Bollettino Unione Matematica Italiana

N. A. Chernyavskaya, L. A. Shuster

Integral Inequalities for the Principal Fundamental System of Solutions of a Homogeneous Sturm-Liouville Equation

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **5** (2012), n.2, p. 423–448.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_2_423_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012), 423-448

Integral Inequalities for the Principal Fundamental System of Solutions of a Homogeneous Sturm-Liouville Equation and their Applications

N. A. CHERNYAVSKAYA - L. A. SHUSTER

Abstract. – *We consider the equation*

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{1}$$

where $f \in L_p(\mathbb{R})$, $p \in [1, \infty]$ $(L_\infty(\mathbb{R}) := C(\mathbb{R}))$ *and*

$$0 \le q \in L_1^{\text{loc}}(\mathbb{R}); \quad \exists a > 0 : \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)dt > 0. \tag{2}$$

(Condition (2) *guarantees correct solvability of* (1) *in class* $L_p(\mathbb{R})$, $p \in [1, \infty]$.*) Let* y *be a solution of* (1) *in class* $L_p(\mathbb{R})$, $p \in [1, \infty]$, *and* θ *some non-negative and continuous function in* $\mathbb{R}$. *We find minimal additional requirements to* θ *under which for a given* $p \in [1, \infty]$, *there exists an absolute positive constant* $c(p)$ *such that the following inequality holds:*

$$\sup_{x\in\mathbb{R}} \theta(x)|y(x)| \le c(p)\|f\|_{L_p(\mathbb{R})}, \quad \forall f \in L_p(\mathbb{R}).$$

1. – Introduction

In the present paper, we consider the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in L_p$ $(L_p(\mathbb{R}) := L_p)$, $p \in [1, \infty]$ $(L_\infty(\mathbb{R}) := C(\mathbb{R}))$ and

$$0 \le q \in L_1^{\text{loc}} \quad (L_1^{\text{loc}}(\mathbb{R}) := L_1^{\text{loc}}). \tag{1.2}$$

A solution y of (1.1) is understood as any function, absolutely continuous together with its derivative, satisfying (1.1) almost everywhere in $\mathbb{R}$. In addition, we assume that equation (1.1) is correctly solvable in L_p, $p \in [1, \infty]$. The latter requirement means that

I) for every function $f \in L_p$ there exists a unique solution of (1.1) $y \in L_p$;

II) there exists an absolute constant $c(p) \in (0,\infty)$ such that the solution of (1.1) $y \in L_p$ satisfies the inequality

$$\|y\|_p \le c(p)\|f\|_p, \quad \forall f \in L_p \quad (\|f\|_{L_p} := \|f\|_p) \tag{1.3}$$

(see [12, Ch. III, § 6, no. 2]).

Note that a precise requirement for the function q to guarantee I)-II) is as follows:

$$\exists a > 0 : \quad q_0(a) = \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} q(t)dt > 0 \tag{1.4}$$

(see [5]). Therefore throughout the sequel we assume that conditions (1.2) and (1.4) hold and do not mention them in the formulations. Another convention is that the letters c, $c(\,\cdot\,)$ stand for absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations. Finally, the symbol y will everywhere denote the solution of (1.1) from the class L_p which corresponds, according to I), to the function $f \in L_p$, $p \in [1,\infty]$.

Our general goal consists in investigating the behaviour of solutions of (1.1) in the uniform metric. In particular, we study possibilities for strengthening the following inequality (see [8]):

$$\sup_{x\in\mathbb{R}} |y(x)| \le c(p)\|f\|_p, \quad \forall f \in L_p. \tag{1.5}$$

Let us now go over to precise statements. Suppose that we are given some continuous and non-negative function $\theta(x)$ for $x \in \mathbb{R}$ and a number $p \in [1,\infty]$. We have to find a minimal additional requirement to the function θ under which the following estimate holds:

$$\sup_{x\in\mathbb{R}} \theta(x)|y(x)| \le c(p)\|f\|_p, \quad \forall f \in L_p. \tag{1.6}$$

(Below for brevity, we say "the problem (1.6)" or "the question (1.6)".)

Let us now describe in general terms a solution of the aforementioned problem. For this, we need the following lemmas.

LEMMA 1.1 [3]. – *There exists a fundamental system of solutions (FSS) $\{u,v\}$ of the equation*

$$z''(x) = q(x)z(x), \quad x \in R \tag{1.7}$$

which possesses the following properties:

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0, \quad x \in \mathbb{R}, \tag{1.8}$$

$$v'(x)u(x) - u'(x)v(x) = 1, \quad x \in \mathbb{R}, \tag{1.9}$$

$$\lim_{x\to-\infty}\frac{v(x)}{u(x)}=\lim_{x\to\infty}\frac{u(x)}{v(x)}=0, \tag{1.10}$$

$$\int_{-\infty}^{0}\frac{dt}{v^2(t)}=\int_{0}^{\infty}\frac{dt}{u^2(t)}=\infty,\quad \int_{0}^{\infty}\frac{dt}{v^2(t)}<\infty,\quad \int_{-\infty}^{0}\frac{dt}{u^2(t)}<\infty. \tag{1.11}$$

Moreover, properties (1.8)-(1.11) *determine the FSS* $\{u,v\}$ *uniquely up to positive constant mutually inverse factors.*

An FSS $\{u,v\}$ with properties (1.8)-(1.11) is called a principal FSS (PFSS) of (1.1)

LEMMA 1.2 [10]. – *We have*

$$y(x)=\int_{-\infty}^{\infty}G(x,t)f(t)dt,\quad x\in\mathbb{R} \tag{1.12}$$

where $G(x,t)$ *is the Green function of equation* (1.1):

$$G(x,t)=\begin{cases}u(x)v(t), & x\ge t\\ u(t)v(x), & x\le t.\end{cases} \tag{1.13}$$

We come to our original problem. Below, in connection with (1.6), the following assertion is of great importance.

THEOREM 1.3. – *Inequality* (1.6) *holds if and on ly if* $\sigma_p<\infty$. *Here*

$$\sigma_p=\sup_{x\in\mathbb{R}}(\theta(x)\sigma_p(x)), \tag{1.14}$$

$$\sigma_p(x)=\begin{cases}u(x)v(x), & \textit{if } p=1\\ u(x)J_{p'}(x)^{1/p'}+v(x)I_{p'}(x)^{1/p'}, & \textit{if } p\in(1,\infty)\\ \displaystyle\int_{-\infty}^{\infty}G(x,t)dt, & \textit{if } p=\infty,\end{cases} \tag{1.15}$$

$$J_{p'}(x)=\int_{-\infty}^{x}v^{p'}(t)dt,\qquad I_{p'}(x)=\int_{x}^{\infty}u^{p'}(t)dt \tag{1.16}$$

and, finally, $p'=p(p-1)^{-1}$.

Condition (1.14) is, obviously, implicit and therefore needs clarification and further investigation. Towards this end, we need an additional function d and Otelbaev's inequalities.

LEMMA 1.4 [6, 9]. – *For every $x \in \mathbb{R}$, there exists a unique solution in $d \geq 0$ of the equation*

$$\int_0^{\sqrt{2}d} \int_{x-t}^{x+t} q(\xi) d\xi dt = 2. \tag{1.17}$$

Denote this solution by $d(x)$, $x \in \mathbb{R}$. The function $d(x)$ is positive and differentiable for all $x \in \mathbb{R}$, and $|d'(x)| \leq 1/\sqrt{2}$, $x \in \mathbb{R}$.

THEOREM 1.5 [6]. – *Let $\rho(x) = u(x)v(x)$, $x \in \mathbb{R}$. Then Otelbaev's inequalities hold:*

$$\frac{d(x)}{2\sqrt{2}} \leq \rho(x) \leq \sqrt{2} d(x), \quad x \in \mathbb{R}. \tag{1.18}$$

REMARK 1.6. – The function d was introduced in [2]. Two-sided, sharp by order estimates for the function ρ were first obtained by M. Otelbaev (see [17]) (under some additional requirements to q and with another, more complicated, auxiliary function). Therefore, all inequalities of type (1.18) (see, e.g., [3]) are called Otelbaev's inequalities.

Let us come back to (1.14). By merely matching (1.15) and (1.18), we arrive at the following statement.

THEOREM 1.7. – *For $p = 1$ inequality* (1.6) *holds if and only if $h_1 < \infty$. Here*

$$h_1 = \sup_{x \in \mathbb{R}} (\theta(x) d(x)). \tag{1.19}$$

Note that usually the function d admits sharp by order two-sided estimates (see Section 5) which, together with (1.19), lead (for $p = 1$) to a complete solution of problem (1.6) for concrete equations (1.1) (see Section 5). Further, it is clear that if also for $p > 1$ we obtain sharp by order two-sided estimates for the function $\sigma_p(x)$, then, as in Theorem 1.7, problem (1.6) is solved also for $p > 1$. To get such inequalities, we write the function $\sigma_p(x)$, $x \in \mathbb{R}$, for $p > 1$ in a different form (see (1.18))

$$\sigma_p(x) = \rho(x) \left\{ \left[\frac{J_{p'}(x)}{v(x)^{p'}} \right]^{1/p'} + \left[\frac{I_{p'}(x)}{u(x)^{p'}} \right]^{1/p'} \right\}, \quad x \in \mathbb{R}. \tag{1.20}$$

From (1.18) and (1.19) it follows that the required estimates will be obtained when we find analogous inequalities for the values from (1.20) appearing in brackets.

THEOREM 1.8. – *For every* $p \in (1, \infty)$, *we have*

$$J_{p'}(x) \geq c(p)^{-1} v^{p'}(x) d(x), \qquad I_{p'}(x) \geq c^{-1}(p) u^{p'}(x) d(x), \qquad x \in \mathbb{R}. \tag{1.21}$$

Thus in (1.21) we establish a possible order of the values $J_{p'}(x)$ and $I_{p'}(x)$, $x \in \mathbb{R}$. Therefore, in view of Theorem 1.8, by applying Theorems 1.3 and 1.5, we obtain the following preliminary conclusion.

THEOREM 1.9. – *Suppose that for some* $p \in (1, \infty)$ *we have the estimates*

$$c^{-1}(p) v(x) d(x)^{1/p'} \leq J_{p'}^{1/p'}(x) \leq c(p) v(x) d(x)^{1/p'}, \quad x \in \mathbb{R}, \tag{1.22}$$

$$c^{-1}(p) u(x) d(x)^{1/p'} \leq I_{p'}^{1/p'}(x) \leq c(p) u(x) d(x)^{1/p'}, \quad x \in \mathbb{R}. \tag{1.23}$$

Then inequality (1.6) *holds if and only if* $h_p < \infty$.

Here

$$h_p = \sup_{x \in \mathbb{R}} \theta(x) d(x)^{2-\frac{1}{p}}. \tag{1.24}$$

Thus, it only remains to find necessary and sufficient conditions under which estimates (1.22)-(1.23) hold. In Section 3, we present the results of the investigation of this particular problem.

Note in addition that, to the best of our knowledge, inequalities (1.22)-(1.23) are new. Obviously, they can be used in problems of estimating the solution of (1.1) in weight spaces as well as in the problem of the behaviour of the solution of (1.1) at infinity. Some of these applications will be presented in a forthcoming paper.

For the reader's convenience, we describe the structure of the paper. In Section 2, we collect facts used in the proofs. Section 3 contains a description of the results that were not listed above, along with some comments. Section 4 contains all the proofs. Finally, in Section 5, we consider examples of the solutions of problem (1.6) for concrete equations (1.1) as well as some technical assertions.

2. – Preliminaries

LEMMA 2.1 [6]. – *For a given* $x \in \mathbb{R}$, *consider the equations in* $d \geq 0$:

$$\int_0^{\sqrt{2}d} \int_{x-t}^{x} q(\xi) d\xi dt = 1, \qquad \int_0^{\sqrt{2}d} \int_{x}^{x+t} q(\xi) d\xi dt = 1. \tag{2.1}$$

Each of the equations (2.1) *has a unique finite positive solution.*

Further, we denote the solutions of (2.1) by $d_1(x)$, $d_2(x)$, respectively.

THEOREM 2.2 [6]. – *For $x \in \mathbb{R}$, we have the inequalities*

$$\frac{1}{\sqrt{2}} \le \frac{v'(x)}{v(x)} d_1(x) \le \sqrt{2}, \qquad \frac{1}{\sqrt{2}} \le \frac{|u'(x)|}{u(x)} d_2(x) \le \sqrt{2}. \tag{2.2}$$

THEOREM 2.3 [13, 3, 9]. – *The PFSS $\{u, v\}$ of equation (1.7) satisfies the Davis-Harrell representations:*

$$u(x) = \sqrt{\rho(x)} \exp\left(-\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right), \qquad v(x) = \sqrt{\rho(x)} \exp\left(\frac{1}{2} \int_{x_0}^{x} \frac{d\xi}{\rho(\xi)} \right) \tag{2.3}$$

where $\rho(x) = u(x)v(x)$, $x \in \mathbb{R}$, x_0 is a unique solution of the equation $u(x) = v(x)$ in $\mathbb{R}$. In addition, we have

$$\frac{v'(x)}{v(x)} = \frac{1 + \rho'(x)}{2\rho(x)}, \qquad \frac{u'(x)}{u(x)} = -\frac{1 - \rho'(x)}{2\rho(x)}, \qquad x \in \mathbb{R}, \tag{2.4}$$

$$|\rho'(x)| < 1, \qquad x \in \mathbb{R}. \tag{2.5}$$

REMARK 2.4. – Representations (2.3) (in a slightly different form) were found in [13]; see [3] for a generalization; relations (2.4) and (2.5) were used in [9].

DEFINITION 2.5 [7]. – *Suppose we are given $x \in \mathbb{R}$, a positive function κ, a sequence $\{x_n\}_{n \in N'}$, $N' = \{\pm 1, \pm 2, \ldots\}$. Consider the segments $\Delta_n = [\Delta_n^-, \Delta_n^+]$, $\Delta_n^{\pm} = x_n \pm \kappa(x_n)$. We say that the sequence of segments $\{\Delta_n\}_{n=1}^{\infty}$ (resp. $\{\Delta_n\}_{n=-\infty}^{-1}$ forms an $\mathbb{R}(x, \kappa)$-covering of $[x, \infty)$ (resp. $(-\infty, x]$) if the following conditions hold:*

1) $\Delta_n^+ = \Delta_{n+1}^-$ *for* $n \ge 1$ (*resp.* $\Delta_{n-1}^+ = \Delta_n^-$ *for* $n \le -1$);
2) $\Delta_1^- = x, \ \bigcup_{n \ge 1} \Delta_n = [x, \infty)$ (*resp.* $\Delta_{-1}^+ = x, \ \bigcup_{n \le -1} \Delta_n = (-\infty, x]$).

LEMMA 2.6 [7]. – *Suppose that a positive continuous function κ for $x \in \mathbb{R}$ satisfies the condition*

$$\lim_{t \to \infty} (t - \kappa(t)) = \infty \qquad (\text{resp. } \lim_{t \to -\infty} (t + \kappa(t)) = -\infty).$$

Then for every $x \in \mathbb{R}$ there is an $\mathbb{R}(x, \kappa)$-covering of $[x, \infty)$ (resp. $\mathbb{R}(x, \kappa)$-covering of $(-\infty, x]$).

REMARK 2.7. – Assertions similar to Lemma 2.6 were first used by Otelbaev (see [14]). Note that some technical assertions are in Section 5 in the course of exposition.

3. – Results

We want to emphasize that the theorems from Section 1 are note restated here (except for Case A) of Theorem 3.7). Therefore in the sequel we only present our investigation of inequalities (1.22)-(1.23) and the main result of the paper. In addition, since estimates (1.22)-(1.23) perhaps are of intrinsic interest, we reconsider them here as an object of independent study, without any connection to the problem of (1.6).

THEOREM 3.1. – *Suppose that for some* $s \in [1, \infty)$ *the following inequalities hold:*

$$c^{-1}(s)v^s(x)d(x) \le J_s(x) \le c(s)v^s(x)d(x), \quad x \in \mathbb{R}, \tag{3.1}$$

$$c^{-1}(s)u^s(x)d(x) \le I_s(x) \le c(s)u^s(x)d(x), \quad x \in \mathbb{R}. \tag{3.2}$$

where $J_s(x) = J_{p'}(x)\big|_{p'=s}$, $I_s(x) = I_{p'}(x)\big|_{p'=s}$, $x \in \mathbb{R}$ (*see* (1.16)). *Then there exists a constant* $c \ge 1$ *such that for all* $x, t \in \mathbb{R}$, *we have the estimate*

$$\rho(t) \le c\rho(x) \exp\left(\frac{s}{s+2}\left|\int_x^t \frac{d\xi}{\rho(\xi)}\right|\right). \tag{3.3}$$

Unfortunately, we have not succeeded in obtaining an unconditional converse to Theorem 3.1. On the other hand, in the next assertion we propose a sufficient condition for inequalities (3.1)-(3.2) to hold which is “as near as possible” to the necessary condition (3.1).

THEOREM 3.2. – *Suppose we are given* $s \in [1, \infty)$. *If there are constants* $\delta \in \left(0, \dfrac{s}{s+2}\right)$, $c \ge 1$ *and* $x_0 \ge 1$ *such that in the domain* D

$$D = \{x, t \in \mathbb{R} : t \le x \le -x_0\} \cup \{x, t \in \mathbb{R} : t \ge x \ge x_0\}, \tag{3.4}$$

we have the estimate

$$\rho(t) \le c\rho(x) \exp\left(\left(\frac{s}{s+2} - \delta\right)\left|\int_x^t \frac{d\xi}{\rho(\xi)}\right|\right), \quad x, t \in D, \tag{3.5}$$

then inequalities (3.1)-(3.2) *hold.*

Using Theorem 3.2, one can deduce various sufficient conditions for (3.1)-(3.2) to hold; in particular, such conditions are obtained in the next two theorems. The first one is of special interest since its statement does not depend on the parameter $s \in [1, \infty)$.

THEOREM 3.3. – *Suppose the following condition holds (see* (1.17)*):*

$$\lim_{|x|\to\infty} d(x)\left[\int_0^{\sqrt{2}d(x)} (q(x+t)-q(x-t))dt\right]=0. \tag{3.6}$$

Then for any $s\in[1,\infty)$*, inequalities* (3.1)-(3.2) *hold.*

To formulate the second, more subtle, condition for estimates (3.1)-(3.2) to hold, we need a new definition.

DEFINITION 3.4. – *Suppose that for a given function* q *there exist* $a\geq 1, b>0$ *and* $x_0\geq 1$ *such that for all* $|x|\geq x_0$*, we have the inequalities:*

$$a^{-1}d(x)\leq d(t)\leq ad(x) \qquad \text{if} \qquad |t-x|\leq bd(x) \tag{3.7}$$

(*see* (1.17)). *Then the value*

$$\gamma(s)=a^2\exp\left(-\frac{\sqrt{2}s}{s+2}\frac{b}{a}\right), \qquad s\in[1,\infty) \tag{3.8}$$

is called an exponent of the function q *corresponding to the number* s.

THEOREM 3.5. – *Suppose we are given a function* q *and a number* $\gamma_0>1$. *Then for every* $s\in[1,\infty)$ *there exists an exponent* $\gamma(s)$ *of this function such that*

$$\gamma(s)=\gamma_0.$$

We can now formulate the second condition for inequalities (3.1)-(3.2) *to hold.*

THEOREM 3.6. – *Let a function* q *be given. If for a given* $s\in[1,\infty)$ *at least one of its exponents* $\gamma(s)$ *is less than* 1, *then estimates* (3.1)-(3.2) *hold.*

We now state the main result of the paper.

THEOREM 3.7. – *Suppose we are given a function* q *and a continuous and non-negative for* $x\in\mathbb{R}$ *function* $\theta(x)$. *Then the following assertions hold:*

A) *for* $p=1$, *inequality* (1.6) *holds if and only if* $h_1<\infty$ *(see* (1.24)*);*

B) *for the validity of inequality* (1.6) *for* $p\in(1,\infty)$ *it is necessary and, under the condition that at least one exponent* $\gamma(p')$ *of the function* q *is less than* 1, *also sufficient that* $h_p<\infty$ *(see* (1.24)*);*

C) *for the validity of inequality* (1.6) *for* $p=\infty$ *it is necessary and, under the condition that at least one exponent* $\gamma(1)$ *of the function* q *is less than* 1, *also* *sufficient that* $h_\infty < \infty$. *Here*

$$h_\infty = \sup_{x\in\mathbb{R}} \theta(x)d^2(x). \tag{3.9}$$

D) *Let* $p \in [1, \infty]$. *For the validity of inequality* (1.6) *it is necessary and, under the condition* (3.6), *also sufficient that* $h_p < \infty$.

REMARK 3.8. – For the sake of completeness, we restate Theorem 1.7 in Case A) of Theorem 3.7.

REMARK 3.9. – Inequalities (3.7) and a scheme for their application were introduced by Otelbaev (see [14, 16]).

4. – Proofs

PROOF OF THEOREM 1.3 (*Necessity*). – We treat the cases 1) $p = 1$, 2) $p \in (1, \infty)$, 3) $p = \infty$ separately.

1) *Case* $p = 1$.

Let us check the implication: (1.6) $\Rightarrow \sigma_1 < \infty$. Fix $x \in \mathbb{R}$ and set in (1.1)

$$f(t) := f_x(t) = \begin{cases} v'(t), & t \le x \\ 0, & t > x. \end{cases}$$

Then $\|f_x\|_1 = v(x)$, and

$$\begin{aligned} y(x) &= u(x)\int_{-\infty}^{x} v(\xi)f_x(\xi)d\xi + v(x)\int_{x}^{\infty} u(\xi)f_x(\xi)d\xi \\ &= u(x)\int_{-\infty}^{x} v(\xi)v'(\xi)d\xi = \frac{u(x)\cdot v(x)^2}{2} \end{aligned}$$

(see Lemmas 1.1 and 1.2). By (1.6), we now have

$$\frac{\theta(x)u(x)v^2(x)}{2} = \theta(x)|y(x)| \le \sup_{t\in\mathbb{R}} \theta(t)|y(t)| \le c\|f_x\|_1 = cv(x), \quad \mathbb{R} \quad \Rightarrow$$

$$\theta(x)\rho(x) = \theta(x)u(x)v(x) \le 2c, \quad x \in \mathbb{R} \quad \Rightarrow \quad \sigma_1 < \infty.$$

2) *Case* $p \in (1, \infty)$.

Let us check the implication: (1.6) $\Rightarrow \sigma_p < \infty$. Fix $x \in \mathbb{R}$ and set in (1.1)

$$f(t) = f_x(t) = \begin{cases} v(t)^{p'-1}, & t \le x \\ 0, & t > x \end{cases} \Rightarrow$$

$$\|f_x\|_p = \left[\int_{-\infty}^{x} |f_x(t)|^p dt\right]^{1/p} = \left[\int_{-\infty}^{x} v(t)^{p(p'-1)} dt\right]^{1/p} = \left[\int_{-\infty}^{x} v(t)^{p'} dt\right]^{1/p}.$$

Further (see (1.12)),

$$y(x) = u(x)\int_{-\infty}^{x} v(t)f_x(t)dt + v(x)\int_{x}^{\infty} u(t)f_x(t)dt = u(x)\int_{-\infty}^{x} v(t)\cdot v^{p'-1}(t)dt$$

$$= u(x)\int_{-\infty}^{x} v(t)^{p'} dt.$$

Using (1.6), we get

$$\theta(x)u(x)\int_{-\infty}^{x} v^{p'}(t)dt = \theta(x)|y(x)| \le \sup_{t\in\mathbb{R}} \theta(t)|y(t)| \le c(p)\|f_x\|_p = c(p)\left[\int_{-\infty}^{x} v(t)^{p'} dt\right]^{1/p}$$

$$\Rightarrow \quad \theta(x)u(x)J_{p'}^{1/p'}(x) \le c(p) < \infty, \quad x \in \mathbb{R}.$$

Similarly, we get the inequality

$$\theta(x)v(x)I_{p'}^{1/p'}(x) \le c(p) < \infty, \quad x \in \mathbb{R}.$$

Hence $\sigma_p(x) \le 2c(p), \quad x \in \mathbb{R} \;\Rightarrow\; \sigma_p < \infty.$

3) *Case* $p = \infty$.

Let us check the implication: (1.6) $\Rightarrow \sigma_\infty$. Set $f(t) = f_0(t) \equiv 1$, $t \in \mathbb{R}$ in (1.1). Then from (1.12) for $x \in \mathbb{R}$, it follows that

$$y(x) = \int_{-\infty}^{\infty} G(x,t)f_0(t)dt = \int_{-\infty}^{\infty} G(x,t)dt = \sigma_\infty(x) \quad \Rightarrow$$

$$\theta(x)\sigma_\infty(x) = \theta(x)|y(x)| \le \sup_{t\in\mathbb{R}} \theta(t)|y(t)| \le c(\infty)\|f_0\|_{C(\mathbb{R})} = c(\infty) < \infty \quad \Rightarrow \quad \sigma_\infty < \infty.$$

PROOF OF THEOREM 1.3 (*Sufficiency*). – We treat the cases 1) $p = 1$, 2) $p \in (1,\infty)$, 3) $p = \infty$ separately.

1) *Case* $p = 1$.

In the following relations, we use (1.12) and (1.18):

$$\sup_{x\in\mathbb{R}}\theta(x)|y(x)| \le \sup_{x\in\mathbb{R}}\theta(x)\left[u(x)\int_{-\infty}^{x} v(t)|f(t)|dt + v(x)\int_{x}^{\infty} u(t)|f(t)|dt\right]$$

$$\le \sup_{x\in\mathbb{R}}\theta(x)u(x)v(x)\left[\int_{-\infty}^{x}|f(t)|dt + \int_{x}^{\infty}|f(t)|dt\right] = \sigma_1\cdot\|f\|_1 \;\Rightarrow\; (1.6).$$

2) *Case* $p \in (1,\infty)$.

Below we use (1.12) and Hölder's inequality:

$$\sup_{x\in\mathbb{R}}\theta(x)|y(x)| \le \sup_{x\in\mathbb{R}}\left[\theta(x)u(x)\int_{-\infty}^{x} v(t)|f(t)|dt + \theta(x)v(x)\int_{x}^{\infty} u(t)|f(t)|dt\right]$$

$$\le \sup_{x\in\mathbb{R}}\theta(x)\left[u(x)J_{p'}^{1/p'}(x)\|f\|_p + v(x)I_{p'}^{1/p'}(x)\|f\|_p\right] = \sigma_p\cdot\|f\|_p \;\Rightarrow\; (1.6).$$

3) *Case* $p = \infty$.

In the following estimate we use (1.12):

$$\sup_{x\in\mathbb{R}} y(x)|\theta(x)| \le \sup_{x\in\mathbb{R}}\theta(x)\int_{-\infty}^{\infty} G(x,t)|f(t)|dt$$

$$\le \sup_{x\in\mathbb{R}}\theta(x)\int_{-\infty}^{\infty} G(x,t)dt\cdot\|f\|_{C(\mathbb{R})} = \sigma_\infty\cdot\|f\|_{C(\mathbb{R})} \;\Rightarrow\; (1.6).$$

□

PROOF OF THEOREM 1.7. – The assertion is a direct consequence of relations (1.14), (1.15), (1.18) and Theorem 1.3. □

PROOF OF THEOREM 1.8. – We need some auxiliary facts.

LEMMA 4.1. – *For all* $x \in \mathbb{R}$ *we have the estimates*

$$c^{-1}d(x) \le d(t) \le cd(x) \quad \text{if} \quad |t-x| \le d(x). \tag{4.1}$$

PROOF. – Let $t \in [x-d(x), x+d(x)]$. Below we use Lagrange's formula and Lemma 1.4:

$$|d(t)-d(x)| = |d'(\theta)|\,|t-x| \le \frac{|t-x|}{\sqrt{2}} \le \frac{d(x)}{\sqrt{2}} \quad\Rightarrow\quad (4.1).$$

□

LEMMA 4.2. – *For $x \in \mathbb{R}$ and $t \in [x - d(x), x + d(x)]$, we have the estimates*

$$c^{-1}v(x) \leq v(t) \leq cv(x) \quad and \quad c^{-1}u(x) \leq u(t) \leq cu(x). \tag{4.2}$$

PROOF. – For $x \in \mathbb{R}$ and $\xi \in [x - d(x), x + d(x)]$ from (2.5), (2.4), (1.18) and (4.1), it follows that

$$\frac{v'(\xi)}{v(\xi)} = \frac{1 + \rho'(\xi)}{2\rho(\xi)} \leq \frac{1}{\rho(\xi)} \leq \frac{c}{d(\xi)} \leq \frac{c}{d(x)} \quad \Rightarrow$$

$$\left. \begin{aligned} ln\frac{v(x + d(x))}{v(x)} &= \int_x^{x+d(x)} \frac{v'(\xi)}{v(\xi)} d\xi \leq c \int_x^{x+d(x)} \frac{d\xi}{d(x)} = c \\ ln\frac{v(x)}{v(x - d(x))} &= \int_{x-d(x)}^{x} \frac{v'(\xi)}{v(\xi)} d\xi \leq c \int_{x-d(x)}^{x} \frac{d\xi}{d(x)} = c \end{aligned} \right\} \quad \Rightarrow$$

$$\frac{1}{c} \leq \frac{v(x - d(x))}{v(x)}, \qquad \frac{v(x + d(x))}{v(x)} \leq c, \quad x \in \mathbb{R}.$$

These estimates together with Lemma 1.1 imply (4.2). □

Inequalities (1.21) are proved in the same way, using (4.2). Let us check, say, the first one:

$$J_{p'}(x) = \int_{-\infty}^{x} v^{p'}(t)dt \geq \int_{x-d(x)}^{x} v^{p'}(t)dt \geq c^{-1}v(x)^{p'}d(x).$$

□

PROOF OF THEOREM 1.9. – The assertion immediately follows from (1.22)-(1.23), (1.18) and Theorem 1.3. □

PROOF OF THEOREM 3.1. – We need some additional information.

DEFINITION 4.3. – *We say that finite positive functions φ and ψ defined in the interval (a, b) $(-\infty \leq a < b \leq \infty)$ are weakly equivalent (and write $\varphi(x) \asymp \psi(x)$, $x \in (a, b)$) if for all $x \in (a, b)$, we have the inequalities:*

$$c^{-1}\varphi(x) \leq \psi(x) \leq c\varphi(x). \tag{4.3}$$

LEMMA 4.4 *Suppose we are given a finite positive function f defined in $\mathbb{R}$. Then this function is weakly equivalent in $\mathbb{R}$ to some nondecreasing (non-increasing) finite positive function if and only if there is a constant $c \geq 1$ such that* *for all $x \in \mathbb{R}$, we have the estimate*

$$\sup_{t\le x} f(t) \le cf(x) \quad \Big(\sup_{t\ge x} f(t) \le cf(x)\Big). \tag{4.4}$$

REMARK 4.5. – The problem solved in Lemma 4.4 was suggested to the authors by Prof. I.R. Liflyand during their discussion of the paper [11].

PROOF OF LEMMA 4.4 (*Necessity*). – Both assertions of the lemma are checked in the same way; therefore, we only consider the first one.

So suppose we are given a finite positive nondecreasing function φ defined in $\mathbb{R}$ and $f(x) \asymp \varphi(x)$, $x \in \mathbb{R}$:

$$c^{-1}\varphi(x) \le f(x) \le c\varphi(x), \qquad x \in \mathbb{R}. \tag{4.5}$$

Assume that the estimate (4.4) does not hold. Then for every $n \ge 1$, there exist α_n and x_n such that

$$\alpha_n < x_n \qquad \text{and} \qquad f(\alpha_n) \ge nf(x_n), \qquad u = 1,2,\dots . \tag{4.6}$$

Then using (4.5) and (4.6), we obtain the following chain of inequalities:

$$\left.\begin{array}{l} c^{-1}\varphi(\alpha_n) \le f(\alpha_n) \le c\varphi(\alpha_n), \quad n = 1,2,\dots \\ c^{-1}\varphi(x_n) \le f(x_n) \le c\varphi(x_n), \quad n = 1,2,\dots \end{array}\right\} \quad \Rightarrow$$

$$c^{-1}n\varphi(\alpha_n) \le c^{-1}n\varphi(x_n) \le nf(x_n) \le f(\alpha_n) \le c\varphi(\alpha_n) \quad \Rightarrow$$

$$n \le c^2, \quad n = 1,2,\dots, \text{ a contradiction } \Rightarrow (4.4)$$

PROOF OF LEMMA 4.4 (*Sufficiency*). – Suppose that (4.4) holds. Set $\varphi(x) = \sup_{t\le x} f(t)$, $x \in \mathbb{R}$. Then the function φ defined in $\mathbb{R}$ is finite, positive and does not decrease, and $f(x) \asymp \varphi(x)$, $x \in \mathbb{R}$:

$$c^{-1}\varphi(x) = c^{-1}\sup_{t\le x} f(t) \le f(x) \le \sup_{t\le x} f(t) = \varphi(x) \le c\varphi(x), \; x \in \mathbb{R}.$$

□

Let us now prove (3.3). From (1.18), (3.1) and (3.2), it follows that

$$v^s(x)\rho(x) \asymp v^s(x)d(x) \asymp J_s(x), \qquad x \in \mathbb{R}, \quad u^s(x)\rho(x) \asymp u^s(x)d(x) \asymp I_s(x), \quad x \in \mathbb{R}.$$

Here the functions $J_s(x), I_s(x), x \in \mathbb{R}$ for every $s \in [1,\infty)$, are defined in $\mathbb{R}$, finite, positive and do not decrease (increase) in $\mathbb{R}$, respectively. Hence by Lemma 4.4 there exists a constant $c \ge 1$ such that for all $x \in \mathbb{R}$, we have the inequalities

$$v^s(t)\rho(t) \le cv^s(x)\rho(x) \text{ if } t \le x, \; x \in \mathbb{R}, \qquad u^s(t)\rho(t) \le cu^s(x)\rho(x) \text{ if } t \ge x, \; x \in \mathbb{R}.$$

To prove (3.3), it remains to substitute instead of the functions v and u their representations (2.3). □

PROOF OF THEOREM 3.2. – To prove the theorem, we need some auxiliary assertions.

LEMMA 4.6. – *Denote (see Lemma* 1.4 *and* 2.1*)*

$$d_0 = \sup_{x\in\mathbb{R}} d(x), \qquad d_0^{(1)} = \sup_{x\in\mathbb{R}} d_1(x), \qquad d_0^{(2)} = \sup_{x\in\mathbb{R}} d_2(x). \tag{4.7}$$

Then $d_0<\infty$, $d_0^{(1)}<\infty$, $d_0^{(2)}<\infty$.

PROOF. – All three inequalities are checked in the same way. Consider, say, the first one. Assume the contrary. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that (see Lemma 1.4):

$$d(x_n) \ge \sqrt{2}n, \qquad n=1,2,\dots, \qquad |x_n|\to\infty \qquad \text{as} \qquad n\to\infty. \tag{4.8}$$

Then for all $n\ge a$ (see (1.4)), we get (see (1.4)):

$$2 = \int_0^{\sqrt{2}d(x_n)} \int_{x_n-t}^{x_n+t} q(\xi)d\xi dt \ge \int_{\frac{1}{\sqrt{2}}d(x_n)}^{\sqrt{2}d(x_n)} \int_{x_n-t}^{x_n+t} q(\xi)d\xi dt \ge \frac{d(x_n)}{\sqrt{2}} \int_{x_n-\frac{1}{\sqrt{2}}d(x_n)}^{x_n+\frac{1}{\sqrt{2}}d(x_n)} q(\xi)d\xi$$

$$\ge n\int_{x_n-n}^{x_n+n} q(\xi)d\xi \ge n\int_{x_n-a}^{x_n+a} q(\xi)d\xi \ge nq_0(a)\to\infty \ \text{ as } n\to\infty.$$

Contradiction. □

LEMMA 4.7. – *For* $s\in[1,\infty)$*, we have the estimates*

$$J_s(x)\le c\cdot\int_{x-d_0^{(1)}}^{x} v^s(t)dt, \qquad x\in\mathbb{R}, \tag{4.9}$$

$$I_s(x)\le c\cdot\int_{x}^{x+d_0^{(2)}} u^s(t)dt, \qquad x\in\mathbb{R}. \tag{4.10}$$

PROOF. – Both estimates are proved in the same way. We prove (4.10). Let $x\in\mathbb{R}$, $k\ge1$. Below we use (4.7), (2.2) and (1.8):

$$\begin{aligned}\frac{k-1}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\int_{x+d_0^{(2)}}^{x+kd_0^{(2)}} \frac{d\xi}{d_0^{(2)}} \le \frac{1}{\sqrt{2}}\int_{x+d_0^{(2)}}^{x+kd_0^{(2)}} \frac{d\xi}{d_2(\xi)}\\ &\le \int_{x+d_0^{(2)}}^{x+kd_0^{(2)}} \frac{|u'(\xi)|}{u(\xi)}d\xi = -\int_{x+d_0^{(2)}}^{x+kd_0^{(2)}} \frac{u'(\xi)}{u(\xi)}d\xi = ln\frac{u(x+d_0^{(2)})}{u(x+kd_0^{(2)})}\\ &\Rightarrow \quad u(x+kd_0^{(2)})\le e^{-\frac{k-1}{\sqrt{2}}}u(x+d_0^{(2)}), \quad x\in\mathbb{R},\ k\ge1.\end{aligned} \tag{4.11}$$

The following relations are based on (1.8) and (4.11):

$$\begin{aligned}
I_s(x) &= \int_x^\infty u^s(t)dt = \int_x^{x+d_0^{(2)}} u^s(t)dt + \sum_{k=1}^\infty \int_{x+kd_0^{(2)}}^{x+(k+1)d_0^{(2)}} u^s(t)dt \\
&= \int_x^{x+d_0^{(2)}} u^s(t)dt \left\{ 1 + \sum_{k=1}^\infty \left[\int_{x+kd_0^{(2)}}^{x+(k+1)d_0^{(2)}} u^s(t)dt \right] \cdot \left[\int_x^{x+d_0^{(2)}} u^s(t)dt \right]^{-1} \right\} \\
&\le \int_x^{x+d_0^{(2)}} u^s(t)dt \cdot \left\{ 1 + \sum_{k=1}^\infty \left[u^s(x+kd_0^{(2)}) \cdot d_0^{(2)} \right] \cdot \left[u^s(x+d_0^{(2)}) d_0^{(2)} \right]^{-1} \right\} \\
&= \int_x^{x+d_0^{(2)}} u^s(t)dt \left\{ 2 + \sum_{k=2}^\infty \left[\frac{u(x+kd_0^{(2)})}{u(x+d_0^{(2)})} \right]^s \right\} \\
&\le \int_x^{x+d_0^{(2)}} u^s(t)dt \left\{ 2 + \sum_{k=2}^\infty e^{-\frac{k-1}{\sqrt{2}}s} \right\} = c \int_x^{x+d_0^{(2)}} u^s(t)dt.
\end{aligned}$$

□

Let us now go over to (3.1)-(3.2). These inequalities are proved in the same way, and therefore we only prove (3.1). Moreover, it is clear that only the upper estimate of (3.1) needs a proof (see (1.21)).

In order to prove it, consider three separate cases (see (3.4)):

1) $x \le -x_0$; 2) $x \ge x_0 + d_0^{(1)}$; 3) $x \in [-x_0, x_0 + d_0^{(1)}]$.

1) *Case* $x \le -x_0$.

In the following relations, we use (2.3), (3.5) and (1.18):

$$\begin{aligned}
J_s(x) &= \int_{-\infty}^x v^s(t)dt = \int_{-\infty}^x \rho(t)^{\frac{s}{2}} \exp\left(\frac{s}{2} \int_{x_0}^t \frac{d\xi}{\rho(\xi)} d\xi \right) dt \\
&= \frac{v^s(x)}{\rho(x)^{s/2}} \cdot \int_{-\infty}^x \rho(t)^{\frac{s}{2}+1} \cdot \frac{1}{\rho(t)} \exp\left(-\frac{s}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \\
&\le cv^s(x)\rho(x) \cdot \int_{-\infty}^x \exp\left[\left(\frac{s}{2} - \frac{(s+2)}{2}\delta \right) \int_t^x \frac{d\xi}{\rho(\xi)} \right] \cdot \frac{1}{\rho(t)} \exp\left(-\frac{s}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \\
&\le cv^s(x)d(x) \int_{-\infty}^x \frac{1}{\rho(t)} \exp\left(-\frac{s+2}{2}\delta \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt = cv^s(x)d(x) \Rightarrow (3.1).
\end{aligned}$$

2) *Case* $x \geq x_0 + d_0^{(1)}$.

In the following relations, we use (4.9), (2.3), (3.5) and (1.18):

$$J_s(x) = \int_{-\infty}^{x} v^s(t)dt \leq c \int_{x-d_0^{(1)}}^{x} v^s(t)dt = c \int_{x-d_0^{(1)}}^{x} \rho(t)^{s/2} \exp\left(\frac{s}{2}\int_{x_0}^{t} \frac{d\xi}{\rho(\xi)}\right) dt$$

$$= c\frac{v^s(x)}{\rho(x)^{s/2}} \cdot \int_{x-d_0^{(1)}}^{x} \rho(t)^{\frac{s}{2}+1} \cdot \frac{1}{\rho(t)} \exp\left(-\frac{s}{2}\int_{t}^{x} \frac{d\xi}{\rho(\xi)}\right) dt$$

$$\leq cv^s(x)\rho(x) \cdot \int_{x-d_0^{(1)}}^{x} \frac{1}{\rho(t)} \exp\left(-\frac{s+2}{2}\delta\int_{t}^{x} \frac{d\xi}{\rho(\xi)}\right) dt \leq cv(x)^s d(x) \;\Rightarrow\; (3.1).$$

3) *Case* $x \in [-x_0, x_0 + d_0^{(1)}]$.

It is easy to see that the function

$$f(x) = J_s(x)(v^s(x)d(x))^{-1}, \qquad x \in [-x_0, x_0 + d_0^{(1)}]$$

is continuous on the whole segment under consideration (see Lemmas 1.1 and 1.4), and therefore it is bounded. Hence (3.1) is also true for $x \in [-x_0, x_0 + d_0^{(1)}]$. □

PROOF OF THEOREM 3.3. – From (1.17) it is easy to deduce the relations (see [4]):

$$d'(x) = \frac{1}{\sqrt{2}} \left[\int_0^{\sqrt{2}d(x)} (q(x-t) - q(x+t))dt \right] \cdot \left[\int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(t)dt \right]^{-1}, \quad x \in \mathbb{R}$$

$$2 \leq \sqrt{2}d(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(t)dt, \quad x \in \mathbb{R}$$

which immediately imply

$$|d'(x)| \leq \frac{d(x)}{2} \left| \int_0^{\sqrt{2}d(x)} (q(x+t) - q(x-t))dt \right|, \quad x \in \mathbb{R}. \tag{4.12}$$

From (4.12) and (3.6), it follows that for a given $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon)$ such that

$$|d'(x)| \leq \varepsilon \qquad \text{for all} \qquad |x| \geq x_0. \tag{4.13}$$

From (4.13) and (1.18) we now obtain the estimates

$$-\frac{\varepsilon\sqrt{2}}{\rho(\xi)} \leq \frac{d'(\xi)}{d(\xi)} \leq \frac{\sqrt{2}\varepsilon}{\rho(\xi)}, \qquad |\xi| \geq x_0. \tag{4.14}$$

Fix δ, any number in the interval $\left(0, \frac{s}{s+2}\right)$, and set

$$\varepsilon = \frac{1}{\sqrt{2}}\left(\frac{s}{s+2} - \delta\right). \tag{4.15}$$

Then from (4.14) and (4.15) it follows that in the domain D (see (3.4)) where x_0 is taken from (4.13)), the following inequality holds:

$$d(t) \le d(x) \exp\left[\left(\frac{s}{s+2} - \delta\right)\left|\int_x^t \frac{d\xi}{\rho(\xi)}\right|\right]. \tag{4.16}$$

Further, using (4.16) and (1.18) we conclude that in the domain D, condition (3.5) holds. The theorem now follows from Theorem 3.2. □

PROOF OF THEOREM 3.5. – We need the following lemma.

LEMMA 4.8. – *Let* $\varepsilon \in [0,1]$. *Then for any* $x \in \mathbb{R}$ *we have the inequalities*

$$(1-\varepsilon)d(x) \le d(t) \le (1+\varepsilon)d(x) \quad \textit{if} \quad |t-x| \le \sqrt{2}\varepsilon d(x). \tag{4.17}$$

PROOF. – From Lagrange's formula and Lemma 1.4, it follows that

$$|d(t) - d(x)| = |d'(\theta)|\,|t-x| \le \frac{|t-x|}{\sqrt{2}} \le \varepsilon d(x) \quad \Rightarrow (4.17).$$

□

REMARK 4.9. – Inequalities similar to (4.17) were introduced by Otelbaev (see [14]).

Let us now turn to the proof of the theorem. From (4.17) it follows that

$$(1-\varepsilon)d(x) \le d(t) \le (1+\varepsilon)d(x) \le (1-\varepsilon)^{-1}d(x) \quad \text{for} \quad |t-x| \le \sqrt{2}\varepsilon d(x), \quad \varepsilon \in (0,1).$$

This means that in (3.7) we can take $a = (1-\varepsilon)^{-1}$, $b = \sqrt{2}\varepsilon$, $\varepsilon \in (0,1)$. Thus for a given $s \in [1,\infty)$, the function q has the exponent $\gamma(s)$ where (see (3.8)):

$$\gamma(x) = \frac{1}{(1-\varepsilon)^2}\exp\left(-\frac{2s}{s+2}\varepsilon(1-\varepsilon)\right) := \varphi(\varepsilon), \quad \varepsilon \in (0,1).$$

It is not hard to see that the function $\varphi(\varepsilon)$, $\varepsilon \in (0,1)$ has the properties $\varphi(0+0) = 1$, $\varphi(1-0) = +\infty$, $\varphi'(\varepsilon) > 0$ for $\varepsilon \in (0,1)$. Thus for any $\gamma_0 > 1$ there is $\varepsilon_0 \in (0,1)$ such that $\gamma(s) = \varphi(\varepsilon_0) = \gamma_0$. □

PROOF OF THEOREM 3.6. – Clearly, only upper estimates need a proof (see (1.21)). So suppose that for some $a \ge 1$ and $b > 0$ we have (3.7) and $\gamma(s) < 1$ for a given $s \in [1,\infty)$ (see (3.8)). Below we will show that condition (3.5) then holds and hence inequalities (3.1)-(3.2) are valid because of Theorem 3.5. We maintain the convention that within the present proof, the letter a, b, and x_0 exclusively stand for parameters from (3.7) and, in addition, the notation of the following Lemma 4.10 is kept.

LEMMA 4.10. – *Let $x, t \in \mathbb{R}$ be given such that either $t \le x \le -x_0$ or $t \ge x \ge x_0$. Denote by $\{\Delta_n\}_{n=-\infty}^{-1}$ and $\{\Delta_n\}_{n=1}^{\infty}$, $\mathbb{R}(x, bd)$-covering of $(-\infty, x]$ and $[x, \infty)$, respectively. Then if $t \in \Delta_n$, $n \in N' = \{\pm 1, \pm 2, \dots\}$, the following inequalities hold:*

$$a^{-2|n|} d(x) \le d(t) \le a^{2|n|} d(x), \tag{4.18}$$

$$\int_t^x \frac{d\xi}{\rho(\xi)} \ge \sqrt{2}\,\frac{b}{a}(|n|-1), \quad \text{if} \quad t \le x \le -x_0, \tag{4.19}$$

$$\int_x^t \frac{d\xi}{\rho(\xi)} \ge \sqrt{2}\,\frac{b}{a}(n-1), \quad \text{if} \quad t \ge x \ge x_0. \tag{4.20}$$

PROOF. – Let $t \ge x \ge x_0$ (for $t \le x \le -x_0$ all the inequalities of the lemma are proved in the same way). First note that from (3.7) and Definition 2.5, we get the estimates

$$\left.\begin{aligned} &\frac{1}{a} \le \frac{d(t)}{d(x_m)} \le a \quad \text{if} \quad t \in \Delta_m, \quad m \ge 1 \\ &\frac{1}{a} \le \frac{d(x_m)}{d(\Delta_m^-)} \le a \quad \text{if} \quad m \ge 1, \end{aligned}\right\} \Rightarrow \tag{4.21}$$
$$\frac{1}{a^2} \le \frac{d(t)}{d(\Delta_m^-)} \le a^2 \quad \text{for} \quad t \in \Delta_m, \quad m \ge 1$$

From (4.21) and Definition 2.5, we obtain (4.18) for $n = m = 1$. In addition, if in (4.21) we set $t = \Delta_m^+$, we get the estimates

$$\frac{1}{a^2} \le \frac{d(\Delta_m^+)}{d(\Delta_m^-)} \le a^2 \qquad \text{for} \qquad m \ge 1. \tag{4.22}$$

Let us now obtain the upper estimate of (4.18) for $n \ge 2$ (the lower estimate of (4.18) is obtained in the same way). Below we use (4.22), (4.21) and Definition 2.5:

$$\frac{d(t)}{d(x)} = \frac{d(t)}{d(\Delta_1^-)} = \frac{d(\Delta_1^+)}{d(\Delta_1^-)} \cdot \frac{d(\Delta_2^+)}{d(\Delta_2^-)} \cdots \frac{d(\Delta_{n-1}^+)}{d(\Delta_{n-1}^-)} \cdot \frac{d(t)}{d(\Delta_n^-)} \le \underbrace{a^2 \cdot a^2 \dots a^2}_{n \text{ times}} = a^{2n} \quad \Rightarrow \quad (4.18).$$

Let us now check (4.20) (the estimate (4.19) is proved in the same way). For $n = 1$ this inequality is obvious. Let $n \ge 2$. Below we use (1.18), Definition 2.5

and (3.7):

$$\int_x^t \frac{d\xi}{\rho(\xi)} \ge \frac{1}{\sqrt{2}}\int_x^t \frac{d\xi}{d(\xi)} = \frac{1}{\sqrt{2}}\sum_{k=1}^{n-1}\int_{\Delta_k}\frac{d\xi}{d(\xi)} + \frac{1}{\sqrt{2}}\int_{\Delta_n^-}\frac{d\xi}{d(\xi)}$$

$$\ge \frac{1}{\sqrt{2}}\sum_{k=1}^{n-1}\int_{\Delta_k}\frac{d(x_k)}{d(\xi)}\frac{d\xi}{d(x_k)} \ge \frac{1}{\sqrt{2}}\sum_{k=1}^{n-1}\frac{2b}{a} = \frac{\sqrt{2}b}{a}(n-1).$$

We now show that if $\gamma(s)<1$, then (3.5) holds. Let $t \ge x \ge x_0$ and $t \in \Delta_n$, $n \ge 1$ (the case $t \le x \le -x_0$ is treated in a similar way). Since $\gamma(s)<1$, there is $\delta \in \left(0, \frac{s}{s+2}\right)$ such that

$$\gamma(s)\exp\left(\frac{\sqrt{2}b}{a}\,\frac{s}{s+2}\delta\right)<1 \quad\Rightarrow\quad \text{(see (3.8))}$$

$$a^2 < \exp\left(\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)\right), \quad 0<\delta<\frac{s}{s+2} \tag{4.23}$$

By (4.23), for any $n \ge 1$ we get

$$a^{2n} \le \exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)n\right] = \exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta\right]\cdot\exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)(n-1)\right]$$

$$= v\exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)(n-1)\right], \qquad v = \exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)\right]. \tag{4.24}$$

Since $t \in \Delta_n$, $n \ge 1$, from (1.18), (4.18) and (4.24) it follows that

$$\frac{\rho(t)}{\rho(x)} \le 4\frac{d(t)}{d(x)} \le 4a^{2n} \le 4v\exp\left[\sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)(n-1)\right] \tag{4.25}$$

and from (4.20) it obviously follows that

$$(1-\delta)\frac{s}{s+2}\int_x^t\frac{d\xi}{\rho(\xi)} \ge \sqrt{2}\,\frac{b}{a}\,\frac{s}{s+2}(1-\delta)(n-1). \tag{4.26}$$

Finally, from (4.25) and (4.26) we get

$$\rho(t) \le 4v\rho(x)\exp\left[(1-\delta)\frac{s}{s+2}\int_x^t\frac{d\xi}{\rho(\xi)}\right] = c\rho(x)\exp\left[\left(\frac{s}{s+2}-\delta(s)\right)\int_x^t\frac{d\xi}{\rho(\xi)}\right] \tag{4.27}$$

where $c = 4v > 1$, $\delta(s) = \frac{s\delta}{s+2}$, $t \ge x \ge x_0$. Since (4.27) coincides with (3.5), it remains to refer to Theorem 3.2.

PROOF OF THEOREM 3.7. – The assertion of the theorem is a direct consequence of Theorems 1.9, 3.3 and 3.6. □

5. – Examples

In this section, we study two special cases of equation (1.1):

$$-y''(x) + e^{|x|}y(x) = f(x), \quad x \in \mathbb{R}, \tag{5.1}$$

$$-y''(x) + (e^{|x|} + h(x))y(x) = f(x), \quad x \in \mathbb{R}, \tag{5.2}$$

where

$$h(x) = e^{|x|}\cos(e^{\alpha|x|}), \quad x \in \mathbb{R}, \quad \alpha \in (0, \infty). \tag{5.3}$$

Throughout the sequel equation (5.2) is viewed as a perturbed equation (5.1) where the perturbation h is defined by (5.3).

Our general goal is to determine what is the influence of the perturbation h on the solution of problem (1.6) for equation (5.2) while $\alpha \in (0, \infty)$ changes. The following four items contain a list of main results concerning this question together with some comments:

1) Equation (5.1) is correctly solvable in L_p, $p \in [1, \infty]$.

2) For $p \in [1, \infty]$ the solutions of equation (5.1) satisfy the following estimate:

$$\sup_{x \in \mathbb{R}} e^{(1-\frac{1}{2p})|x|}|y(x)| \leq c(p)\|f\|_p, \quad \forall f \in L_p. \tag{5.4}$$

3) Equation (5.2) is correctly solvable in L_p, $p \in [1, \infty]$ for any $\alpha \in (0, \infty)$. Thus equation (5.2) "inherits" correct solvability in L_p, $p \in [1, \infty]$ from equation (5.1), in spite of the perturbation h. Since our main goal is related to (5.4), we can now formulate our problem more precisely: find all $\alpha \in (0, \infty)$ such that the solutions of equation (5.2) satisfy the same estimate (5.4) as the solutions of (5.1).

4) Let $p \in [1, \infty]$. The solutions of (5.2) satisfy inequality (5.4) for $\alpha \geq \frac{1}{2}$. For $\alpha \in \left(0, \frac{1}{2}\right)$, estimate (5.4) does not hold.

REMARK 5.1. – Equations (5.1) and (5.2) were already studied in [1]. Therefore, to avoid repetitions, we do not present proofs of the technical facts from that paper where their exposition is too lengthy. Instead of calculations, we present the final results together with key intermediate assertions. We thus mainly describe the logic and methods used for transition from the "implicit" (because of the function d, see (1.17)) Theorem 3.7 to the study of problem (1.6) for concrete equations. All details of the computations can be found in [1].

PROOF OF THE EXAMPLES

PROOF OF 1). – Equation (5.1) is correctly solvable in L_p, $p \in [1, \infty]$ because (see (1.4))

$$q_0(1) = \inf_{x \in \mathbb{R}} \int_{x-1}^{x+1} e^{|t|} dt \geq \inf_{x \in \mathbb{R}} \int_{x-1}^{x+1} 1 dt = 2 > 0.$$

PROOF OF 2). – We need an auxiliary function introduced by Otelbaev (see [9, 14]). For a given $x \in \mathbb{R}$, consider the equation in $d \geq 0$:

$$F(d) = 2, \qquad F(d) = d \int_{x-d}^{x+d} q(\xi) d\xi. \tag{5.5}$$

LEMMA 5.2 [9]. – *For every $x \in \mathbb{R}$, equation* (5.5) *has a unique finite positive solution.*

Below we denote the solution of (5.5) by $\hat{d}(x)$, $x \in \mathbb{R}$. Our interest in the function $\hat{d}$ is explained in the following lemma.

LEMMA 5.3 [6]. – *For $x \in \mathbb{R}$ we have*

$$\frac{\hat{d}(x)}{\sqrt{2}} \leq d(x) \leq \sqrt{2} \hat{d}(x). \tag{5.6}$$

Thus to obtain sharp by order two-sided estimates for the function d, it is enough to get such estimates for the function $\hat{d}$. To solve the latter problem, one usually uses the following fact.

LEMMA 5.4 [9]. – *For every $x \in \mathbb{R}$, the inequality $\eta \geq \hat{d}(x)$ $(0 \leq \eta \leq \hat{d}(x))$ holds if and only if $F(\eta) \geq 2$ $(F(\eta) \leq 2)$.*

The following proposition exemplifies an application of Lemma 5.4.

LEMMA 5.5 [8, 9]. – *Suppose the function q is representable in the form*

$$q = q_1 + q_2 \tag{5.7}$$

where $q_1(x)$ is positive and continuous everywhere in and is twice differentiable for $|x| \gg 1$, and $q_2 \in L_1^{\text{loc}}$. Denote

$$\mathcal{A}(x) = \left[0, 2q_1(x)^{-1/2}\right], \quad x \in \mathbb{R}, \tag{5.8}$$

(5.9)
$$\kappa_1(x) = \frac{1}{q_1(x)^{3/2}} \sup_{t\in\mathcal{A}(x)} \left| \int_{x-t}^{x+t} q_1''(\xi)d\xi \right|, \quad |x| \gg 1,$$

(5.10)
$$\kappa_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t\in\mathcal{A}(x)} \left| \int_{x-t}^{x+t} q_2(\xi)d\xi \right|, \quad x \in \mathbb{R}.$$

Then if $\kappa_1(x) \to 0$, $\kappa_2(x) \to 0$ *as* $|x| \to \infty$, *we have*

(5.11)
$$\hat{d}(x) \asymp q_1(x)^{-1/2}, \qquad x \in \mathbb{R}.$$

Let us now turn to 2). In (5.7) set

(5.12)
$$q(x) = q_1(x) = e^{|x|}, \qquad q_2(x) \equiv 0, \qquad x \in \mathbb{R}.$$

Then by Lemmas 5.5 and 5.3, we get

(5.13)
$$d(x) \asymp \hat{d}(x) \asymp e^{-\frac{1}{2}|x|}, \qquad x \in \mathbb{R}.$$

From (5.13) it follows that in case (5.12), condition (3.6) holds. Indeed, for $x \gg 1$ (the case $x \to -\infty$ is treated in a similar way), we have

$$d(x)\cdot \left| \int_0^{\sqrt{2}d(x)} (e^{x+t} - e^{x-t})dt \right| \le ce^{x/2} \left| \int_0^{\sqrt{2}d(x)} (e^t - e^{-t})dt \right|$$
$$\le ce^{x/2} \left| \int_0^{\sqrt{2}d(x)} tdt \right| = ce^{x/2}d^2(x) \le ce^{-x/2} \to 0,$$
$$\text{as} \quad x \to \infty \quad \Rightarrow \quad (3.6)$$

Inequality (5.4) now follows from Theorem 3.7(D) and (5.12).

PROOF OF 3). – This assertion is a consequence of the following lemma.

LEMMA 5.6 [1]. – *Suppose that the function* q *is representable in the form* (5.7) *where* $0 < q_1 \in L_1^{\mathrm{loc}}$ *and* $q_2 \in L_1^{\mathrm{loc}}$. *If the following conditions*

α) *there exists* $a_0 > 0$ *such that* $\inf x \in \mathbb{R} \int_{x-a_0}^{x+a_0} q_1(t)dt > 0$,

β) $\tau(a) \to 0$ *as* $a \to \infty$ *where*

$$\tau(a) = \sup_{x\in\mathbb{R}} \left[\int_{x-a}^{x+a} q_2(\xi)d\xi \right] \cdot \left[\int_{x-a}^{x+a} q_1(t)dt \right]^{-1}, \quad a \ge a_0$$

hold, then there exists $a_1 \ge a_0$ *such that* $q_0(a_1) > 0$ *(see* (1.4)*).*

REMARK 5.7. – In [1], Lemma 5.6 was applied to equation (5.2) with $q_1(x) = e^{|x|}$, $q_2(x) = h(x)$, $x \in \mathbb{R}$.

PROOF OF 4). – We need two more lemmas.

LEMMA 5.8. – *Let $q_1(x)$ be a positive continuous function in $\mathbb{R}$, continuously differentiable for $|x| \gg 1$, such that*

$$\lim_{|x|\to\infty} \frac{q_1'(x)}{q_1(x)^{3/2}} = 0, \tag{5.14}$$

$$q_1^0 > 0, \qquad q_1^0 = \inf_{x\in\mathbb{R}} q_1(x). \tag{5.15}$$

Then for any $\beta \geq 1$, there exists $x_1 = x_1(\beta)$ such that for all $|x| \geq x_1$ and $t \in \omega(x)$,

$$\omega(x) = \left[x - \frac{\beta}{\sqrt{q_1(x)}}, x + \frac{\beta}{\sqrt{q_1(x)}}\right], \quad x \in \mathbb{R}, \tag{5.16}$$

we have the inequalities

$$\left(1 - \frac{1}{2\beta}\right)^2 \leq \frac{q_1(t)}{q_1(x)} \leq \left(1 - \frac{1}{2\beta}\right)^{-2}. \tag{5.17}$$

PROOF. – Fix $\beta \geq 1$. Then by (5.14) and (5.15), there exist $x_0 = x_0(\beta)$ and $x_1(\beta) \gg x_0(\beta)$ such that

$$\frac{|q_1'(\xi)|}{q_1(\xi)^{3/2}} \leq \frac{1}{\beta^2} \qquad \text{if} \qquad |\xi| \geq x_0, \tag{5.18}$$

$$\omega(x) \cap [-x_0, x_0] = \emptyset \quad \text{if} \qquad |x| \geq x_1. \tag{5.19}$$

Let $|x| \geq x_1$ and $t \in \omega(x)$. By (5.18) and (5.19), we have

$$\left|\frac{1}{\sqrt{q_1(t)}} - \frac{1}{\sqrt{q_1(x)}}\right| = \frac{1}{2}\left|\int_x^t \frac{q_1'(\xi)}{q_1(\xi)^{3/2}} d\xi\right| \leq \frac{1}{2}\int_x^t \frac{|q_1'(\xi)|}{q_1^{3/2}(\xi)} d\xi \leq \frac{|t-x|}{2\beta^2} \leq \frac{1}{2\beta}\frac{1}{\sqrt{q_1(x)}} \quad \Rightarrow$$

$$\left(1 - \frac{1}{2\beta}\right)\frac{1}{\sqrt{q_1(x)}} \leq \frac{1}{\sqrt{q_1(t)}} \leq \left(1 + \frac{1}{2\beta}\right)\frac{1}{\sqrt{q_1(x)}}, \quad t \in \omega(x), \quad |x| \geq x_1.$$

The last estimates imply (5.17). □

LEMMA 5.9. – *Suppose that for a given function q there exists a function q_1 such that the following two conditions hold:*

1) *the function q_1 such that conditions of Lemma* 5.8;
2) *the following inequalities hold (see* (1.17)*):*

$$\frac{c^{-1}}{\sqrt{q_1(x)}} \leq d(x) \leq \frac{c}{\sqrt{q_1(x)}}, \qquad x \in \mathbb{R}. \tag{5.20}$$

Then for every given $\beta \geq 1$ *there exists* $x_1 = x_1(\beta)$ *such that for all* $|x| \geq x_1$ *and* $t \in \tilde{\omega}(x)$ *where*

$$\tilde{\omega}(x) = \left[x - \frac{\beta}{c} d(x), x + \frac{\beta}{c} d(x)\right], \quad x \in \mathbb{R}. \tag{5.21}$$

the following inequalities hold:

$$\frac{1}{c^2}\left(1 - \frac{1}{2\beta}\right) \leq \frac{d(t)}{d(x)} \leq c^2\left(1 - \frac{1}{2\beta}\right)^{-1}. \tag{5.22}$$

Proof. – From (5.20) we obtain the inclusion $\tilde{w}(x) \subseteq \omega(x)$, $x \in \mathbb{R}$ (see (5.16)). Together with Lemma 5.8, this implies that for a given $\beta \geq 1$ there exists $x_1 = x_1(\beta)$ such that for all $|x| \geq x_1$ and $t \in \tilde{\omega}(x)$ the following inequalities hold:

$$\frac{1}{c^2}\left(1 - \frac{1}{2\beta}\right) \leq \frac{1}{c^2}\sqrt{\frac{q_1(x)}{q_1(t)}} \leq \frac{d(t)}{d(x)} \leq c^2\sqrt{\frac{q_1(x)}{q_1(t)}} \leq c^2\left(1 - \frac{1}{2\beta}\right)^{-1} \quad \Rightarrow \quad (5.22)$$

□

Let us now show that for $\alpha \geq \frac{1}{2}$, the estimate (5.4) holds. In (5.7), set (see (5.3))

$$q = q_1 + q_2, \qquad q_1(x) = e^{|x|}, \qquad q_2(x) = h(x), \qquad x \in \mathbb{R}. \tag{5.23}$$

Then by Lemma 5.5 for $\alpha > 1/2$, we get (see [1])

$$\hat{d}(x) \asymp \frac{1}{\sqrt{q_1(x)}} = e^{\frac{1}{2}|x|}, \qquad x \in \mathbb{R}. \tag{5.24}$$

In [1], Lemma 5.3 was applied to establish (5.24) also for $\alpha = \frac{1}{2}$. In view of (5.6) and (5.24), this gives

$$d(x) \asymp \hat{d}(x) \asymp \frac{1}{\sqrt{q_1(x)}} = e^{-\frac{|x|}{2}}, \qquad x \in \mathbb{R}, \qquad \alpha \geq \frac{1}{2}. \tag{5.25}$$

From (5.25) and Lemmas 5.8 and 5.9, it now follows that in (5.23) inequalities (5.22) hold. Therefore to compute the exponent $\gamma(s)$, $s \in [1, \infty)$ of the function q from (5.23) (see (3.8)), one can take

$$a = c^2\left(1 - \frac{1}{2\beta}\right)^{-1}, \qquad b = \frac{\beta}{c}, \qquad \forall \beta \geq 1$$

and then for $s \in [1, \infty)$, we have

$$\gamma(s) = c^4\left(1 - \frac{1}{2\beta}\right)^{-2} \exp\left(-\frac{\sqrt{2}s}{s+2}\frac{\beta}{c^3}\left(1 - \frac{1}{2\beta}\right)\right), \qquad \forall \beta \geq 1.$$

□

This implies that regardless of $s \in [1, \infty)$, there exists $\beta \gg 1$ such that $\gamma(s) < 1 - \delta$, $\delta \in (0, 1)$ for all $s \in [1, \infty)$. By Theorem 3.7 in case (5.23), this implies (5.4) for all $p \in [1, \infty]$, $\alpha \geq \frac{1}{2}$.

Consider now the case $\alpha \in \left(0, \frac{1}{2}\right)$. Below we show that for every such α relation (5.24), and hence (5.25), do not hold, i.e.,

$$\varliminf_{|x|\to\infty} d(x)e^{\frac{1}{2}|x|} = \infty. \tag{5.26}$$

Then using (5.26), Theorem 1.9 and the necessary part of Theorem 3.7, we conclude that in case (5.23), for all $p \in [1, \infty]$ and $\alpha \in \left(0, \frac{1}{2}\right)$, inequality (5.4) does not hold either because

$$\varliminf_{|x|\to\infty} e^{\left(1-\frac{1}{2p}\right)|x|} d(x)^{2-\frac{1}{p}} = \varliminf_{|x|\to\infty} \left[e^{\frac{|x|}{2}} d(x)\right]^{2-\frac{1}{p}} = \infty \quad \Rightarrow \quad h_p = \infty.$$

To realize this program, we need one more lemma.

LEMMA 5.10 [1]. – *Suppose that the following conditions hold:*

1) *the function q has roots in points x_k, $k = 1, 2, \dots$ and $|x_k| \to \infty$ as $k \to \infty$;*

2) *the function q is absolutely continuous in $\mathbb{R}$ together with its derivatives $q^{(i)}$, $i = 1, 2, 3$ and $q''(x_k) \neq 0$ for all $k \gg 1$.*

Denote

$$A_k = \left[0, 4\sqrt[4]{\frac{1}{q''(x_k)}}\right], \quad \sigma_k = \sup_{t \in A_k} \left| \int_{x-t}^{x+t} q^{(4)}(\xi) d\xi \right|, \quad \delta_k = \frac{\sigma_k}{q''(x_k)^{5/4}}, \quad k \gg 1.$$

Then if $\delta_k \to 0$ as $k \to \infty$, we have

$$\hat{d}(x_k) = \sqrt[4]{\frac{6}{q''(x_k)}}(1 + \varepsilon_k), \qquad |\varepsilon_k| \leq c\delta_k, \qquad k \gg 1. \tag{5.27}$$

Let us apply this lemma to the case (5.23). Clearly

$$q(x_k) = 0 \quad \text{for} \quad x_k = \frac{ln((2k+1)\pi)}{\alpha}, \quad k \gg 1$$

(below it is enough to consider only positive roots of the function q). In [1], it was shown that Lemma 5.10 implies the relation

$$\hat{d}(x_k) \asymp (2k+1)^{-\frac{1+2\alpha}{4\alpha}}, \qquad k \gg 1. \tag{5.28}$$

Equality (5.26) now follows from (5.6), (5.28) and the following obvious relations:

$$\begin{aligned}\lim_{k\to\infty} e^{\frac{x_k}{2}} \cdot (2k+1)^{-\frac{1+2\alpha}{4\alpha}} &= \lim_{k\to\infty} c(\alpha)(2k+1)^{\frac{1}{2\alpha}-\frac{1+2\alpha}{4\alpha}} \\ &= c(\alpha) \lim_{k\to\infty} (2k+1)^{\frac{1-2\alpha}{4\alpha}} = \infty, \qquad \forall \alpha \in \left(0, \frac{1}{2}\right).\end{aligned}$$

Thus assertion 4) is completely proved.

REFERENCES

[1] N. Chernyavskaya - N. El-Natanov - L. Shuster, *Weighted estimates for solutions of a Sturm-Liouville equation in the space $L_1(R)$*, to appear in Proc. Royal Soc. Edinburgh.

[2] N. Chernyavskaya - L. Shuster, *On the WKB-method*, Diff. Uravnenija, **25** (10) (1989), 1826-1829.

[3] N. Chernyavskaya - L. Shuster, *Estimates for the Green function of a general Sturm-Liouville operator and their applications*, Proc. Amer. Math. Soc., **127** (1999), 1413-1426.

[4] N. Chernyavskaya - L. Shuster, *Asymptotics on the diagonal of the Green function of a Sturm-Liouville operator and its applications*, J. of London Math. Soc., **61** (2) (2000), 506-530.

[5] N. Chernyavskaya - L. Shuster, *A criterion for correct solvability of the Sturm-Liouvile equation in $L_p(R)$*, Proc. Amer. Math. Soc., **130** (4) (2002), 1043-1054.

[6] N. Chernyavskaya - L. Shuster, *Classification of initial data for the Riccati equation*, Bollettino dela Unione Matematica Italiana, 8, 5-B (2002), 511-525.

[7] N. Chernyavskaya - L. Shuster, *Conditions for correct solvability of a simplest singular boundary value problem of general form, I*, Z. Anal. Anwend., **25** (2006), 205-235.

[8] N. Chernyavskaya - L. Shuster, *An asymptotic majorant for solutions of Sturm-Liouville equations in $L_p(R)$*, Proc. Edinb. Math. Soc., **50** (2007), 87-114.

[9] N. Chernyavskaya - L. Shuster, *Davies-Harrell representations, Otelbaev's inequalities and properties of solutions of Riccati equations*, J. Math. Anal. Appl., **334** (2007), 998-1021.

[10] N. Chernyavskaya - L. Shuster, *A criterion for correct solvability in $L_p(R)$ of a general Sturm-Liouville equation*, J. London Math. Soc. (2), **80** (2009), 99-120.

[11] N. Chernyavskaya - L. Shuster, *Weight estimates for solutions of linear singular differential equations of the first order and the Everitt-Giertz problem*, Advances in Differential Equations, to appear.

[12] R. Courant, *Partial Differential Equations*, John Wiley & Sons, New York, 1962.

[13] E. B. Davies - E. M. Harrell, *Conformally flat Riemannian metrices, Schrödinger operators and semi-classical approximation*, J. Diff. Eq., **66** (2) (1987), 165-188.

[14] K. Mynbaev - M. Otelbaev, *Weighted Function Spaces and the Spectrum of Differential Operators*, Nauka, Moscow, 1988.

[15] R. Oinarov, *Properties of a Sturm-Liouville operator in L_p*, Izvestiya Akad. Nauk Kazakh. SSR, **1** (1990), 43-47.

[16] M. Otelbaev, *On smoothness of solutions of differential equations*, Izv. Akad. Nauk Kazah. SSR, **5** (1977), 45-48.

[17] M. Otelbaev, *A criterion for the resolvent of a Sturm-Liouville operator to be a kernel*, Math. Notes, **25** (1979), 296-297.

N. A. Chernyavskaya: Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, 84105, Israel
E-mail: nina@math.bgu.ac.il

L. A. Shuster: Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, Israel
E-mail: miriam@macs.biu.ac.il

Received June 29, 2011 and in revised form January 26, 2012