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Eigenfunctions of the Laplace-Beltrami Operator, and Isoperimetric and Isocapacitary Inequalities (*)

ANDREA CIANCHI

1. – Introduction

This is a slightly expanded version of a talk that I delivered in the occasion of the XIX Congress of the Italian Mathematical Society held in Bologna in September 2011. I wish to thank the Scientific Committee and the Organizing Committee of the Congress for their kind invitation, of which I am sincerely honored.

We shall discuss some methods of geometric nature in the study of qualitative and quantitative aspects of eigenvalue problems for the Laplace operator, and some of its generalizations. Among various issues of relevance on this topic, two questions will be focused. On the one hand, information on the spectrum of the Laplacian, and, in particular, on its discreteness, will be provided. On the other hand, criteria for the regularity of eigenfunctions, and specifically their integrability and boundedness, will be illustrated. The results to be presented are the fruits of a collaboration with V. G. Maz'ya.

A prototypal instance of the problems at hand concerns the Laplace operator in a bounded open subset Ω of $\mathbb{R}^n$ coupled with (homogenous) Dirichlet boundary conditions. The Dirichlet Laplacian on Ω, denoted by Δ_Ω^D, is the semi-definite self-adjoint operator in the Hilbert space $L^2(M)$ associated with the closed bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \tag{1.1}$$

which is defined for u and v in the Sobolev space $W_0^{1,2}(\Omega)$.

The spectrum of the operator Δ_Ω^D is discrete, namely its continuous spectrum is empty, and all the eigenvalues have finite multiplicity. This is a classics in spectral theory – see e.g. [RS].

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In a PDE's setting, the eigenvalue problem for the Laplacian under Dirichlet boundary conditions takes the form

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{su } \partial\Omega\,. \end{cases} \tag{1.2}$$

Recall that a function $u \in W_0^{1,2}(\Omega)$ is an eigenfunction of problem (1.2) associated with the eigenvalue $\lambda \in \mathbb{R}$ if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx \tag{1.3}$$

for every test function $v \in W_0^{1,2}(\Omega)$.

Standard regularity results ensure that any eigenfunction u of problem (1.2) is bounded in Ω. Moreover, a sharp estimate is known for any $L^q(\Omega)$ norm of u in terms of its $L^2(\Omega)$ norm. A result of [PR] and [Chi] tells us that, if $q \in (2, \infty]$, then

$$\|u\|_{L^q(\Omega)} \leq C(n, q, \lambda) \|u\|_{L^2(\Omega)} \tag{1.4}$$

for every eigenfunction u of (1.2) associated with the eigenvalue λ, where

$$C(n, q, \lambda) = n\omega_n^{\frac{1}{q}-\frac{1}{2}} \lambda^{\frac{(q-2)n}{4q}} \frac{\left(\int_0^{j_{\frac{n}{2}-1}} r^{n-1+q-\frac{qn}{2}} J_{\frac{n}{2}-1}(r)^q dr \right)^{\frac{1}{q}}}{\left(\int_0^{j_{\frac{n}{2}-1}} r J_{\frac{n}{2}-1}(r)^2 dr \right)^{\frac{1}{2}}} ,$$

$j_{\frac{n}{2}-1}$ is the first positive zero of the Bessel function $J_{\frac{n}{2}-1}$, and $\omega_n = \pi^{\frac{n}{2}} / \Gamma\left(\frac{n}{2} + 1\right)$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Moreover, equality holds in (1.4) if and only if Ω is a ball, and λ is the smallest Dirichlet eigenvalue in Ω.

A key tool in the proof of inequality (1.4) is the standard isoperimetric inequality in $\mathbb{R}^n$ due to De Giorgi [D], which, loosely speaking, amounts to saying that the ball minimizes perimeter among all sets in $\mathbb{R}^n$ of given measure. More precisely, on denoting by $|E|$ the Lebesgue measure of a set E in $\mathbb{R}^n$, and by $P(E)$ its perimeter (in the sense of geometric measure theory [Ma7, Zi]), the isoperimetric inequality in $\mathbb{R}^n$ asserts that

$$n\omega_n^{1/n} |E|^{1/n'} \leq P(E) \tag{1.5}$$

for every measurable set E in $\mathbb{R}^n$ having finite measure.

Variants in this basic setting lead to different conclusions, whose proofs require other tools. This is the case, for instance, when eigenvalue problems for the Laplacian under Neumann boundary conditions are considered.

The Neumann Laplacian in an open set Ω of finite measure in $\mathbb{R}^n$, denoted by Δ_Ω^N, is the semi-definite self-adjoint operator in the Hilbert space $L^2(M)$ associated with the same closed bilinear form $a(u,v)$ as in (1.1), but now defined for u and v in the whole Sobolev space $W^{1,2}(\Omega)$.

The Neumann counterpart of (1.2) is problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{su } \partial\Omega\,. \end{cases} \tag{1.6}$$

An eigenfunction u of the Laplacian under homogeneous Neumann boundary conditions is a solution to (1.6) for some $\lambda \in \mathbb{R}$. This amounts to requiring that $u \in W^{1,2}(\Omega)$, and fulfills the equality

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx \tag{1.7}$$

for every test function $v \in W^{1,2}(\Omega)$.

Unlike that of Δ_Ω^D, the spectrum of Δ_Ω^N need not be discrete if Ω is irregular. Moreover, the (possible) eigenfunctions u of (1.6) may not belong to $L^q(\Omega)$ for $q > 2$.

Thus, the question arises of minimal regularity conditions on Ω for the spectrum of Δ_Ω^N to be discrete, or for an estimate of the form

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{L^2(\Omega)} \tag{1.8}$$

to hold for some $q \in (2, \infty]$, for some constant $C = C(\Omega, q, \lambda)$, and for any eigenfunction u of (1.6) associated with the eigenvalue λ.

One purpose of this exposition to point out how these issues can be addressed on making use of a variant of the isoperimetric inequality (1.5), called the relative isoperimetric inequality in Ω, where subsets E of Ω are considered, and the $(n-1)$-dimensional Hausdorff measure of just the inner part of the boundary of E comes into play. An inequality involving a suitable capacity – the condeser capacity – of subsets of Ω, instead of their perimeter, will be shown to be even more effective in dealing with very iregular domains.

In fact, our discussion will be carried out in a broader framework, where the open set Ω is possibly replaced with a more general noncompact n-dimensional Riemannian manifold M. Accordingly, the Laplace-Beltrami operator on M will be taken into account.

The analysis of spectral problems for the Laplace-Beltrami operator on Riemannian manifolds is a very classical issue. We do not even attempt to provide an exhaustive bibliography on this matter; let us just mention the reference monographs [Cha, BGM], and the papers [Bou, Br, BD, Che, CGY, DS, Do1, Do2, Es, Ga, Gr2, HSS, JMS, Na, SS, So, SZ, Ya]. Most contributions to this topic regard compact manifolds.

Our main focus will be instead on the case when

$$M \text{ need not be compact,}$$

and

$$\mathcal{H}^n(M) < \infty ,$$

an assumption which will be kept in force throughout. Here, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on M, namely the volume measure associated with the Riemannian metric on M. We shall also assume that M is connected.

Minimal conditions on the geomerty of the manifold M for the discreteness of the spectrum of the Laplace-Beltrami operator, and for $L^q(M)$ or $L^\infty(M)$ estimates for eigenfunctions will be presented.

Information on the regularity of the geometry of M will be retained either through the isocapacitary function ν_M, or the isoperimetric function I_M of M. They are the largest functions of the measure of subsets of M which can be estimated by the capacity, or by the perimeter of the relevant subsets, respectively. Loosely speaking, the asymptotic behavior of ν_M and I_M at 0 accounts for the regularity of the geometry of the noncompact manifold M: decreasing this regularity causes $\nu_M(s)$ and $I_M(s)$ to decay faster to 0 as s goes to 0. The inequalities associated with ν_M and I_M are called the isocapacitary inequality and the isoperimetric inequality on M, respectively.

Let us mention that the isoperimetric function of open sets in $\mathbb{R}^n$ was introduced in [Ma1], and employed in the study of Sobolev inequalities [Ma1], and in a priori estimates for solutions to elliptic boundary value problems [Ma2, Ma6]. Isocapacitary functions were introduced and used in [Ma1, Ma3, Ma4, Ma5, Ma7] in the characterization of Euclidean Sobolev embeddings. Later investigations and applications of the isoperimetric function, as well as extensions to the case of Riemannian manifolds, can be found e.g. in [CGL, Ga, Gr1, Gr2, GP, Kl].

Both the conditions in terms of ν_M, and those in terms of I_M, that will be imposed for the spectrum to be discrete, or for the validity of eigenfunction estimates are sharp in the class of manifolds M with prescribed asymptotic behavior of ν_M and I_M at 0. Each one of these two approaches has its own advantages. The isoperimetric function I_M has an apparent geometric nature, and it is usually easier to investigate. Although the isocapacitary function can be less simple to be analyzed, its use is in a sense more appropriate, since it not only implies the results involving I_M, but also applies in situations where the latter does not. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.

The results outlined in the present note are the object of the papers [CM2] and [CM3], to which we refer for a more detailed exposition and for proofs. Let us mention that methods relying upon the use of isoperimetric and isocapacitary functions have also been successfully applied to deal with existence [ACMM] and regularity [CM1, CM4] of solutions to Neumann boundary value problems in irregular domains and with irregular data.

2. – Perimeter and capacity

Let E be a measurable subset of M. Its perimeter $P(E)$ can be defined as

$$P(E) = \mathcal{H}^{n-1}(\partial^* E)\,,$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure on M induced by its Riemannian metric. Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when E is sufficiently regular, for instance an n-dimensional Riemannian submanifold (with boundary) of M. In the special case when M is an open subset Ω of $\mathbb{R}^n$, and $E \subset \Omega$, we have that $P(E) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$.

The isoperimetric function $I_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty]$ of M is defined as

$$I_M(s) = \inf\{P(E) : s \le \mathcal{H}^n(E) \le \mathcal{H}^n(M)/2\} \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]\,. \tag{2.1}$$

The isoperimetric inequality

$$I_M(\mathcal{H}^n(E)) \le P(E) \qquad \text{if } 0 < \mathcal{H}^n(E) \le \mathcal{H}^n(M)/2\,, \tag{2.2}$$

is a straightforward consequence of definition (2.1). The function I_M is explicitly known only for manifolds in special classes [BC, CF, CGL, GP, MHH, Kl, MJ, Pi, Ri], including Euclidean balls and spheres [BuZa, Ci2, Ma7]. Qualitative and quantitative information on I_M is however available under fairly general assumptions – see e.g. [BuZa, Ci1, HK, KM, La, Ma7]. In particular, the fact that M is connected ensures that

$$I_M(s) > 0 \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2]. \tag{2.3}$$

The asymptotic behavior of I_M at 0 depends on the regularity of the geometry of M. For instance, if M is compact, or if it is an open set in $\mathbb{R}^n$ with Lipschitz boundary, then

$$I_M(s) \approx s^{1/n'} \qquad \text{near } 0. \tag{2.4}$$

Here, and in what follows, the notation

$$f \approx g \qquad \text{near } 0 \tag{2.5}$$

for functions $f, g : (0, \infty) \to [0, \infty)$ means that there exist positive constants c_1, c_2 and s_0 such that

$$c_1 g(c_1 s) \le f(s) \le c_2 g(c_2 s) \qquad \text{if } s \in (0, s_0). \tag{2.6}$$

The notion of capacity is related to that of the Sobolev space $W^{1,2}(M)$. The latter is defined as

$$W^{1,2}(M) = \{u \in L^2(M) : \quad u \text{ is weakly differentiable on } M \text{ and } |\nabla u| \in L^2(M)\}\,.$$

Here, ∇u denotes the gradient of u on M, and $|\nabla u|$ is its length in the Riemannian metric of M. We adopt the notation $W^{1,2}_0(M)$ for the closure in $W^{1,2}(M)$ of the set of smooth compactly supported functions on M.

The standard capacity of a set $E \subset M$ can be defined as

$$(2.7)\quad C(E) = \inf\left\{\int_M |\nabla u|^2\,dx : u \in W^{1,2}_0(M), u \geq 1 \text{ in some open neighbourhood of } E\right\}.$$

A property is said to hold quasi everywhere in M, briefly q.e., if it is fulfilled outside a set of capacity zero.

Each function $u \in W^{1,2}(M)$ has a precise representative $\widetilde{u}$ which is quasi continuous, in the sense that for every $\varepsilon > 0$, there exists a set $A \subset M$, with $C(A) < \varepsilon$, such that the restriction of $\widetilde{u}$ to $M \setminus A$ is continuous. The function $\widetilde{u}$ is unique, up to subsets of capacity zero. In what follows, we assume that any function $u \in W^{1,2}(M)$ is precisely represented. A standard result in the theory of capacity ensures that, for every set $E \subset M$,

$$(2.8)\qquad C(E) = \inf\left\{\int_M |\nabla u|^2\,dx : u \in W^{1,2}_0(M), u \geq 1 \text{ q.e. in } E\right\}$$

(see e.g. [MZ, Corollary 2.25]).

A variant of the capacity $C(E)$, of use in our applications, is the condenser capacity $C(E,G)$ of the sets $E \subset G \subset M$, which is defined as

$$(2.9)\quad C(E,G) = \inf\left\{\int_M |\nabla u|^2\,dx : u \in W^{1,2}(M),\ u \geq 1 \text{ q.e. in } E \text{ and } u \leq 0 \text{ q.e. in } M \setminus G\right\}.$$

Accordingly, the isocapacitary function $\nu_M : [0, \mathcal{H}^n(M)/2] \to [0,\infty]$ of M is given by

$$(2.10)\quad \nu_M(s) = \inf\{C(E,G) : E \text{ and } G \text{ are measurable subsets of } M \text{ such that } E \subset G \subset M \text{ and } s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2\} \quad \text{for } s \in [0, \mathcal{H}^n(M)/2].$$

The ensuing isocapacitary inequality reads

$$(2.11)\qquad \nu_M(\mathcal{H}^n(E)) \leq C(E,G) \qquad \text{if } 0 < \mathcal{H}^{n-1}(E) \leq \mathcal{H}^n(M)/2\,.$$

The function ν_M is clearly non-decreasing. The functions ν_M and I_M are related by the inequality

$$\nu_M(s) \geq \frac{1}{\int\limits_s^{\mathcal{H}^n(M)/2} \frac{dr}{I_M(r)^2}} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2). \tag{2.12}$$

Let us notice that a reverse inequality in (2.12) does not hold in general, even up to a multiplicative constant.

Combining (2.3) and (2.12) tells us that

$$\nu_M(s) > 0 \qquad \text{for } s \in [0, \mathcal{H}^n(M)/2]. \tag{2.13}$$

When M is compact, or when it is an open set in $\mathbb{R}^n$ with Lipschitz boundary, one can exploit equations (2.4) and (2.12) to show that

$$\nu_M(s) \approx \begin{cases} s^{\frac{n-2}{n}} & \text{if } n \geq 3, \\ \left(\log \frac{1}{s}\right)^{-1} & \text{if } n = 2, \end{cases} \tag{2.14}$$

near 0.

3. – Discreteness of the spectrum

Given any n-dimensional Riemannian manifold M as in Section 1, we denote by Δ_M the semi-definite self-adjoint Laplace operator in the Hilbert space $L^2(M)$ associated with the closed bilinear form

$$a(u,v) = \int\limits_M \langle \nabla u, \nabla v \rangle \, d\mathcal{H}^n(x),$$

defined for u and v in the Sobolev space $W^{1,2}(M)$. Here, $\langle \cdot \, , \cdot \rangle$ stands for the scalar product induced by the Riemannian metric on M.

This definition of Δ_M includes various special instances. For example, if the space $C_0^\infty(M)$ of smooth compactly supported functions on M is dense in $W^{1,2}(M)$, the operator Δ_M agrees with the Friedrichs extension of the classical Laplacian, regarded as an unbounded operator on $L^2(M)$ with domain $C_0^\infty(M)$. This is certainly the case when M is complete [Ro, St], and, in particular, if M is compact. When M is an open subset Ω of $\mathbb{R}^n$, the operator Δ_Ω agrees with the Neumann Laplacian Δ_Ω^N on Ω.

In the present section we deal with the problem of the discreteness of the spectrum of Δ_M. This property is well known when M is compact, or when M is an open subset of $\mathbb{R}^n$ with finite measure and sufficiently regular boundary. However, the spectrum of Δ_M may be not discrete in general.

Special situations, which are not included in this standard frameworks, have been considered in the literature by ad hoc methods. For instance, conditions for the discreteness of the spectrum of the Laplacian on noncompact complete manifolds with a peculiar structure are the object of [Ba, Bro, DL, Es, Kl1, Kl2].

A necessary and sufficient condition on a Riemannian manifold M for the spectrum of Δ_M to be discrete can be given in terms of its isocapacitary function ν_M.

THEOREM 3.1 (Manifolds with a discrete spectrum). – *The spectrum of Δ_M is discrete if and only if*

$$\lim_{s\to 0}\frac{s}{\nu_M(s)}=0\,. \tag{3.1}$$

Theorem 3.1, combined with inequality (2.12), yields a sufficient condition for the discreteness of the spectrum of Δ_M in terms of the isoperimetric function I_M of M.

COROLLARY 3.2 (Discreteness of the spectrum via I_M). – *Assume that*

$$\lim_{s\to 0}\frac{s}{I_M(s)}=0\,. \tag{3.2}$$

Then the spectrum of Δ_M is discrete.

Let us point out that conditions (3.1) and (3.2), as well as any other criterium involving ν_M or I_M that will exhibited in what follows, are invariant under replacements of ν_M or I_M with functions which are equivalent near 0 in the sense of (2.5).

Observe that in particular, the classical result on the discreteness of the spectrum of the Laplacian on any compact Riemannian manifold M can be recovered either via Theorem 3.1, or via Corollary 3.2, owing to (2.4), or (2.14), respectively.

Although not necessary for a single manifold M, assumption (3.2) is essentially minimal for the spectrum of Δ_M to be discrete in classes of manifolds M with prescribed isoperimetric function I_M. To be more specific, consider any non-decreasing function $I:[0,\infty)\to[0,\infty)$, vanishing only at 0, and such that

$$\frac{I(s)}{s^{\frac{n-1}{n}}}\approx \text{ a non-decreasing function near 0.} \tag{3.3}$$

Then, there exists an n-dimensional Riemannian manifold of revolution M fulfilling

$$I_M(s)\approx I(s) \qquad \text{near } 0, \tag{3.4}$$

[CM3, Proposition 4.3]. Note that assumption (3.3) is required in the light of the fact that (2.4) holds for any compact manifold M, and that $I_M(s)$ cannot decay more slowly to 0 as $s \to 0$ in the noncompact case. Now, if I is such that $\limsup_{s\to 0} \frac{s}{I(s)} > 0$, then

$$\limsup_{s\to 0} \frac{s}{I_M(s)} > 0 \tag{3.5}$$

as well. Owing to Proposition 5.1, Section 5 below, condition (3.5) for the relevant manifold of revolution M is equivalent to

$$\limsup_{s\to 0} \frac{s}{v_M(s)} > 0,$$

and, by Theorem 3.1, the latter implies that the spectrum of Δ_M is not discrete.

A key step, of possible independent interest, in the derivation of Theorem 3.1 is the next theorem, showing the equivalence of condition (3.1) to the compactness of the embedding

$$W^{1,2}(M) \to L^2(M). \tag{3.6}$$

THEOREM 3.3. – *Embedding* (3.6) *is compact if and only if* (3.1) *holds.*

Indeed, a standard result in the theory of positive-definite self-adjoint operators in Hilbert spaces (see e.g. [BS, Chapter 10, Section 1, Theorem 5]) ensures that the discreteness of the spectrum of the operator $-\Delta_M + \mathrm{Id}$, and hence of $-\Delta_M$, on M is equivalent to the compactness of embedding (3.6).

4. – Eigenfunction estimates

We are concerned here with estimates for eigenfunctions of the Laplacian on the manifold M, namely functions $u \in W^{1,2}(M)$ fulfilling

$$\int_M \langle \nabla u \,, \nabla v \rangle \, d\mathcal{H}^n(x) = \lambda \int_M u\, v \, d\mathcal{H}^n(x) \tag{4.1}$$

for some $\lambda \in \mathbb{R}$, and for every test function $v \in W^{1,2}(M)$.

Note that, if M is a complete Riemannian manifold, then (4.1) is equivalent to the weak formulation of the equation

$$\Delta u + \lambda u = 0 \qquad \text{on } M, \tag{4.2}$$

where Δ denotes the Laplace-Beltrami operator on M, called Laplacian in what follows, for simplicity. In the case where M is an open subset Ω of $\mathbb{R}^n$, equation

(4.1) agrees with (1.7), and hence solutions to (4.1) are eigenfunctions of the Neumann problem (1.6).

When M is compact, one easily infers, via local regularity results for elliptic equations, that any eigenfunction u of the Laplacian belongs to $L^\infty(M)$. Explicit bounds, with sharp dependence on the eigenvalue λ, are also available [SS, SZ], and require sophisticated tools from differential geometry and harmonic analysis. If the compactness assumption is dropped, then the membership of u to $W^{1,2}(M)$ only (trivially) implies that $u \in L^2(M)$. Higher integrability of eigenfunctions is not guaranteed anymore. Our aim is to exhibit minimal assumptions on M ensuring $L^q(M)$ bounds for all $q<\infty$, or even $L^\infty(M)$ bounds for eigenfunctions of the Laplacian on M. The results to be presented can be easily extended to linear uniformly elliptic differential operators, in divergence form, with merely measurable coefficients on M. Let us emphasize, however, that they provide nontrivial new information even for the Neumann Laplacian on open subsets of $\mathbb{R}^n$.

4.1 – L^q *estimates for eigenfunctions*

An optimal condition on the decay of ν_M at 0 ensuring $L^q(M)$ estimates for eigenfunctions of the Laplacian on M for $q \in (2, \infty)$ is contained in the following theorem. Interestingly enough, such a condition is independent of q.

THEOREM 4.1 (L^q bounds for eigenfunctions via ν_M). – *Assume that*

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,. \tag{4.3}$$

Then for any $q \in (2, \infty)$ and for any eigenvalue λ, there exists a constant $C = C(\nu_M, q, \lambda)$ such that

$$\|u\|_{L^q(M)} \le C\|u\|_{L^2(M)} \tag{4.4}$$

for every eigenfunction u of the Laplacian on M associated with λ.

An estimate for the constant C in inequality (4.4) can also be provided.

PROPOSITION 4.2. – *Define the function $\Theta : (0, \mathcal{H}^n(M)/2] \to [0, \infty)$ as*

$$\Theta(s) = \sup_{r\in(0,s)} \frac{r}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then inequality (4.4) *holds with*

$$C(\nu_M, q, \lambda) = \frac{C_1}{\left(\Theta^{-1}(C_2/\lambda)\right)^{\frac{1}{2}-\frac{1}{q}}}\,,$$

where $C_1 = C_1(q, \mathcal{H}^n(M))$ and $C_2 = C_2(q, \mathcal{H}^n(M))$ are suitable constants, and Θ^{-1} is the generalized left-continuous inverse of Θ.

EXAMPLE 4.3. – Assume that $n \geq 3$, and there exists $\beta \in [(n-2)/n, 1)$ such that the manifold M fulfils $\nu_M(s) \geq Cs^\beta$ for some positive constant C and for $s \in [0, \mathcal{H}^n(M)/2]$. Then (4.3) holds, and, by Proposition 4.2, for every $q \in (2, \infty)$ there exists an constant $C = C(q, \mathcal{H}^n(M))$ such that

$$\|u\|_{L^q(M)} \leq C\lambda^{\frac{q-2}{2q(1-\beta)}}\|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue λ.

Let us note that, by Theorem 3.3, condition (4.3) turns out to be equivalent to the compactness of the embedding $W^{1,2}(M) \to L^2(M)$. Hence, in particular, the variational characterization of the eigenvalues of the Laplacian on M entails that they certainly exist under (4.3). Observe that condition (4.3) is also equivalent to the discreteness of the spectrum of Δ_M on M provided by Theorem 3.1.

The next result shows that assumption (4.3) is essentially minimal in Theorem 4.1, in the sense that $L^q(M)$ regularity of eigenfunctions may fail under the mere assumption that

$$\nu_M(s) \approx s \qquad \text{near } 0.$$

THEOREM 4.4 (Sharpness of condition (4.3)). – *For any $n \geq 2$ and $q \in (2, \infty]$, there exists an n-dimensional Riemannian manifold M such that*

$$\nu_M(s) \approx s \qquad \textit{near } 0, \tag{4.5}$$

and the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

The manifold mentioned in Theorem 4.4 is a manifold of revolution from a family described in Subsection 5.1 below.

The following criterion for $L^q(M)$ bounds of eigenfunctions in terms of the isoperimetric function I_M can be obtained from Theorem 4.1 and inequality (2.12).

THEOREM 4.5 (L^q bounds for eigenfunctions via I_M). – *Assume that*

$$\lim_{s\to 0} \frac{s}{I_M(s)} = 0\,. \tag{4.6}$$

Then for any $q \in (2, \infty)$ and any eigenvalue λ, there exists a constant $C = C(I_M, q, \lambda)$ such that

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)} \tag{4.7}$$

for every eigenfunction u of the Laplacian on M associated with λ.

An analogue of Theorem 4.4 on the minimality of assumption (4.6) in Theorem 4.5 is contained in the the next result, showing that, for every $q > 2$, eigenfunctions which do not belong to $L^q(M)$ may actually exist when

$$I_M(s) \approx s \qquad \text{near } 0.$$

THEOREM 4.6 (Sharpness of condition (4.6)). – *For any $n \geq 2$ and $q \in (2, \infty]$, there exists an n-dimensional Riemannian manifold M such that*

$$I_M(s) \approx s \qquad \textit{near } 0, \tag{4.8}$$

and the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

4.2 – *Boundedness of eigenfunctions*

The boundedness of eigenfunctions cannot be established via the criterion of Theorem 4.1. This is instead the object of the following result, where a slight strengthening of assumption (4.3) is shown to yield $L^\infty(M)$ estimates for eigenfunctions of the Laplacian on M.

THEOREM 4.7 (Boundedness of eigenfunctions via ν_M). – *Assume that*

$$\int_0 \frac{ds}{\nu_M(s)} < \infty\,. \tag{4.9}$$

Then, for any eigenvalue λ, there exists a constant $C = C(\nu_M, \lambda)$ such that

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)} \tag{4.10}$$

for every eigenfunction u of the Laplacian on M associated with λ.

Information on the constant C appearing in (4.10) is available.

PROPOSITION 4.8. – *Assume that* (4.9) *is in force. Define the function $\Xi : (0, \mathcal{H}^n(M)/2] \to [0, \infty)$ as*

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)} \qquad \textit{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then inequality (4.10) *holds with*

$$C(\nu_M, \lambda) = \frac{C_1}{\big(\Xi^{-1}(C_2/\lambda)\big)^{\frac{1}{2}}}\,,$$

where C_1 and C_2 are suitable absolute constants, and Ξ^{-1} is the generalized left-continuous inverse of Ξ.

EXAMPLE 4.9. – Assume that $n \geq 3$, and there exists $\beta \in [(n-2)/n, 1)$ such that the manifold M fulfils $\nu_M(s) \geq Cs^{\beta}$ for some positive constant C and for $s \in [0, \mathcal{H}^n(M)/2]$. Then (4.9) holds, and, by Proposition 4.8, there exists an absolute constant C such that

$$\|u\|_{L^\infty(M)} \leq C\lambda^{\frac{1}{2(1-\beta)}}\|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue λ.

Condition (4.9) in Theorem 4.7 is essentially sharp for the boundedness of eigenfunctions of the Laplacian on M. In particular, it cannot be relaxed to (4.3), although the latter ensures $L^q(M)$ estimates for every $q < \infty$. Indeed, under some mild qualification, Theorem 4.10 below asserts that given (up to equivalence) any isocapacitary function fulfilling (4.3) but not (4.9), there exists a manifold M with the prescribed isocapacitary function on which the Laplacian has an unbounded eigenfunction.

A precise statement of this result involves the notion of function of class Δ_2. Recall that a non-decreasing function $f : (0, \infty) \to [0, \infty)$ is said to belong to the class Δ_2 near 0 if there exist constants c and s_0 such that

$$f(2s) \leq cf(s) \qquad \text{if } 0 < s \leq s_0. \tag{4.11}$$

THEOREM 4.10 (Sharpness of condition (4.9)). – *Let ν be a non-decreasing function, vanishing only at* 0*, such that*

$$\lim_{s\to 0} \frac{s}{\nu(s)} = 0\,,$$

but

$$\int_0 \frac{ds}{\nu(s)} = \infty\,.$$

Assume in addition that $\nu \in \Delta_2$ near 0*, and that either $n \geq 3$ and*

$$\frac{\nu(s)}{s^{\frac{n-2}{n}}} \approx \text{ a non-decreasing function near } 0, \tag{4.12}$$

or $n = 2$ and there exists $\alpha > 0$ such that

$$\frac{\nu(s)}{s^{\alpha}} \approx \text{ a non-decreasing function near } 0. \tag{4.13}$$

Then, there exists an n-dimensional Riemannian manifold M fulfilling

$$\nu_M(s) \approx \nu(s) \qquad \text{near } 0, \tag{4.14}$$

and such that the Laplacian on M has an unbounded eigenfunction.

Assumption (4.12) or (4.13) in Theorem 4.10 is explained by the behavior (2.14) of ν_M when M is compact, and the fact that $\nu_M(s)$ cannot decay more slowly to 0 as $s \to 0$ in general. The assumption that $\nu \in \Delta_2$ near 0 is due to technical reasons.

A condition on I_M, parallel to (4.9), ensuring the boundedness of eigenfunctions of the Laplacian on M follows from Theorem 4.7 and inequality (2.12).

THEOREM 4.11 (Boundedness of eigenfunctions via I_M). – *Assume that*

$$\int_0 \frac{s}{I_M(s)^2}\, ds < \infty\,. \tag{4.15}$$

Then, for any eigenvalue λ, there exists a constant $C = C(I_M, \lambda)$ such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{4.16}$$

for every eigenfunction u of the Laplacian on M associated with λ.

Our last result tell us that the gap between condition (4.15), ensuring $L^\infty(M)$ bounds for eigenfunctions, and condition (4.6), yielding $L^q(M)$ bounds for any $q < \infty$, cannot be essentially filled.

THEOREM 4.12 (Sharpness of condition (4.15)). – *Let I be a non-decreasing function, vanishing only at* 0, *such that*

$$\lim_{s\to 0} \frac{s}{I(s)} = 0\,,$$

but

$$\int_0 \frac{s}{I(s)^2}\, ds = \infty\,.$$

Assume in addition that

$$\frac{I(s)}{s^{\frac{n-1}{n}}} \approx \text{ a non-decreasing function near } 0. \tag{4.17}$$

Then, there exists an n-dimensional Riemannian manifold M fulfilling

$$I_M(s) \approx I(s) \qquad \textit{near } 0,$$

and such that the Laplacian on M has an unbounded eigenfunction.

Observe that assumption (4.17) agrees with (3.3), on which we commented in Section 3.

5. – Applications

We conclude with applications of the results of the preceding sections to two classes of Riemannian manifolds. The former consists of Riemannian manifolds of revolution, the latter of a family of surfaces in $\mathbb{R}^3$, each one containing a sequence of clustering submanifolds. The analysis of the surfaces from such a family will demonstrate the advantage in the use of isocapacitary inequalities, instead of the more standard isoperimetric inequalities, in the problems under consideration.

5.1 – *Manifolds of revolution*

Let $L \in (0, \infty]$, and let $\varphi : [0, L) \to [0, \infty)$ be a function in $C^1([0, L))$, such that

$$\varphi(r) > 0 \quad \text{for } r \in (0, L), \tag{5.1}$$

$$\varphi(0) = 0\,, \qquad \text{and} \qquad \varphi'(0) = 1\,. \tag{5.2}$$

Here, φ' denotes the derivative of φ. Given $n \geq 2$, we call n-dimensional manifold of revolution M associated with φ the ball in $\mathbb{R}^n$ given, in polar coordinates, by $\{(r, \omega) : r \in [0, L), \omega \in \mathbb{S}^{n-1}\}$ and endowed with the Riemannian metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2\,, \tag{5.3}$$

where $d\omega^2$ stands for the standard metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$. Owing to our assumptions on φ, the metric (5.3) is of class $C^1(M)$.

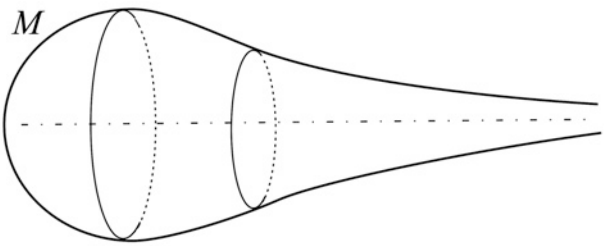

Fig. 1. – A manifold of revolution.

Under an additional convexity assumption near infinity, for the manifolds of this family conditions (4.3) and (4.6), and conditions (4.9) and (4.15) turn out to coincide, and can be formulated in terms of φ.

PROPOSITION 5.1. – *Let* $L \in (0, \infty]$ *and let* $\varphi : [0, L) \to [0, \infty)$ *be a function in* $C^1([0, L))$ *fulfilling* (5.1) *and* (5.2) *and such that:*

(i) $\lim_{r\to L} \varphi(r) = 0$;

(ii) *there exists* $L_0 \in (0, L)$ *such that* φ *is decreasing and convex in* (L_0, L);

(iii) $\int_0^L \varphi(\rho)^{n-1}\, d\rho < \infty$.

Then the metric of the n-dimensional manifold of revolution M built upon φ *is of class* $C^1(M)$, *and* $\mathcal{H}^n(M) < \infty$. *Moreover:*

(i) *Conditions* (4.3), (4.6), *and*

$$\lim_{r\to L} \left(\int_R^r \frac{d\rho}{\varphi(\rho)^{n-1}} \right) \left(\int_r^L \varphi(\rho)^{n-1} d\rho \right) = 0$$

are equivalent. Here, R is any number in $(0, L)$.

(ii) *Conditions* (4.9), (4.15), *and*

$$\int^L \left(\frac{1}{\varphi(r)^{n-1}} \int_r^L \varphi(\rho)^{n-1} d\rho \right) dr < \infty$$

are equivalent.

A characterization of those manifolds of revolution on which the spectrum of Δ_M is discrete follows from Theorem 3.1 and Proposition 5.1.

PROPOSITION 5.2. – *Let L and* φ *be as in the statement of Theorem* 5.1. *Let M be the n-dimensional manifold of revolution built upon* φ. *Then the spectrum of* Δ_M *is discrete if and only if*

$$\lim_{r\to L} \left(\int_R^r \frac{d\rho}{\varphi(\rho)^{n-1}} \right) \left(\int_r^L \varphi(\rho)^{n-1} d\rho \right) = 0$$

for any $R \in (0, L)$.

Estimates for eigenfunctions of the Laplacian on manifolds of revolution M are the content of the next result.

PROPOSITION 5.3. – *Let L and* φ *be as in the statement of Theorem* 5.1. *Let M be the n-dimensional manifold of revolution built upon* φ.

(i) *Assume that*

$$\lim_{r\to L} \left(\int_R^r \frac{d\rho}{\varphi(\rho)^{n-1}} \right) \left(\int_r^L \varphi(\rho)^{n-1} d\rho \right) = 0$$

for any $R \in (0, L)$*. Then for every* $q \in (2, \infty)$ *and every eigenvalue* λ *of the Laplacian on* M*, there exists a constant* $C = C(\varphi, q, \lambda)$ *such that*

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)}$$

for every eigenfunction u *associated with* λ*.*

(ii) *Assume that*

$$\int^L \left(\frac{1}{\varphi(r)^{n-1}} \int_r^L \varphi(\rho)^{n-1} d\rho \right) dr < \infty.$$

Then for every eigenvalue λ *of the Laplacian on* M *there exists a constant* $C = C(\varphi, \lambda)$ *such that*

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}$$

for every eigenfunction u *associated with* λ*.*

Propositions 5.2 and 5.3 tell us that, as far as manifolds of revolution are concerned, methods based on the use of the isoperimetric function and of the isocapacitary function lead to equivalent results on the discreteness of the spectrum of Δ_M and eigenfunction estimates.

Let us specialize the results of Propositions 5.2 and 5.3 to the one-parameter family of manifolds of revolution M whose profile $\varphi : [0, \infty) \to [0, \infty)$ satisfies

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r. \tag{5.4}$$

One can show that

$$I_M(s) \approx s\big(\log(1/s)\big)^{1-1/\alpha} \qquad \text{near } 0, \tag{5.5}$$

and

$$\nu_M(s) \approx \left(\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{I_M(r)^2} \right)^{-1} \approx s\big(\log(1/s)\big)^{2-2/\alpha} \qquad \text{near } 0. \tag{5.6}$$

An application of Proposition 5.2 ensures that the spectrum of Δ_M is discrete if and only if

$$\alpha > 1. \tag{5.7}$$

Owing to Part (i) of Proposition 5.3, under (5.7) all eigenfunctions of the Laplacian on M belong to $L^q(M)$. Moreover, by (5.6) and Proposition 4.2, there exist constants $C_1 = C_1(q)$ and $C_2 = C_2(q)$ such that

$$\|u\|_{L^q(M)} \leq C_1 e^{C_2 \lambda^{\frac{\alpha}{2\alpha-2}}} \|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian associated with the eigenvalue λ.

On the other hand, Part (ii) of Proposition 5.3 tells us that the relevant eigenfunctions are bounded under the more stringent assumption that

$$\alpha > 2. \tag{5.8}$$

In particular, owing to (5.6) and Proposition 4.8,

$$\|u\|_{L^\infty(M)} \leq C_1 e^{C_2 \lambda^{\frac{\alpha}{\alpha-2}}} \|u\|_{L^2(M)}$$

for some absolute constants C_1 and C_2 and for every eigenfunction u associated with λ.

5.2 – *A family of manifolds with clustering submanifolds*

Here, we are concerned with a class of noncompact surfaces M in $\mathbb{R}^3$, which are reminiscent of a planar domain appearing in an example of [CH]. The relevant surfaces contain a sequence of mushroom-shaped submanifolds $\{N^k\}$ clustering at some point, which does not belong to the surface (Figure 2).

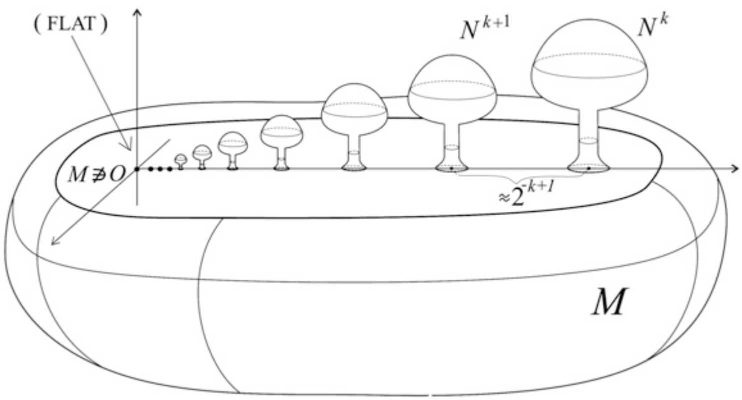

Fig. 2. – A manifold with a family of clustering submanifolds.

Let us emphasize that the submanifolds $\{N^k\}$ are not scalings of each other. Roughly speaking, the diameter of the head and the length of the neck of N^k decay to 0 as 2^{-k} when $k \to \infty$, whereas the width of the neck of N^k decays to 0 as $\sigma(2^{-k})$, where σ is a function such that

$$\lim_{s \to 0} \frac{\sigma(s)}{s} = 0. \tag{5.9}$$

The isoperimetric and isocapacitary functions of M depend on the behavior of σ at 0 in a way described in the next proposition. Roughly speaking, a faster decay to 0 of the function $\sigma(s)$ as $s \to 0$ results in a faster decay to 0 of $I_M(s)$ and $\nu_M(s)$, and hence in a surface M with a more irregular geometry.

PROPOSITION 5.4. – *Let M be the surface described above. Assume that $\sigma : [0,\infty) \to [0,\infty)$ is an increasing function of class Δ_2 near* 0, *such that*

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \textit{is non-increasing} \tag{5.10}$$

for some $\beta > 0$.

(i) *If*

$$\frac{s^2}{\sigma(s)} \quad \textit{is non-decreasing},$$

then

$$I_M(s) \approx \sigma(s^{1/2}) \qquad \textit{near } 0. \tag{5.11}$$

(ii) *If*

$$\frac{s^3}{\sigma(s)} \quad \textit{is non-decreasing},$$

then

$$\nu_M(s) \approx \sigma(s^{1/2})s^{-\frac{1}{2}} \qquad \textit{near } 0. \tag{5.12}$$

The operator Δ_M for the surfaces of this family can be described as follows.

PROPOSITION 5.5. – *Let M be the surface described above, with σ satisfying* (5.9). *Then*

$$\overline{C_0^\infty(M)} = W^{1,2}(M).$$

Hence, the operator Δ_M agrees with the Friedrichs extension of the Laplacian on M.

The next proposition relies upon the criterion for the discreteness of the spectrum of Δ_M in terms of the isocapacitary function of M given in Theorem 3.1.

PROPOSITION 5.6. – *Let M be the surface described above. Assume that $\sigma \in \Delta_2$ and fulfills* (5.10), *and that the function $\dfrac{s^3}{\sigma(s)}$ is monotonic. Then the spectrum of Δ_M (which, by Proposition* 5.5, *agrees with the Friedrichs extension of the Laplacian on M) is discrete if only if*

$$\lim_{s\to 0} \frac{s^3}{\sigma(s)} = 0\,. \tag{5.13}$$

Owing to equation (5.12), one can derive the following conditions for bounds of eigenfunctions of problem (4.1) via Theorems 4.1 and 4.7, whose criteria involve the isocapacitary function ν_M.

PROPOSITION 5.7. – *Let M be the surface described above with $\sigma \in \Delta_2$.*

(i) *Assume that*

$$\lim_{s\to 0} \frac{s^3}{\sigma(s)} = 0\,. \tag{5.14}$$

Then any eigenfunction of the Laplacian on M belongs to $L^q(M)$ for any $q<\infty$.

(ii) *Assume that*

$$\int_0 \frac{s^2}{\sigma(s)}\, ds < \infty\,. \tag{5.15}$$

Then any eigenfunction of the Laplacian on M is bounded.

Note that assumptions (5.14) and (5.15) are weaker than parallel assumptions which follow from an application of Theorems 4.5 and 4.11, and equation (5.11), and read

$$\lim_{s\to 0} \frac{s^2}{\sigma(s)} = 0\,, \tag{5.16}$$

and

$$\int_0 \frac{s^3}{\sigma(s)^2}\, ds < \infty\,, \tag{5.17}$$

respectively. For instance, if $b > 1$ and

$$\sigma(s) = s^b \qquad \text{for } s > 0,$$

then (5.14) and (5.15) amount to $b<3$, whereas (5.16) and (5.17) are equivalent to the more stringent condition that $b<2$. This shows that the use of ν_M can actually yield the discreteness of the spectrum of the Laplacian, and regularity of eigenfunctions, for manifolds where the criteria involving I_M do not apply.

Since, by (5.12), $\nu_M(s) \approx s^{\frac{b-1}{2}}$, from Examples 4.3 and 4.9 we deduce that there exists a constant $C = C(q)$ such that

$$\|u\|_{L^q(M)} \le C\lambda^{\frac{q-2}{q(3-b)}}\|u\|_{L^2(M)}$$

for every $q \in (2, \infty]$ and for any eigenfunction u of the Laplacian associated with the eigenvalue λ. Observe that the existence of such eigenfunction follows from condition (4.3), as explained in the comments following Theorem 4.1.

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