+\varepsilon^2\rho\ell_{\varepsilon\tau}\sigma_{\varepsilon\tau}^2\big((\delta v_n,v_n)-(\delta v_1,v_1)\big).
\end{aligned}
$$

Finally, using again summation by parts for the remaining mixed terms, we check as in the time-continuous case that the quadratic form Q_ε is uniformly convex. Hence, the existence of unique minimizers follows by the Direct Method. □

5.2 – *Discrete Euler-Lagrange system*

As in Section 3.1 the unique minimizer $(u_0, \ldots, u_n)$ of the time-discrete functional $W_{\varepsilon\tau}$ solves the corresponding Euler-Lagrange system and we directly compute that

$$0 = \varepsilon^2 \rho \sum_{j=2}^{n} \tau e_j(\delta^2 u_j, \delta^2 v_j) + \varepsilon \nu \sum_{j=2}^{n-1} \tau e_{j+1}(\delta u_j, \delta v_j) + \sum_{j=2}^{n-2} \tau e_{j+2} \langle \mathrm{D}E(u_j), v_j \rangle \quad \forall (v_0, \ldots, v_n) \in \mathcal{K}_\tau(0,0). \tag{5.3}$$

Let us now proceed as in the continuous case. First of all, we sum-by-parts and obtain that

$$\rho \sum_{j=2}^{n} \tau e_j(\delta^2 u_j, \delta^2 v_j) = \rho \sum_{j=2}^{n} e_j(\delta^2 u_j, \delta v_j) - \rho \sum_{j=2}^{n} e_j(\delta^2 u_j, \delta v_{j-1})$$

$$= \rho e_n(\delta^2 u_n, \delta v_n) - \rho e_2(\delta^2 u_2, \delta v_1) - \rho \sum_{j=2}^{n-1} \tau(\delta(e\delta^2 u)_{j+1}, \delta v_j)$$

$$\overset{\rho\delta v_1 = 0}{=} \rho e_n(\delta^2 u_n, \delta v_n) - \rho \sum_{j=2}^{n-1} (\delta(e\delta^2 u)_{j+1}, v_j) + \rho \sum_{j=2}^{n-1} (\delta(e\delta^2 u)_{j+1}, v_{j-1})$$

$$= \rho e_n(\delta^2 u_n, \delta v_n) - \rho(\delta(e\delta^2 u)_n, v_{n-1}) + \rho(\delta(e\delta^2 u)_3, v_1) + \rho \sum_{j=2}^{n-2} \tau(\delta^2(e\delta^2 u)_{j+2}, v_j)$$

$$\overset{v_0 = 0}{=} \rho e_n(\delta^2 u_n, \delta v_n) - \rho(\delta(e\delta^2 u)_n, v_{n-1}) + \rho \sum_{j=2}^{n-2} \tau(\delta^2(e\delta^2 u)_{j+2}, v_j),$$

where we have used that $\rho\delta v_1 = 0$ and $v_0 = 0$. Similarly, we obtain

$$\nu \sum_{j=2}^{n-1} \tau e_{j+1}(\delta u_j, \delta v_j) = \nu e_n(\delta u_{n-1}, v_{n-1}) - \nu \sum_{j=2}^{n-2} \tau(\delta(e_{j+2} \delta u_{j+1}), v_j).$$

Next, by means of the definition of e_j in (5.1) and some tedious computations we check that

$$\rho\delta^2(e\delta^2 u)_{j+2} = \rho e_{j+2}\Big(\delta^4 u_{j+2} - \frac{2}{\varepsilon}\delta^3 u_{j+1} + \frac{1}{\varepsilon^2}\delta^2 u_j\Big),$$

$$\rho\delta(e_{j+2}\delta u_{j+1}) = \rho e_{j+2}\Big(\delta^2 u_{j+1} - \frac{1}{\varepsilon}\delta u_j\Big),$$

and rewrite relation (5.3) in the equivalent form

$$0 = \sum_{j=2}^{N-2} \tau e_{j+2}\Big(\rho\varepsilon^2\delta^4 u_{j+2} - 2\rho\varepsilon\delta^3 u_{j+1} + \rho\delta^2 u_j - \varepsilon\nu\delta^2 u_{j+1} + \nu\delta u_j, v_j\Big) + \langle \mathrm{D}E(u_j), v_j\rangle$$
$$+\varepsilon^2\rho\Big(e_n(\delta^2 u_n, \delta v_n) - (\delta(e\delta^2 u)_n, v_{n-1})\Big) + \varepsilon\nu e_n(\delta u_{n-1}, v_{n-1}).$$

This holds for arbitrary $v \in \mathcal{K}_\tau(0,0)$ and we have therefore proved the following.

LEMMA 5.2 (Discrete Euler-Lagrange system). – *Let* $(u_0, \ldots, u_n) \in \mathcal{K}_\tau(u^0, u^1)$ *be the unique minimizer of the discrete* WIDE *functional* $W_{\varepsilon\tau}$. *Then,* $(u_0, \ldots, u_n)$ *solves*

$$\varepsilon^2\rho\delta^4 u_{j+2} - 2\varepsilon\rho\delta^3 u_{j+1} + \rho\delta^2 u_j - \varepsilon\nu\delta^2 u_{j+1} + \nu\delta u_j + \mathrm{D}E(u_j) = 0, \tag{5.4}$$

subject to the initial and final conditions

$$u_0 = u^0, \qquad \rho\delta u_1 = \rho u^1, \qquad \text{and} \tag{5.5}$$

$$\varepsilon^2\rho\delta^2 u_n = 0, \qquad \varepsilon^2\rho\delta^3 u_n = \varepsilon\rho\delta^2 u_{n-1} + \varepsilon\nu\delta u_{n-1}. \tag{5.6}$$

Equations (5.4)-(5.6) are the discrete analogue of equations (3.5)-(3.7).

5.3 – *Discrete estimate*

The argument of Subsection 4.1 can be made rigorous at the time-discrete level. We present here a time-discrete version of estimate (4.5) by using the time-discrete Euler-Lagrange system (5.4). Namely, we aim at proving the following.

LEMMA 5.3 (Discrete estimate). – *Let* $(u_0, \ldots, u_n)$ *minimize the discrete* WIDE *functional* $W_{\varepsilon\tau}$ *over* $\mathcal{K}_\tau(u^0, u^1)$. *Then, for all* ε *and* τ *sufficiently small we have*

$$(\rho+\nu)\sum_{j=2}^{n-2} \tau|\delta u_j|^2 + \sum_{j=2}^{n-2} \tau E(u_j) \le C. \tag{5.7}$$

PROOF. – Let us assume from the very beginning that $\rho > 0$ throughout this proof. Indeed, the case $\rho = 0$ (and correspondingly $\nu > 0$) is already detailed in

[17]. We argue by mimicking at the discrete level the estimate of Subsection 4.1. Namely, we shall perform the following:

$$\sum_{k=2}^{n-2} \tau(5.4)\big|_{v=\delta u_k-u^1} + \sum_{k=2}^{n-2} \tau\left(\sum_{i=1}^{k} \tau(5.4)\big|_{v=\delta u_i-u^1}\right). \tag{5.8}$$

At first, let us test the time-discrete Euler-Lagrange equation in (5.4) on $v = \tau(\delta u_j - u^1)$ and sum for $j = 2, \ldots, k \le n-2$ in order to get that

$$\begin{aligned} &\varepsilon^2\rho\sum_{j=2}^{k}\tau(\delta^4 u_{j+2},\delta u_j-u^1)-2\varepsilon\rho\sum_{j=2}^{k}\tau(\delta^3 u_{j+1},\delta u_j-u^1)-\varepsilon\nu\sum_{j=2}^{k}\tau(\delta^2 u_{j+1},\delta u_j-u^1)\\ &\quad+\rho\sum_{j=2}^{k}\tau(\delta^2 u_j,\delta u_j-u^1)+\nu\sum_{j=2}^{k}\tau(\delta u_j,\delta u_j-u^1)+\sum_{j=2}^{k}\tau\langle \mathrm{D}E(u_j),\delta u_j-u^1\rangle=0. \end{aligned} \tag{5.9}$$

We now treat separately all terms in the above left-hand side. The fourth-order-in-time term can be handled as follows.

$$\begin{aligned} \varepsilon^2\rho\sum_{j=2}^{k}\tau(\delta^4 u_{j+2},\delta u_j-u^1) &= \varepsilon^2\rho\sum_{j=2}^{k}(\delta^3 u_{j+2}-\delta^3 u_{j+1},\delta u_j-u^1)\\ &= \varepsilon^2\rho(\delta^3 u_{k+2},\delta u_k-u^1)-\varepsilon^2\rho(\delta^3 u_3,\delta u_2-u^1)-\varepsilon^2\rho\sum_{j=3}^{k}\tau(\delta^3 u_{j+1},\delta^2 u_j)\\ &= \varepsilon^2\rho(\delta^3 u_{k+2},\delta u_k-u^1)-\varepsilon^2\rho\sum_{j=2}^{k}\tau(\delta^3 u_{j+1},\delta^2 u_j)\\ &= \varepsilon^2\rho(\delta^3 u_{k+2},\delta u_k-u^1)-\frac{\varepsilon^2\rho}{2}|\delta^2 u_{k+1}|^2+\frac{\varepsilon^2\rho}{2}|\delta^2 u_2|^2+\frac{\varepsilon^2\rho}{2}\sum_{j=2}^{k}|\delta^2 u_{j+1}-\delta^2 u_j|^2, \end{aligned} \tag{5.10}$$

where we also used that $\delta u_1 = u^1$. Next, we treat the third-order-in-time term of (5.9) somehow similarly as

$$-2\varepsilon\rho\sum_{j=2}^{k}\tau(\delta^3 u_{j+1},\delta u_j-u^1) = -2\varepsilon\rho(\delta^2 u_{k+1},\delta u_k-u^1)+2\varepsilon\rho\sum_{j=2}^{k}\tau|\delta^2 u_j|^2. \tag{5.11}$$

As for the remaining discrete time derivatives in (5.9) we compute

$$\rho\sum_{j=2}^{k}\tau(\delta^2 u_j,\delta u_j-u^1)=\frac{\rho}{2}\sum_{j=2}^{k}|\delta u_j-\delta u_{j-1}|^2+\frac{\rho}{2}|\delta u_k-u^1|^2, \tag{5.12}$$

$$\varepsilon\nu\sum_{j=2}^{k}\tau(\delta^2 u_{j+1},\delta u_j-u^1)=\frac{\varepsilon\nu}{2}|\delta u_{k+1}-u^1|^2-\frac{\varepsilon\nu}{2}|\delta u_2-u^1|^2-\frac{\varepsilon\nu}{2}\sum_{j=2}^{k}|\delta u_{j+1}-\delta u_j|^2, \tag{5.13}$$

$$\nu\sum_{j=2}^{k}\tau(\delta u_j,\delta u_j-u^1)=\nu\sum_{j=2}^{k}\tau|\delta u_j|^2-\nu\sum_{j=2}^{k}\tau(\delta u_j,u^1). \tag{5.14}$$

By using the λ-convexity of F we obtain for the last term in (5.9)

$$\begin{aligned}\sum_{j=2}^{k}\tau\langle \mathrm{D}E(u_j),\delta u_j-u^1\rangle &\overset{\delta u_1=u^1}{=}\sum_{j=1}^{k}\tau\langle \mathrm{D}E(u_j),\delta u_j-u^1\rangle\\ &\geq E(u_k)-E(u^0)-\sum_{j=1}^{k}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\frac{\lambda}{2}\sum_{j=1}^{k}|u_j-u_{j-1}|^2.\end{aligned}\tag{5.15}$$

We now recollect computations (5.10)-(5.15) into equation (5.9) in order to deduce that

$$\begin{aligned}&\varepsilon^2\rho(\delta^3u_{k+2},\delta u_k-u^1)-\frac{\varepsilon^2\rho}{2}|\delta^2u_{k+1}|^2-2\varepsilon\rho(\delta^2u_{k+1},\delta u_k-u^1)+2\varepsilon\rho\sum_{j=2}^{k}\tau|\delta^2u_j|^2\\ &+\frac{\rho}{2}|\delta u_k-u^1|^2-\frac{\varepsilon\nu}{2}|\delta u_{k+1}-u^1|^2+\nu\sum_{j=2}^{k}\tau|\delta u_j|^2+E(u_k)+\frac{\lambda\tau}{2}\sum_{j=2}^{k}\tau|\delta u_j|^2\\ &\leq C+\sum_{j=1}^{k}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\nu\sum_{j=2}^{k}\tau(\delta u_j,u^1).\end{aligned}\tag{5.16}$$

Let us now move to the consideration of the second term in (5.8). We multiply (5.16) by τ and take the sum for $k=2,\ldots,n-2$ getting

$$\begin{aligned}&\varepsilon^2\rho\sum_{k=2}^{n-2}\tau(\delta^3u_{k+2},\delta u_k-u^1)-\frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau|\delta^2u_{k+1}|^2-2\varepsilon\rho\sum_{k=2}^{n-2}\tau(\delta^2u_{k+1},\delta u_k-u^1)\\ &+2\varepsilon\rho\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta^2u_j|^2+\frac{\rho}{2}\sum_{j=2}^{n-2}\tau|\delta u_k-u^1|^2-\frac{\varepsilon\nu}{2}\sum_{k=2}^{n-2}\tau|\delta u_{k+1}-u^1|^2\\ &+\nu\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta u_j|^2+\sum_{k=2}^{n-2}\tau E(u_k)+\frac{\lambda\tau}{2}\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta u_j|^2\\ &\leq C+\sum_{k=2}^{n-2}\tau\sum_{j=1}^{k}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\nu\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau(\delta u_j,u^1).\end{aligned}\tag{5.17}$$

By summing by parts, we can write the first term in (5.17) in the following way:

$$\begin{aligned}&\varepsilon^2\rho\sum_{k=2}^{n-2}\tau(\delta^3u_{k+2},\delta u_k-u^1)=\varepsilon^2\rho\sum_{k=2}^{n-2}(\delta^2u_{k+2}-\delta^2u_{k+1},\delta u_k-u^1)\\ &=\varepsilon^2\rho(\delta^2u_n,\delta u_{n-2}-u^1)-\varepsilon^2\rho(\delta^2u_3,\delta u_2-u^1)-\varepsilon^2\rho\sum_{k=3}^{n-2}\tau(\delta^2u_{k+1},\delta^2u_k)\\ &=-\varepsilon^2\rho\sum_{k=2}^{n-2}\tau(\delta^2u_{k+1},\delta^2u_k)\end{aligned}\tag{5.18}$$

where we have used the initial condition $\rho\delta u_1 = \rho u^1$ and the final condition $\rho\delta^2 u_n = 0$. Moreover, we can also compute that

(5.19)

$$
\begin{aligned}
&-2\varepsilon\rho\sum_{k=2}^{n-2}\tau(\delta^2 u_{k+1},\delta u_k - u^1) = 2\varepsilon\rho\sum_{k=2}^{n-2}(\delta u_k - \delta u_{k+1},\delta u_k - u^1)\\
&= \varepsilon\rho|\delta u_2 - u^1|^2 + \varepsilon\rho\sum_{k=2}^{n-3}|\delta u_{k+1}-\delta u_k|^2 - \varepsilon\rho|\delta u_{n-2}-u^1|^2 + 2\varepsilon\rho(\delta u_{n-2}-\delta u_{n-1},\delta u_{n-2}-u^1)\\
&= \varepsilon\rho|\delta u_2 - u^1|^2 + \varepsilon\rho\sum_{k=2}^{n-3}|\delta u_{k+1}-\delta u_k|^2 - \varepsilon\rho|\delta u_{n-2}-u^1|^2 - 2\varepsilon\rho\tau(\delta^2 u_{n-1},\delta u_{n-2}-u^1)\\
&\geq -\frac{3}{2}\varepsilon\rho|\delta u_{n-2}-u^1|^2 - 2\varepsilon\rho\tau^2|\delta^2 u_{n-1}|^2.
\end{aligned}
$$

Furthermore, we observe that

$$
\begin{aligned}
2\varepsilon\rho&\sum_{k=2}^{n-2}\tau|\delta^2 u_k|^2 - \varepsilon^2\rho\sum_{k=2}^{n-2}\tau(\delta^2 u_{k+1},\delta^2 u_k) - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau|\delta^2 u_{k+1}|^2\\
&\geq 2\varepsilon\rho\sum_{k=2}^{n-2}\tau|\delta^2 u_k|^2 - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau|\delta^2 u_{k+1}|^2 - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau|\delta^2 u_k|^2 - \frac{\varepsilon^2\rho}{2}\sum_{k=2}^{n-2}\tau|\delta^2 u_{k+1}|^2\\
&\geq \rho\left(2\varepsilon - \frac{3}{2}\varepsilon^2\right)\sum_{k=2}^{n-2}\tau|\delta^2 u_k|^2 - \varepsilon^2\rho\tau|\delta^2 u_{n-1}|^2.
\end{aligned}
\tag{5.20}
$$

Let us now write estimate (5.16) by choosing $k = n-2$ and taking advantage of the final boundary conditions in (5.6)

$$
\begin{aligned}
&\varepsilon\nu(\delta u_{n-1},\delta u_{n-2}-u^1) - \varepsilon\rho(\delta^2 u_{n-1},\delta u_{n-2}-u^1) - \frac{\varepsilon^2\rho}{2}|\delta^2 u_{n-1}|^2\\
&+2\varepsilon\rho\sum_{j=2}^{n-2}\tau|\delta^2 u_j|^2 + \frac{\rho}{2}|\delta u_{n-2}-u^1|^2 - \frac{\varepsilon\nu}{2}|\delta u_{n-1} - u^1|^2 + \nu\sum_{j=2}^{n-2}\tau|\delta u_j|^2 + E(u_{n-2})\\
&\leq C + \sum_{j=1}^{n-2}\tau\langle \mathrm{D}E(u_j),u^1\rangle + \nu\sum_{j=2}^{n-2}\tau(\delta u_j,u^1).
\end{aligned}
\tag{5.21}
$$

Note that due to the final conditions in (5.6) the following identity holds:

$$
-\rho\delta^2 u_{n-1} = \frac{\tau\nu}{\tau+\varepsilon}\delta u_{n-1}.
$$

Hence, we have that

$$
\begin{aligned}
&\varepsilon\nu(\delta u_{n-1},\delta u_{n-2}-u^1)-\varepsilon\rho(\delta^2 u_{n-1},\delta u_{n-2}-u^1)\\
&\quad=\varepsilon\nu|\delta u_{n-1}|^2-\varepsilon\nu(\delta u_{n-1},\delta u_{n-1}-\delta u_{n-2})-\varepsilon\nu(\delta u_{n-1},u^1)+\frac{\tau\varepsilon\nu}{\tau+\varepsilon}(\delta u_{n-1},\delta u_{n-2}-u^1)\\
&\quad=\varepsilon\nu|\delta u_{n-1}|^2-\varepsilon\nu\tau(\delta u_{n-1},\delta^2 u_{n-1})-\varepsilon\nu\left(1+\frac{\tau}{\tau+\varepsilon}\right)(\delta u_{n-1},u^1)+\varepsilon\nu\left(\frac{\tau}{\tau+\varepsilon}\right)|\delta u_{n-1}|^2\\
&\qquad-\frac{\varepsilon\nu\tau}{\tau+\varepsilon}(\delta u_{n-1},\delta u_{n-1}-\delta u_{n-2})\\
&\quad=\varepsilon\nu\left(1+\frac{\tau}{\tau+\varepsilon}+\frac{\nu\tau^2}{\rho(\tau+\varepsilon)}+\frac{\nu\tau^3}{\rho(\tau+\varepsilon)^2}\right)|\delta u_{n-1}|^2-\varepsilon\nu\left(1+\frac{\tau}{\tau+\varepsilon}\right)(\delta u_{n-1},u^1)\\
&\quad\geq\nu\left(\varepsilon-\frac{\varepsilon^2}{2}\right)|\delta u_{n-1}|^2-2\nu|u^1|^2.
\end{aligned}
$$

Therefore, by recalling that $\rho>0$ we have from (5.21) that

$$
\begin{aligned}
&\left(\frac{\nu\varepsilon}{4}-\frac{\nu\varepsilon^2}{2}-\frac{\varepsilon^2\tau}{\rho}-\frac{2\varepsilon\tau^2\nu^2}{\rho}\right)|\delta u_{n-1}|^2+2\varepsilon\rho\sum_{j=2}^{n-2}\tau|\delta^2 u_j|^2\\
&+\frac{\rho}{2}|\delta u_{n-2}-u^1|^2+\nu\sum_{j=2}^{n-2}\tau|\delta u_j|^2+E(u_{n-2})\\
&\leq C+\sum_{j=1}^{n-2}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\nu\sum_{j=2}^{n-2}\tau(\delta u_j,u^1)
\end{aligned}
\tag{5.22}
$$

where we have used that

$$
\begin{aligned}
&\frac{\varepsilon\nu}{2}|\delta u_{n-1}-u^1|^2\leq\frac{3}{4}\varepsilon\nu|\delta u_{n-1}|^2+C,\\
&-\frac{\varepsilon^2\rho}{2}|\delta^2 u_{n-1}|^2=-\frac{\varepsilon^2\nu^2}{2\rho}\left(\frac{\tau}{\tau+\varepsilon}\right)^2|\delta u_{n-1}|^2\geq-\frac{\varepsilon^2\nu^2}{2\rho}|\delta u_{n-1}|^2.
\end{aligned}
$$

By taking the sum of (5.22) and (5.17), using the equalities (5.18), (5.19), (5.20), and recalling the fact that $\rho>0$ we obtain that

$$
\begin{aligned}
&\rho\left(\frac{1}{2}-\frac{3}{2}\varepsilon\right)|\delta u_{n-2}-u^1|^2+\left(\frac{\nu\varepsilon}{4}-\frac{\nu\varepsilon^2}{2}-\frac{\varepsilon^2\tau}{\rho}-\frac{2\varepsilon\tau^2\nu^2}{\rho}-\frac{3\varepsilon\nu\tau}{4}\right)|\delta u_{n-1}|^2\\
&+\nu(1-\varepsilon)\sum_{j=2}^{n-2}\tau|\delta u_j|^2+E(u_{n-2})+2\varepsilon\rho\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta^2 u_j|^2+\rho\left(2\varepsilon-\frac{3\varepsilon^2}{2}\right)\sum_{k=2}^{n-2}\tau|\delta^2 u_k|^2\\
&+\frac{\rho}{2}\sum_{k=2}^{n-2}\tau|\delta u_k-u^1|^2+\left(\nu-\frac{\lambda^-\tau}{2}\right)\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta u_j|^2+\sum_{k=2}^{n-2}\tau E(u_k)\\
&\leq C+\sum_{j=1}^{n-2}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\nu\sum_{j=2}^{n-2}\tau(\delta u_j,u^1)+\sum_{k=2}^{n-2}\tau\sum_{j=1}^{k}\tau\langle \mathrm{D}E(u_j),u^1\rangle+\nu\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau(\delta u_j,u^1).
\end{aligned}
$$

As ε and τ are assumed to be small, by using the growth conditions in (2.1) and Young's inequality we readily get the estimate. $\square$

5.4 – *Proof of Lemma* 4.1

In order to conclude the proof of Lemma 4.1 we need to show that the time-discrete energy estimate in Lemma 5.3 passes to the limit as $\tau \to 0$ (for fixed $\varepsilon > 0$). To this aim, we check the discrete-to-continuous Γ-convergence $W_{\varepsilon\tau} \xrightarrow{\Gamma} W_\varepsilon$ with respect to the weak topology on $\mathcal{U}$ (see [5, 7] for relevant definitions and results on Γ-convergence).

For all vectors $(w_0, \ldots, w_n)$, we indicate by $\overline{w}_\tau$ and w_τ its backward constant and piecewise affine interpolants on the partition, respectively. Namely,

$$\overline{w}_\tau(0) = w_\tau(0) := w_0,\ \overline{w}_\tau(t) := w_i,\ w_\tau(t) := \alpha_i(t)w_i + (1-\alpha_i(t))w_{i-1}$$
$$\text{for } t \in ((i-1)\tau, i\tau],\ i = 1, \ldots, n$$

where we have used the auxiliary functions

$$\alpha_i(t) := (t-(i-1)\tau)/\tau \text{ for } t \in ((i-1)\tau, i\tau],\ i = 1, \ldots, n.$$

With these definitions we can reformulate the estimate in Lemma 5.3 as

$$(\rho + \nu) \int_\tau^{T-2\tau} |\partial_t u_\tau^\varepsilon|^2 \,\mathrm{d}t + \int_\tau^{T-2\tau} E(\overline{u}_\tau^\varepsilon) \,\mathrm{d}t \leq C \tag{5.23}$$

where u_τ^ε and $\overline{u}_\tau^\varepsilon$ denote the interpolants associated with the minimizer $(u_0, \ldots, u_n)$ of the discrete WIDE functional $W_{\varepsilon\tau}$.

LEMMA 5.4 (Discrete-to-continuous Γ-convergence). – *Let*

$$\widehat{\mathcal{V}}_\tau := \{u : [0,T] \to V \cap X \ :\ u \text{ is piecewise affine on the time partition}\}$$

and define the functionals $\overline{W}_\varepsilon$, $\overline{W}_{\varepsilon\tau} : \mathcal{V} \to [0, \infty]$ *as*

$$\overline{W}_\varepsilon(u) := \begin{cases} W_\varepsilon(u) & \text{if } u \in \mathcal{K}(u^0, u^1), \\ \infty & \text{elsewhere}, \end{cases}$$

$$\overline{W}_{\varepsilon\tau}(u) := \begin{cases} W_{\varepsilon\tau}(u(0), u(\tau), \ldots, u(T)) & \text{if } u \in \widehat{\mathcal{V}}_\tau \cap \mathcal{K}(u^0, u^1), \\ \infty & \text{elsewhere}. \end{cases}$$

Then, $\overline{W}_{\varepsilon\tau}$ *Γ-converges to* $\overline{W}_\varepsilon$ *with respect to both the strong and the weak topology of* $\mathcal{V}$.

Before going on, let us remark that

$$e_\tau,\ \overline{e}_\tau,\ \overline{e}_\tau(\,\cdot\,+\tau),\ \overline{e}_\tau(\,\cdot\,+2\tau) \to \left(t \mapsto \mathrm{e}^{-t/\varepsilon}\right) \text{ strongly in } \mathrm{L}^\infty(0,T), \tag{5.24}$$

the convergence of e_τ being actually strong in $W^{1,\infty}(0,T)$.

PROOF. – The proof is classically divided into (i) proving the Γ-liminf inequality and (ii) checking the existence of a recovery sequence (see [7, 5]).

Ad (i). Assume to be given a sequence $u_\tau \in \widehat{\mathcal{V}}_\tau$ such that $u_\tau \rightharpoonup u$ with respect to the weak topology on $\mathcal{V}$ and $\liminf\limits_{\tau\to 0} \overline{W}_{\varepsilon\tau}(u_\tau) < \infty$. Let us denote by $\tilde{u}_\tau \in \mathrm{C}^1([0,T];V\cap X)$ the piecewise-quadratic-in-time interpolant of $u_i := u_\tau(i\tau)$, $i = 0,\dots,n$, defined by the relations

$$\tilde{u}_\tau(t) := u_\tau(t) \quad \text{for} \quad t \in [0,\tau]$$
$$\text{and} \quad \partial_t \tilde{u}_\tau = \alpha_\tau(t)\partial_t u_\tau(t) + (1-\alpha_\tau(t))\partial_t u_\tau(t-\tau) \quad \text{for } t \in (\tau, T]$$

where we have used the notation $\alpha_\tau(t) := \alpha_i(t)$ for $t \in ((i-1)\tau, i\tau]$, $i = 1,\dots,n$. We preliminarily observe that

$$\partial_t \tilde{u}_\tau(t) = \partial_t u_\tau(t-\tau) + \tau\alpha_\tau(t)\partial_{tt}\tilde{u}_\tau(t) \quad \forall t \in (\tau, T]. \tag{5.25}$$

Since $\liminf\limits_{\tau\to 0} \overline{W}_{\varepsilon\tau}(u_\tau) < \infty$ we can extract a not relabeled subsequence such that we have $u_\tau(0) = u^0$ and

$$\limsup_{\tau\to 0}\left(\frac{\varepsilon^2\rho}{2}\int_\tau^T \overline{e}_\tau|\partial_{tt}\tilde{u}_\tau|^2\,\mathrm{d}t + \frac{\varepsilon\nu}{2}\int_\tau^{T-\tau}\overline{e}_\tau(\,\cdot\,+\tau)|\partial_t u_\tau|^2\,\mathrm{d}t + \int_\tau^{T-2\tau}\overline{e}_\tau(\,\cdot\,+2\tau)E(\overline{u}_\tau)\,\mathrm{d}t\right) < \infty.$$

Then, owing to convergences 5.24 we have that, for small τ,

$$\rho\int_\tau^T |\partial_{tt}\tilde{u}_\tau|^2\,\mathrm{d}t + \nu\int_\tau^{T-\tau}|\partial_t u_\tau|^2\,\mathrm{d}t + \int_\tau^{T-2\tau} E(\overline{u}_\tau)\,\mathrm{d}t \le C.$$

Hence, by using the growth conditions (2.1) and by possibly further extracting a not relabeled subsequence (and considering standard projections for $t > T - 2\tau$) we have that

$$\overline{u}_\tau \rightharpoonup u \text{ weakly in } \mathrm{L}^p(0,T;X), \tag{5.26}$$
$$\overline{u}_\tau \rightharpoonup u \text{ weakly in } \mathrm{L}^2(0,T;V), \tag{5.27}$$
$$u_\tau \rightharpoonup u \text{ weakly in } \mathrm{H}^1(0,T;H), \tag{5.28}$$
$$\tilde{u}_\tau \rightharpoonup v \text{ weakly in } \mathrm{H}^2(0,T;H), \tag{5.29}$$
$$\rho\partial_t\tilde{u}_\tau \rightharpoonup \rho v_t \text{ strongly in } \mathrm{C}^0(0,T;H). \tag{5.30}$$

Indeed, we have that $v = u$. In order to check this fix $w \in \mathrm{L}^2(0,T;H)$ and compute that

$$\rho\int_0^T(\partial_t\tilde u_\tau - u_t, w)\,\mathrm{d}t = \rho\int_0^\tau(\partial_t u_\tau - u_t, w)\,\mathrm{d}t + \rho\int_\tau^T(\partial_t u_\tau(\,\cdot\,-\tau) + \tau\alpha_\tau\partial_{tt}\tilde u_\tau - u_t, w)\,\mathrm{d}t$$

$$= \rho\int_0^T(\partial_t u_\tau - u_t, w)\,\mathrm{d}t + \rho\int_\tau^T(\partial_t u_\tau(\,\cdot\,-\tau) - \partial_t u_\tau + \tau\alpha_\tau\partial_{tt}\tilde u_\tau, w)\,\mathrm{d}t$$

$$= \rho\int_0^T(\partial_t u_\tau - u_t, w)\,\mathrm{d}t - \rho\tau\int_\tau^T(1-\alpha_\tau)(\partial_{tt}\tilde u_\tau, w)\,\mathrm{d}t \to 0$$

where we have used (5.25), (5.28), $|\alpha_\tau|\le 1$, and the boundedness of $\sqrt{\rho}\,\partial_{tt}\tilde u$ in $\mathrm{L}^2(0,T;H)$. Namely, $\rho\partial_t\tilde u_\tau \to \rho u_t$ weakly in $\mathrm{L}^2(0,T;H)$ and $v=u$. In particular, owing to convergence (5.30) we have proved that $\rho u^1 = \rho\partial_t\tilde u_\tau(0) = \rho u_t(0)$ and $u\in\mathcal{K}(u^0,u^1)$.

Eventually, we exploit the convergences in (5.24) and (5.26)-(5.29) in order to get that

$$\int_0^T \mathrm{e}^{-t/\varepsilon}\frac{\varepsilon^2\rho}{2}|u_{tt}|^2\,\mathrm{d}t \le \liminf_{\tau\to 0}\int_\tau^T \overline{e}_\tau\frac{\varepsilon^2\rho}{2}|\partial_{tt}\tilde u_\tau|^2\,\mathrm{d}t = \liminf_{\tau\to 0}\sum_{j=2}^n \tau e_j\frac{\varepsilon^2\rho}{2}|\delta^2 u_j|^2,$$

$$\int_0^T \mathrm{e}^{-t/\varepsilon}\frac{\varepsilon\nu}{2}|u_t|^2\,\mathrm{d}t \le \liminf_{\tau\to 0}\int_\tau^{T-\tau} \overline{e}_\tau(\,\cdot\,+\tau)\frac{\varepsilon\nu}{2}|\partial_t u_\tau|^2\,\mathrm{d}t = \liminf_{\tau\to 0}\sum_{j=2}^{n-1} \tau e_{j+1}\frac{\varepsilon\nu}{2}|\delta u_j|^2,$$

$$\int_0^T \mathrm{e}^{-t/\varepsilon}E(u)\,\mathrm{d}t \le \liminf_{\tau\to 0}\int_\tau^{T-2\tau} \overline{e}_\tau(\,\cdot\,+2\tau)E(\overline{u}_\tau)\,\mathrm{d}t = \liminf_{\tau\to 0}\sum_{j=2}^{n-2} \tau e_{i+2}E(u_i).$$

In particular, these three inequalities ensure that

$$\begin{aligned}\overline{W}_\varepsilon(u) &\le \liminf_{\tau\to 0}\left(\sum_{j=2}^n \tau e_j\frac{\varepsilon^2\rho}{2}|\delta^2 u_j|^2 + \sum_{j=2}^{n-1}\tau e_{j+1}\frac{\varepsilon\nu}{2}|\delta u_j|^2 + \sum_{j=2}^{n-2}\tau e_{j+2}E(u_j)\right)\\ &= \liminf_{\tau\to 0} W_{\varepsilon\tau}(u_0,\ldots,u_n) = \liminf_{\tau\to 0}\overline{W}_{\varepsilon\tau}(u_\tau),\end{aligned}$$

which is the desired Γ-lim inf inequality.

Ad (ii). Let us define the *backward floating mean operator* M_τ on $\mathrm{L}^1(0,T;H)$ by setting

$$M_\tau(u)(t) := \begin{cases} u^0 & \text{for } t\in[0,\tau)\\ \dfrac{1}{\tau}\displaystyle\int_{t-\tau}^t u(s)\,\mathrm{d}s & \text{for } t\in[\tau,T].\end{cases}$$

Then, let $u\in\mathcal{K}(u^0,u^1)$ be fixed and define $(u_0,\ldots,u_n)$ by

$$u_0 = u^0, \quad \rho u_1 = \rho u^0 + \tau\rho u^1, \quad u_i = M_\tau(u)(i\tau) \quad \text{for } i = 2,\ldots,n.$$

We denote by u_τ and $\overline{u}_\tau$ the piecewise affine and constant interpolants, respectively, associated with $(u_0,\dots,u_n)$.

We aim to show that u_τ is a recovery sequence for u. Indeed, we clearly have that $\overline{u}_\tau$ converges strongly to u in $\mathrm{L}^2(0,T;V)\cap\mathrm{L}^p(0,T;X)$, while u_τ converges at least weakly to u in $\mathrm{L}^2(0,T;V)\cap\mathrm{L}^p(0,T;X)$. Moreover, one can check that

$$
\begin{aligned}
\int_0^T |\partial_t u_\tau - u_t|^2\,\mathrm{d}t &= \int_0^\tau |u^1-u_t|^2\,\mathrm{d}t + \sum_{i=2}^n \int_{(i-1)\tau}^{i\tau}\left|\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\big(u(s)-u(s-\tau)\big)\,\mathrm{d}s - u_t\right|^2\mathrm{d}t\\
&= \int_0^\tau |u^1-u_t|^2\,\mathrm{d}t + \sum_{i=2}^n \int_{(i-1)\tau}^{i\tau}\left|\fint_{(i-1)\tau}^{i\tau} M_\tau(u_t)\,\mathrm{d}s - u_t\right|^2\mathrm{d}t
\end{aligned}
\tag{5.31}
$$

Hence, as one has that $M_\tau(u_t)\to u_t$ strongly in $\mathrm{L}^2(0,T;H)$, we conclude that $u_\tau\to u$ strongly in $\mathrm{H}^1(0,T;H)$. In particular, we have verified that $u_\tau \rightharpoonup u$ weakly in $\mathcal{U}$

Next, we exploit the λ-convexity of F and compute that

$$
\begin{aligned}
\sum_{i=2}^{n-2} \tau e_{i+2}E(u_i) &= \sum_{i=2}^{n-2}\int_{(i-1)\tau}^{i\tau} e_{i+2}\big(E(\overline{u}_\tau)-E(u)\big) + e_{i+2}E(u)\,\mathrm{d}t\\
&\le \int_\tau^{T-2\tau} \overline{e}_\tau(\,\cdot\,+2\tau)\Big(\langle A\overline{u}_\tau,\overline{u}_\tau-u\rangle + \langle F(\overline{u}_\tau),\overline{u}_\tau-u\rangle - \frac{\lambda}{2}|\overline{u}_\tau-u|^2 + E(u)\Big)\,\mathrm{d}t
\end{aligned}
$$

In particular, by taking the lim sup as $\tau\to0$ and recalling that $\overline{u}_\tau\to u$ strongly in $\mathrm{L}^2(0,T;V)\cap\mathrm{L}^p(0,T;X)$ and the convergences (5.24), we have that

$$
\limsup_{\tau\to0}\left(\sum_{i=2}^{n-2}\tau e_{j+2}E(u_i)\right) \le \int_0^T \mathrm{e}^{-t/\varepsilon}E(u)\,\mathrm{d}t. \tag{5.32}
$$

Next, we deal with the second-order derivatives in time like we did in (5.31). We compute

$$
\begin{aligned}
\rho\int_\tau^T |\overline{\delta^2 u}_\tau - u_{tt}|^2\,\mathrm{d}t &= \rho\sum_{i=2}^n\int_{(i-1)\tau}^{i\tau}\left|\frac{u_i-2u_{i-1}+u_{i-2}}{\tau^2} - u_{tt}\right|^2\mathrm{d}t\\
&= \rho\sum_{i=2}^n\int_{(i-1)\tau}^{i\tau}\left|\frac{1}{\tau^3}\int_{(i-1)\tau}^{i\tau}\big(u-u(\,\cdot\,-\tau)\big)\,\mathrm{d}s - \frac{1}{\tau^3}\int_{(i-2)\tau}^{(i-1)\tau}\big(u-u(\,\cdot\,-\tau)\big)\,\mathrm{d}s - u_{tt}\right|^2\mathrm{d}t\\
&= \rho\sum_{i=2}^n\int_{(i-1)\tau}^{i\tau}\left|\frac{1}{\tau^3}\int_{(i-1)\tau}^{i\tau}\left(\int_{s-\tau}^{s}\big(u_t-u_t(\,\cdot\,-\tau)\big)\,\mathrm{d}r\right)\mathrm{d}s - u_{tt}\right|^2\mathrm{d}t\\
&= \rho\sum_{i=2}^n\int_{(i-1)\tau}^{i\tau}\left|\fint_{(i-1)\tau}^{i\tau} M_\tau\big(M_\tau(u_{tt})\big)\,\mathrm{d}s - u_{tt}\right|^2\mathrm{d}t \to 0
\end{aligned}
\tag{5.33}
$$

where the convergence to 0 is ensured by the fact that $M_\tau(M_\tau(u_{tt})) \to u_{tt}$ strongly in $\mathrm{L}^2(0,T;H)$. In particular, the convergence in (5.33) shows that (see the proof of (i) for the definition of $\tilde{u}_\tau$)

$$\partial_{tt}\tilde{u}_\tau \to u_{tt} \quad \text{strongly in} \quad \mathrm{L}^2(0,T;H).$$

Finally, combining (5.31)-(5.33) we have proved that

$$\begin{aligned}
\overline{W}_\varepsilon(u) &= \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\frac{\varepsilon^2\rho}{2}|u_{tt}|^2 + \frac{\varepsilon\nu}{2}|u_t|^2 + E(u)\right)\mathrm{d}t \\
&\geq \limsup_{\tau\to 0}\left(\int_\tau^T \overline{e}_\tau\frac{\varepsilon^2\rho}{2}|\partial_{tt}\tilde{u}_\tau|^2\,\mathrm{d}t + \int_\tau^{T-\tau}\overline{e}_\tau(\,\cdot+\tau)\frac{\varepsilon\nu}{2}|\partial_t u_\tau|^2\,\mathrm{d}t + \int_\tau^{T-2\tau}\overline{e}_\tau(\,\cdot+2\tau)E(\overline{u}_\tau)\,\mathrm{d}t\right) \\
&= \limsup_{\tau\to 0}\left(\sum_{i=2}^n \tau e_i\frac{\varepsilon^2\rho}{2}|\delta^2 u_i|^2 + \sum_{i=2}^{n-1}\tau e_{i+1}\frac{\varepsilon\nu}{2}|\delta u_i|^2 + \sum_{i=2}^{n-2}\tau e_{j+2}E(u_i)\right) \\
&= \limsup_{\tau\to 0} W_{\varepsilon\tau}(u_0,\ldots,u_n) \\
&= \limsup_{\tau\to 0} \overline{W}_{\varepsilon\tau}(u_\tau).
\end{aligned}$$

Namely, u_τ is a recovery sequence for u. □

PROOF OF LEMMA 4.1. – The minimizers u_τ^ε of the discrete functional $\overline{W}_{\varepsilon\tau}$ fulfill estimate (5.23) and are hence weakly precompact in $\mathcal{U}$. As $\overline{W}_{\varepsilon\tau}$ Γ-converges to $\overline{W}_\varepsilon$ with respect to the same topology by Lemma 5.4, we have, by the Fundamental Theorem of Γ-convergence (see [7, Ch. 7] and [5, Sect. 1.5]), that $u_\tau^\varepsilon \rightharpoonup u^\varepsilon$ weakly in $\mathcal{U}$, where u^ε is the unique minimizer of $\overline{W}_\varepsilon$. Finally, estimate (5.23) passes to the limit and we have proved Lemma 4.1.

Acknowledgments. U. S. and M. L. are partly supported by FP7-IDEAS-ERC-StG Grant # 200947 *BioSMA*. U. S. acknowledges the partial support of CNR-AV̌CR Grant *SmartMath*, the CNR-JSPS Grant *VarEvol*, and the Alexander von Humboldt Foundation. Furthermore, M. L. thanks the Istituto di Matematica Applicata e Tecnologie Informatiche *Enrico Magenes* in Pavia, where part of the work was conducted, for its kind hospitality.

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Received March 6, 2012 and in revised form August 29, 2012