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Daniele Boffi

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The Immersed Boundary Method for Fluid-Structure Interactions: Mathematical Formulation and Numerical Approximation (*)

DANIELE BOFFI

Abstract. – *The Immersed Boundary Method (IBM) has been introduced by Peskin in the 70's in order to model and approximate fluid-structure interaction problems related to the blood flow in the heart. The original scheme makes use of finite differences for the discretization of the Navier-Stokes equations. Recently, a finite element formulation has been introduced which has the advantage of handling the presence of the solid (modeled via a Dirac delta function) in a more natural way. In this paper we review the finite element formulation of the IBM focusing, in particular, on the choice of the finite element spaces in order to guarantee a suitable mass conservation. Moreover, we present some links with the* fictitious domain *method.*

1. – Introduction

In the 70's Peskin introduced the Immersed Boundary Method (IBM) for the modeling and the approximation of fluid-structure interactions arising from biological problems [33, 34].

When studying fluid-structure interactions, a mathematical model is usually based on distinct equations in the fluid and in the moving solid, which are related through appropriate transmission conditions. One of the most intriguing aspects of any numerical implementation concerns the movement of the computational grids which have to adapt to the evolution of the system.

In view of their numerical approximations, basically all models for the description of fluid-structure interactions can be classified in two families: people refer to *partitioned* schemes when two different solvers are used for the discretization of the fluid and the solid; on the other hand, when fluid and solid are solved simultaneously with a unique solver, a scheme is termed *monolithic*. Moreover, the interaction between the fluid and the structure can be modeled

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using *weakly* or *strongly* coupled approaches. Weakly coupled approaches always lead to partitioned schemes, while strongly coupled approaches can be solved with both partitioned and monolithic strategies. We refer the interested reader to the wide literature on this subject and, in particular to [32, 31, 30, 20] and to the references therein.

Another important issue is related to the effects of the fluid as a so-called *added mass* to the structure interface. In particular, if the solid density is larger than the fluid density, then the added mass effect is negligible and partitioned schemes usually converge in few iterations. On the other hand, when the fluid and solid densities are comparable, then the method may fail to converge. In particular, a simplified one dimensional model for an Arbitrary Lagrangian Eulerian (ALE) scheme has been proved to be unconditionally unstable in this context [17]. In order to circumvent these difficulties, several strategies have been proposed, in particular based on implicit algorithms which are solved iteratively via domain decomposition or algebraic splitting strategies [29, 19, 1, 3, 2]. Other techniques involve discretizations based on operator splitting strategies [27].

The peculiarity of the IBM consists in the fact that the presence of the solid, immersed in the fluid, is modeled via an additional source term to the Navier-Stokes equations describing the fluid evolution. This allows for a fluid resolution on a fixed mesh; moreover, another important feature is that the source term referring to the immersed body is defined by making use of a description of the solid in its reference configuration.

The original version of the IBM makes use of finite differences for the resolution of the Navier-Stokes equations [33, 35, 34]. Since the source term arising from the presence of the solid contains a Dirac delta function concentrated along the solid domain, the finite difference scheme requires a suitable treatment of the delta function, which is one of the major sources of troubles in the IBM. Indeed, a too strong regularization of the delta function gives rise to a severe dissipation.

On the other hand, the finite element method makes use of a variational formulation which allows for a natural treatment of the Dirac delta function. A finite element version of the IBM has been successfully introduced in a series of papers, where several aspects have been considered, ranging from the modeling, to the numerical stability, and to mass conservation of the presented method [11, 13, 12, 14, 28, 8, 7]. The aim of the present paper is twofold: on one side, we review the existing results in a unified framework, with a particular emphasis on the mass conservation property which, to the author's opinion, is one of the most attractive features of the finite element IBM in comparison with the original finite difference version; on the other side, we present a link between the IBM and the fictitious domain approach (see, for instance, [24, 22, 23, 21]) which proves promising for future investigations.

2. – The immersed boundary method

In this section we describe the IBM following the results in [14, 8]. In particular, we shall present in a unified settings the situations when the fluid is two- or three-dimensional and the solid is of codimension zero or one with respect to the fluid.

Let $\Omega \subset \mathbb{R}^d$ be a domain in two or three space dimensions ($d = 2, 3$) and let $\boldsymbol{x}$ denote the Eulerian variable in Ω. In general, the domain Ω represents the space occupied by the fluid and the solid. The fluid velocity and pressure are denoted as usual by $\boldsymbol{u}$ and p, and are functions of $\boldsymbol{x}$ and of the time t.

We denote the solid domain at the time t by $\mathcal{B}_t \subset \mathbb{R}^m$ ($m = d, d-1$) and the reference solid domain by $\mathcal{B}$. The mapping representing the solid position is denoted by $\boldsymbol{X}$, more precisely

$$\boldsymbol{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$$

The Lagrangian variable on $\mathcal{B}$ is denoted by s and the deformation gradient is

$$\mathbb{F} = \frac{\partial \boldsymbol{X}}{\partial s}.$$

A fundamental relationship between $\boldsymbol{u}$ and $\boldsymbol{X}$ is given by the physical condition that the solid moves at the same velocity as the fluid, that is

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{X}}{\partial t}(s, t), \quad \text{with } \boldsymbol{x} = \boldsymbol{X}(s, t).$$

When there are no external forces, the conservation of momenta states

$$\rho \dot{\boldsymbol{u}} = \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \nabla \cdot \boldsymbol{\sigma} \quad \text{in } \Omega,$$

where ρ is the density and $\boldsymbol{\sigma}$ is the stress tensor, which can be defined as follows:

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s & \text{in } \mathcal{B}_t, \end{cases}$$

taking into account the extra stress $\boldsymbol{\sigma}_s$ originated by the elastic component of the stresses (we are assuming the material to be viscoelastic). Moreover, we are considering an incompressible fluid, so that in $\Omega \setminus \mathcal{B}_t$ we have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_f = -p\mathbb{I} + \mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T).$$

Since in general the fluid and solid densities can be different, we consider the following definition:

$$\rho = \begin{cases} \rho_f & \text{in } \Omega \setminus \mathcal{B}_t \\ \rho_s & \text{in } \mathcal{B}_t. \end{cases}$$

In order to write the variational formulation for our model, we need some additional notation.

Let the excess Lagrangian mass density be defined as

$$\delta\rho = \begin{cases} \rho_s - \rho_f & \text{if } m = d \text{ (codimension zero)} \\ t_s(\rho_s - \rho_f) & \text{if } m = d-1 \text{ (codimension one)}, \end{cases}$$

where t_s is the thickness of the structure. Actually, in the codimension-one case, mathematically the structure has no thickness; the rescaled density defined above takes into account the (small) physical thickness.

We shall make use, moreover, of the Piola–Kirchhoff tensor $\mathbb{P}$ which takes into account the change of variables

$$\mathbb{P}(s,t) = |\mathbb{F}(s,t)|\boldsymbol{\sigma}_s(\boldsymbol{X}(s,t),t)\mathbb{F}^{-T}(s,t)$$

so that

$$\int_{\partial\mathcal{P}_t} \boldsymbol{\sigma}_s \boldsymbol{n}\, da = \int_{\partial\mathcal{P}} \mathbb{P}\boldsymbol{N}\, dA \qquad \forall \mathcal{P}_t.$$

The rescaled Piola–Kirchhoff tensor is

$$\tilde{\mathbb{P}} = \begin{cases} \mathbb{P} & \text{if } m = d \text{ (codimension zero)} \\ t_s\mathbb{P} & \text{if } m = d-1 \text{ (codimension one)}. \end{cases}$$

With these definitions, the variational formulation for the IBM reads as follows: given $\boldsymbol{u}_0 \in H^1_0(\Omega)^d$ and $\boldsymbol{X}_0 \in W^{1,\infty}(\mathcal{B})$ such that $\boldsymbol{X}_0(\mathcal{B}) \subset \Omega$, find $\boldsymbol{u}(t) \in H^1_0(\Omega)^d$, $p(t) \in L^2_0(\Omega)$, and $\boldsymbol{X}(t) \in W^{1,\infty}(\mathcal{B})$ for all t such that

$$(1) \qquad \begin{cases} \rho_f \dfrac{d}{dt}(\boldsymbol{u}(t),\boldsymbol{v}) + a(\boldsymbol{u}(t),\boldsymbol{v}) + b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}) & \\ \qquad -(\operatorname{div}\boldsymbol{v}, p(t)) = \langle \boldsymbol{d}(t),\boldsymbol{v}\rangle + \langle \boldsymbol{f}(t),\boldsymbol{v}\rangle & \forall \boldsymbol{v} \in H^1_0(\Omega)^d \\ (\operatorname{div}\boldsymbol{u}(t), q) = 0 & \forall q \in L^2_0(\Omega) \\ \langle \boldsymbol{d}(t),\boldsymbol{v}\rangle = -\delta\rho \displaystyle\int_{\mathcal{B}} \frac{\partial^2 \boldsymbol{X}}{\partial t^2} \boldsymbol{v}(\boldsymbol{X}(s,t))\, ds & \forall \boldsymbol{v} \in H^1_0(\Omega)^d \\ \langle \boldsymbol{f}(t),\boldsymbol{v}\rangle = -\displaystyle\int_{\mathcal{B}} \tilde{\mathbb{P}}(\mathbb{F}(s,t)) : \nabla_s \boldsymbol{v}(\boldsymbol{X}(s,t))\, ds & \forall \boldsymbol{v} \in H^1_0(\Omega)^d \\ \dfrac{\partial \boldsymbol{X}}{\partial t}(s,t) = \boldsymbol{u}(\boldsymbol{X}(s,t),t) & \forall s \in \mathcal{B} \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) & \forall \boldsymbol{x} \in \Omega \\ \boldsymbol{X}(s,0) = \boldsymbol{X}_0(s) & \forall s \in \mathcal{B}. \end{cases}$$

In the previous equation we used the short notation $\nabla_{sym}\boldsymbol{u} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T$ and

$$a(\boldsymbol{u},\boldsymbol{v}) = \mu(\nabla_{sym}\boldsymbol{u}, \nabla_{sym}\boldsymbol{v})$$
$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{\rho_f}{2}((\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v})).$$

We refer the interested reader to [8] where it has been proved that (1) makes sense; in particular the source terms $\boldsymbol{d}$ and $\boldsymbol{f}$ are distributions belonging to H^{-1}.

REMARK 1. – Equation (1) shows one of the most important features of the finite element IBM. The source term $\boldsymbol{f}$ representing the effect of the solid to the fluid can be written in a natural way in the framework of the variational formulation. In the original formulation, this term consisted of a Dirac delta function which needed to be approximated; in many situations this approximation was the cause of artificial diffusion.

3. – Numerical approximation

3.1 – *Space semi-discretization*

Let us consider discrete subspaces $V_h \subset H_0^1(\Omega)^d$ and $Q_h \subset L_0^2$ which satisfy the inf-sup condition [16]. The reference configuration $\mathcal{B}$ of the structure is subdivided into segments, triangles, or tetrahedrons. Let's denote by S_h the discrete space obtained by a piecewise linear approximation of the structure according to this subdivision. The set of edges (or faces) of the mesh of the body is denoted by $\mathcal{E}_h$ and $\boldsymbol{s}_i$ denotes a vertex ($i = 1,\dots,M$). We denote by $[\![\cdot]\!]$ the jump, and in particular we shall make use of the following term

$$[\![\tilde{\mathbb{P}}_h]\!] = \tilde{\mathbb{P}}_h^+\boldsymbol{N}^+ + \tilde{\mathbb{P}}_h^-\boldsymbol{N}^-,$$

where $\tilde{\mathbb{P}}_h = \tilde{\mathbb{P}}(\mathbb{F}_h)$ is piecewise constant, since $\mathbb{F}_h = \nabla_s\boldsymbol{X}_h$ is the gradient of a linear function. Then, the space semi-discretization of problem (1) reads: given $\boldsymbol{u}_{h0} \in V_h$ and $\boldsymbol{X}_{h0} \in S_h$, find $\boldsymbol{u}_h(t) \in V_h$, $p_h(t) \in Q_h$, and $\boldsymbol{X}_h(t) \in S_h$ for all t such that

$$(2)\qquad \begin{cases} \rho_f\dfrac{d}{dt}(\boldsymbol{u}_h(t),\boldsymbol{v}) + a(\boldsymbol{u}_h(t),\boldsymbol{v}) + b(\boldsymbol{u}_h(t),\boldsymbol{u}_h(t),\boldsymbol{v}) & \\ \qquad -(\mathrm{div}\boldsymbol{v}, p_h(t)) = \langle\boldsymbol{d}_h(t),\boldsymbol{v}\rangle + \langle\boldsymbol{f}_h(t),\boldsymbol{v}\rangle & \forall\boldsymbol{v}\in V_h \\ (\mathrm{div}\,\boldsymbol{u}_h(t), q) = 0 & \forall q \in Q_h \\ \langle\boldsymbol{d}_h(t),\boldsymbol{v}\rangle = -\delta\rho\displaystyle\int_{\mathcal{B}}\frac{\partial^2\boldsymbol{X}_h}{\partial t^2}(t)\boldsymbol{v}(\boldsymbol{X}_h(t))\,ds & \forall\boldsymbol{v}\in V_h \\ \langle\boldsymbol{f}_h(t),\boldsymbol{v}\rangle = -\displaystyle\sum_{e\in\mathcal{E}_h}\int_e [\![\tilde{\mathbb{P}}_h]\!](t)\cdot\boldsymbol{v}(\boldsymbol{X}_h(t))\,dA & \forall\boldsymbol{v}\in V_h \\ \boldsymbol{u}_h(\boldsymbol{x},0) = \boldsymbol{u}_{h0}(\boldsymbol{x}) & \forall\boldsymbol{x}\in\Omega \\ \dfrac{\partial\boldsymbol{X}_{hi}}{\partial t}(t) = \boldsymbol{u}_h(\boldsymbol{X}_{hi}(t),t) & \forall i = 1,\dots,M \\ \boldsymbol{X}_{hi}(0) = \boldsymbol{X}_{h0}(\boldsymbol{s}_i) & \forall i = 1,\dots,M. \end{cases}$$

3.2 – *Full discretization*

In order to write a full discrete scheme, we need to approximate the time derivative in the Navier–Stokes equation and to discretize the second time derivative in the definition of the source term $\boldsymbol{d}_h$ arising from the excess Lagrangian mass density. For the sake of the stability properties (better CFL condition), an implicit scheme would be desirable (see, in this context, [13]). We are working in the direction of using a multigrid strategy (in particular, we are considering a full approximation scheme, see [15]) in order to make it possible the solution of our problem with a fully implicit approach [9]. Here we present a semi-implicit approach which has been successfully applied in [8], where a stability analysis has been performed.

Let us denote by Δt the time step and let us mark with the superscript n functions evaluated at time $t_n = n\Delta t$. The solution is performed in several steps. In the first one, the source term $\boldsymbol{f}$ is evaluated as follows:

$$\langle \boldsymbol{f}_h^{n+1}, \boldsymbol{v} \rangle = -\sum_{e \in \mathcal{E}_h} [\![\tilde{\mathbb{P}}_h]\!]^n \cdot \boldsymbol{v}(\boldsymbol{X}_h^n(s,t))\, dA \qquad \forall \boldsymbol{v} \in V_h.$$

Then the Navier-Stokes equations are solved: find $\boldsymbol{u}_h^{n+1} \in V_h$ and $p_h^{n+1} \in Q_h$ such that

$$\rho_f \left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v} \right) + b(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + a(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) - (\mathrm{div}\boldsymbol{v}, p_h^{n+1}) =$$

$$-\delta\rho \int_{\mathcal{B}} \frac{\boldsymbol{u}_h^{n+1}(\boldsymbol{X}_h^n(s)) - \boldsymbol{u}_h^n(\boldsymbol{X}_h^{n-1}(s))}{\Delta t} \cdot \boldsymbol{v}(\boldsymbol{X}_h^n(s))\, ds + \langle \boldsymbol{f}_h^{n+1}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in V_h$$

$$(\mathrm{div}\, \boldsymbol{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h,$$

where the excess Lagrangian mass density has been approximated by taking into account the equation linking the structure and the fluid velocity, so that the second derivative of $\boldsymbol{X}$ has become the first derivative of $\boldsymbol{u}$.

Finally, the structure is moved according to the velocity of the fluid:

$$\frac{\boldsymbol{X}_{hi}^{n+1} - \boldsymbol{X}_{hi}^n}{\Delta t} = \boldsymbol{u}_h^{n+1}(\boldsymbol{X}_{hi}^n) \quad \forall i = 1, \ldots, M.$$

For the reader's convenience, we report in Table 1 the results of the stability analysis performed in [8], where it can be appreciated that for any configuration there exists a stable choice of Δt. This results should be compared to [17], where it is shown that explicit schemes cannot be stable in the framework of ALE discretizations.

TABLE 1. – CFL condition: h_x is the fluid meshsize and h_s is the solid one

space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \leq C h_x h_s$
2	2	$L^n \Delta t \leq C h_x$
3	2	$L^n \Delta t \leq C h_x^2$
3	3	$L^n \Delta t \leq C h_x^2 / h_s$

TABLE 2. – Area loss (%): scheme $Q_2 - P_1$, $\Delta t = 0.01$, $T = 1$.

M	16	32	64	128	256	512	1024
$N = 4$	36.468	35.944	37.166	37.988	38.413	38.635	38.753
$N = 8$	15.951	14.090	13.057	12.809	12.809	12.843	12.867
$N = 16$	20.182	9.015	7.255	7.011	7.105	7.191	7.246
$N = 32$	45.293	9.763	2.788	2.308	2.303	2.325	2.351

4. – Conservation of mass

There is a very easy test to check the mass conservation of our scheme. Let us consider the domain $\Omega =]0,1[\times]0,1[$ and let $\boldsymbol{X}_0$ be a circle of radius R centered at $(1/2, 1/2)$:

$$\boldsymbol{X}_0(s) = \begin{pmatrix} R\cos(s/R) + 0.5 \\ R\sin(s/R) + 0.5 \end{pmatrix}, \quad s \in [0, 2\pi R]$$

Due to the elastic properties of the structure, the circle exerts a normal force pointing towards the center and, since the fluid is incompressible, the fluid does not move and the effect of the force is a jump in the pressure between the interior (high pressure) and the exterior (low pressure). In particular, the area of the domain surrounded by $\boldsymbol{X}$ remains constant.

In a particular configuration, Table 2 shows the area loss with respect to the fluid mesh (N refers to the number of subdivisions of a side of Ω when a uniform mesh of squares is used) and the number M of points used in order to approximate the elastic structure

It is apparent that, when the fluid mesh is refined, more and more points are required in the structure in order to preserve the same mass (see, for instance the first column, where, as N increases, the area loss gets worse with the same number of points $N = 16$). On the other hand, after a threshold, there is no advantage in adding points to the structure if the fluid mesh is not refined: the values corresponding to this threshold are marked with a frame box in the table.

It roughly corresponds to the situation when two solid points are contained in each fluid element.

When such threshold is reached, the quality of mass conservation is driven by the scheme used for the fluid resolution. More precisely, we need to check how the divergence free condition is preserved at discrete level. This is a well known issue which has been the object of a wide literature. In particular, the divergence free condition is imposed at discrete level in a weak sense:

$$\int_\Omega \mathrm{div}\boldsymbol{u}_h q_h \, d\boldsymbol{x} = 0 \qquad \forall q_h \in Q_h. \tag{3}$$

It follows that if the discrete spaces satisfy the inclusion

$$\mathrm{div} V_h \subset Q_h \tag{4}$$

then the divergence free condition is imposed *exactly*, that is $\mathrm{div}\boldsymbol{u}_h = 0$. Unfortunately, there are few choices of spaces V_h and Q_h that satisfy at the same time the inf-sup condition and the inclusion (4); moreover, the most popular of such choices, namely the $P_{k+1} - P_k$ scheme (see [38]) is subjected to severe limitations on the admissible triangulations.

In general a *discontinuous* pressure approximation enjoys more local mass conservation properties: if Q_h contains piecewise constant functions, in particular, equation (3) implies that $\mathrm{div}\boldsymbol{u}_h$ has zero mean value elementwise. In [7] we discussed how to modify some *continuous* pressure schemes in order to enjoy the same property.

Let us consider the Hood-Taylor (HT) family in two and three dimensions [39], and the Bercovier–Pironneau (BP) element in two dimensions [4]. The lowest order HT element in 2D and the BP element are sketched in Figure 1.

Since these elements are based on continuous pressure spaces, in general they do not achieve local conservation of mass. An interesting way of improving the mass conservation for such elements consists in adding an internal degree of

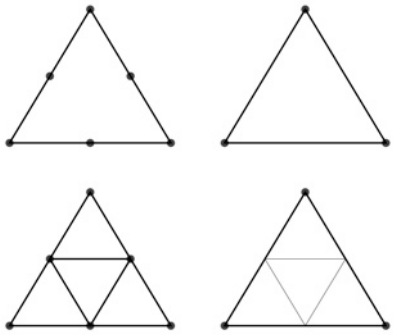

Fig. 1. – Lowest order HT (top) and BP (bottom).

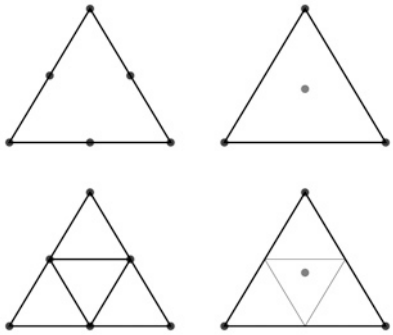

Fig. 2. – Augmented elements: lowest order HT (top) and BP (bottom).

freedom in each element to the pressures. The modified scheme, for a sample of elements, is presented in Figure 2.

Given a triangulation $\mathcal{T}_h$ of the domain Ω, the formal definition of the augmented spaces is given as follows. The *augmented HT* family is defined in two dimensions ($k \geq 1$) as

$$V_h = \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \boldsymbol{v}|_K \in P_{k+1}(K)^2 \ \forall K \in \mathcal{T}_h\}$$

$$Q_h = \{q \in L_0^2(\Omega) : q = q_k + q_0,\ q_k \in C(\Omega),\ q_k|_K \in P_k(K),\ q_0|_K \in P_0 \ \forall K \in \mathcal{T}_h\}$$

and in three dimensions ($k \geq 2$) as

$$V_h = \{\boldsymbol{v} \in H_0^1(\Omega)^3 : \boldsymbol{v}|_K \in P_{k+1}(K)^3 \ \forall K \in \mathcal{T}_h\}$$

$$Q_h = \{q \in L_0^2(\Omega) : q = q_k + q_0,\ q_k \in C(\Omega),\ q_k|_K \in P_k(K),\ q_0|_K \in P_0 \ \forall K \in \mathcal{T}_h\};$$

the *augmented BP* element is defined in two dimensions as

$$V_h = \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \boldsymbol{v}|_K \in P_1(K)^2 \ \forall K \in \mathcal{T}_{h/2}\}$$

$$Q_h = \{q \in L_0^2(\Omega) : q = q_1 + q_0,\ q_1 \in C(\Omega),\ q_1|_K \in P_1(K),\ q_0|_K \in P_0 \ \forall K \in \mathcal{T}_h\}.$$

The idea of adding piecewise constant pressure in order to enhance the mass conservation is not new; we refer in particular to [26, 25, 41, 18, 40, 36, 37]. In [7] we presented a unified proof of stability for a wide class of elements which is based on the stability analysis of [5, 6].

We refer to [7] for several numerical experiments in which the significant improvement of the augmented spaces over the original ones can be appreciated. Here we report in Figure 3 the improvement in the area loss when applied to the model problem described at the beginning of this section.

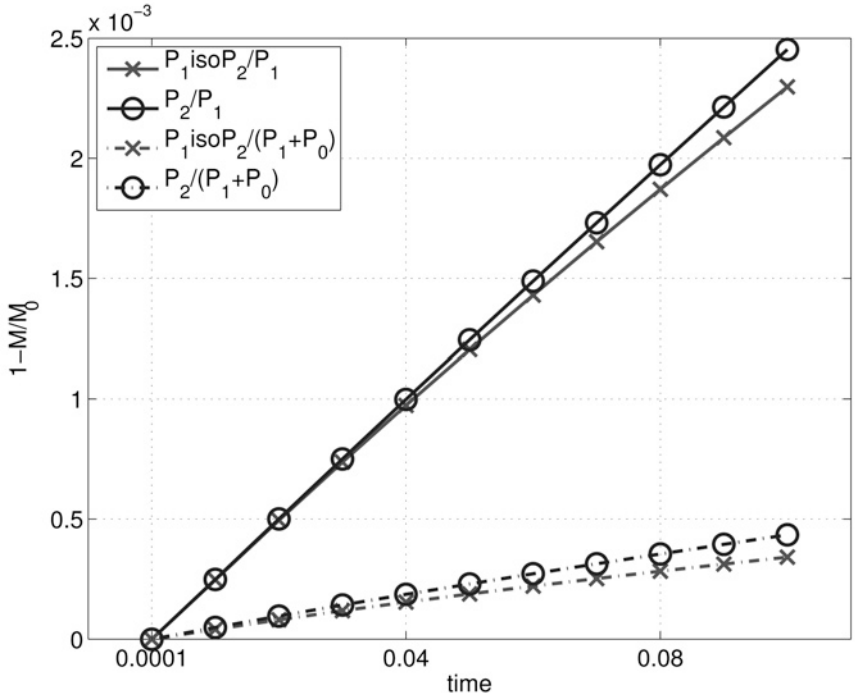

Fig. 3. – Area loss comparison between standard elements and augmented ones.

5. – Immersed boundary method and fictitious domain

The topic presented in this section represents some new ideas which look promising for the analysis and the implementation of a suitable modification of the method presented so far.

The motivation for the present study comes from the analogies that the IBM shares, in a sense to be made more precise, with the fictitious domain approach (see, for instance, [24, 22]). In particular, we shall introduce a formulation of the IBM which has the same structure as the so called fictitious domain method with *distributed Lagrangian multiplier* (see [23, 21]).

We start with a slight modification of the problem presented in equation (1). We observe that the equation that drives the movement of the structure can be written in a variational form in the following way:

$$\left\langle \mu, \boldsymbol{u}(\boldsymbol{X}(\cdot,t),t) - \frac{\partial \boldsymbol{X}(\cdot,t)}{\partial t} \right\rangle = 0 \quad \forall \mu \in (H^1(\mathcal{B})^d)^*.$$

Hence, it turns out that our problem can be restated as follows: find $\boldsymbol{u}(t) \in H_0^1(\Omega)^d$, $p(t) \in L_0^2(\Omega)$, $\boldsymbol{X}(t) \in H^1(\mathcal{B})^d$, and $\lambda \in \Lambda = (H^1(\mathcal{B})^d)^*$ such that

$$
\begin{cases}
\rho \dfrac{d}{dt}(\boldsymbol{u}(t), \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{v}) + b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, p(t)) & \\
\quad = -\delta\rho \displaystyle\int_{\mathcal{B}} \frac{\partial^2 \boldsymbol{X}}{\partial t^2} \boldsymbol{v}(\boldsymbol{X}(s,t))\, ds - \langle \lambda, \boldsymbol{v}(\boldsymbol{X}(\cdot,t)) \rangle & \forall \boldsymbol{v} \in H_0^1(\Omega)^d \\
(\operatorname{div} \boldsymbol{u}(t), q) = 0 & \forall q \in L_0^2(\Omega) \\
\displaystyle\int_{\mathcal{B}} \tilde{\mathbb{P}}(\mathbb{F}(s,t)) : \nabla_s \boldsymbol{v}_s \, ds - \langle \lambda, \boldsymbol{v}_s \rangle = 0 & \forall \boldsymbol{v}_s \in H^1(\mathcal{B})^d \\
\left\langle \mu, \boldsymbol{u}(\boldsymbol{X}(\cdot,t),t) - \dfrac{\partial \boldsymbol{X}(\cdot,t)}{\partial t} \right\rangle = 0 & \forall \mu \in \Lambda \\
\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) & \forall \boldsymbol{x} \in \Omega \\
\boldsymbol{X}(s,0) = \boldsymbol{X}_0(s) & \forall s \in \mathcal{B}.
\end{cases} \tag{5}
$$

The analogies between (5) and the fictitious domain approach with distributed Lagrangian multiplier can be appreciated with the following example which will be deeper analyzed in [10]. Let us consider a simple one dimensional transmission problem: consider a real interval $\Omega = (a, b)$ (which represents the union of the fluid and solid domain) and a subinterval $\Omega_s = (c, d)$ (representing the solid), with $a < c < d < b$. Let ν be a piecewise smooth function (representing a viscosity) which can only jump across the interfaces c and d:

$$
\nu = \begin{cases} \nu_f & \text{in } \Omega_f = \Omega \setminus \Omega_s \\ \nu_s & \text{in } \Omega_s. \end{cases}
$$

We are looking for u such that

$$
\begin{cases} -\nu u'' = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}
$$

In the framework of the fictitious domain approach with distributed Lagrangian multiplier, our problem can be restated as follows: find $U \in V = H_0^1(\Omega)$, $U_s \in V_s = H^1(\Omega_s)$, and $\lambda \in \Lambda = (H^1(\Omega_s))^*$ such that

$$
\begin{cases}
\displaystyle\int_{\Omega} \tilde{\nu} U' v' \, dx + \langle \lambda, v|_{\Omega_s} \rangle = \int_{\Omega} F v \, dx & \forall v \in V \\
\displaystyle\int_{\Omega_s} (\nu_s - \tilde{\nu}) U_s' v_s' \, dx - \langle \lambda, v_s \rangle = \int_{\Omega_s} (f_s - F) v_s \, dx & \forall v_s \in V_s \\
\langle \mu, U|_{\Omega_s} - U_s \rangle = 0 & \forall \mu \in \Lambda,
\end{cases} \tag{6}
$$

where F and $\tilde{\nu}$, restricted to Ω_s coincide with f and ν_s, respectively.

In [10] we show that problem (6) is uniquely solvable and is equivalent in a suitable sense to the original transmission problem. It will be also shown how to approximate problem (6) with appropriate finite elements on a mesh which does not fit with the interface. Hopefully, this will be a good starting point in order to test and analyze the finite element approximation of the modified IBM (5).

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Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia
E-mail: daniele.boffi@unipv.it

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