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Revisiting Pinors and Orientability

LORIANO BONORA - FABIO FERRARI RUFFINO - RAFFAELE SAVELLI

Abstract. – *We study the relations between pin structures on a non-orientable even-dimensional manifold, with or without boundary, and pin structures on its orientable double cover, requiring the latter to be invariant under sheet-exchange. We show that there is not a simple bijection, but that the natural map induced by pull-back is neither injective nor surjective: we thus find the conditions to recover a full correspondence. We then consider the example of surfaces, with detailed computations for the real projective plane, the Klein bottle and the Moebius strip.*

Sunto. – *In questo articolo studiamo le relazioni tra le strutture di pin su una varietà non orientabile di dimensione pari, con o senza bordo, e, dall'altro lato, le strutture di pin sul doppio ricoprimento orientabile, invarianti per lo scambio di foglie. Mostriamo che non c'è una semplice bigezione, come ci si potrebbe attendere, ma che la mappa naturale indotta dal pull-back non è né suriettiva né iniettiva: troviamo quindi le condizioni per recuperare una piena corrispondenza. Consideriamo poi l'esempio delle superfici, con calcoli dettagliati per il piano proiettivo reale, la bottiglia di Klein e il nastro di Moebius.*

1. – Introduction

Given a non-orientable manifold X, we call $\tilde{X}$ its orientable double cover, equipped with the orientation-reversing involution τ such that $\tilde{X}/\tau \simeq X$: we study the relations between pin structures on X and τ-invariant pin structures on $\tilde{X}$. We show that there is not a simple bijection as one might expect. In particular, recalling that there are two inequivalent euclidean pin groups at a fixed dimension, called Pin^+ and Pin^-, there is a natural map induced by pull-back:

$$\Phi : \{\mathrm{pin}^{\pm} \text{ structures on } X\} \longrightarrow \{\tau\text{-invariant } \mathrm{pin}^{\pm} \text{ structures on } \tilde{X}\}$$

but this map is neither injective nor surjective. To make it surjective, we must impose one condition more: if $\tilde{\xi}$ is a $\mathrm{pin}^{\pm}$ structure on $\tilde{X}$ and $\widetilde{d\tau}$ is an equivalence between $\tilde{\xi}$ and $\tau^*\tilde{\xi}$, then $\widetilde{d\tau}^2$ must be the identity, not the sheet-exchange of $\tilde{\xi}$ with respect to the tangent frame bundle $P_O\tilde{X}$. With this requirement we recover surjectivity. Moreover, non-injectivity is due to the fact that two pin$^{\pm}$ structures on X, which can be obtained from one another via the action of $w_1(X) \in H^1(X, \mathbb{Z}_2)$, are pulled back to equivalent structures on $\tilde{X}$. This is clearer if we describe pin structures via the holonomy of the pin connection over 1-cycles: two such structures differ by the holonomy along the cycle whose lift in $\tilde{X}$ is not a cycle any more, but a path joining the two lifts of the same point of X. Thus, pulling them back to $\tilde{X}$ their difference disappears. We will see how to recover such a difference considering the correspondence, analogous to Φ, for pinors as sections of the associated vector bundle, not simply for the pin structures themselves. This gives a global geometrical description of the approach considered in [10], and it is the explicit construction for pinors of what stated in [1] about a generic action of a discrete group on a manifold. We then consider the analogous result for the case of manifolds with boundary.

2. – Preliminaries

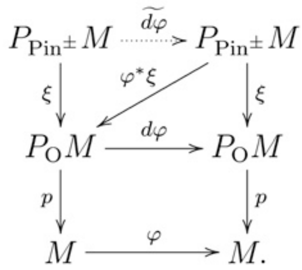
2.1 – *Preliminaries on pinors*

We recall that the group SO(n) has a unique 2-covering Spin(n), while the group O(n) has two inequivalent 2-coverings Pin$^{\pm}$(n), obtained from the Clifford algebras with positive and negative signature respectively, as explained in [8] (for Clifford algebras we use the convention $vw + wv = 2\langle v, w\rangle$, without the minus sign). Let $p^{\pm} : \mathrm{Pin}^{\pm}(n) \to \mathrm{O}(n)$ be such 2-coverings with kernel $\{\pm 1\}$, both restricting to $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. If we fix a the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and we denote by j_1 the reflection with respect to the hyperplane $e_1^{\perp}$, we have that $\mathrm{O}(n) = \langle \mathrm{SO}(n), j_1\rangle$, and $(p^{\pm})^{-1}(\{1, j_1\}) = \{\pm 1, \pm e_1\}$: the latter is isomorphic to $\mathbb{Z}_4$ if $e_1^2 = -1$ and to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $e_1^2 = 1$, that's why in general we get non-isomorphic coverings. For details the reader can see [8]. Similarly to the case of spin structures, there is a simply transitive action of $H^1(M, \mathbb{Z}_2)$ on pin$^{\pm}$ structures on a bundle $E \to M$. In particular, this implies that if there exist both pin$^+$ and pin$^-$ structures, their number is the same. Given a real vector bundle $\pi : E \to M$, the following conditions hold [7]:

- E admits a pin$^+$-structure if and only if $w_2(E) = 0$;
- E admits a pin$^-$-structure if and only if $w_2(E) + w_1(E) \cup w_1(E) = 0$.

As for spin structures, a pin structure on a manifold is by definition a pin structure on its tangent bundle.

Let M be a manifold of dimension $2n$. We say that a pin$^{\pm}$ structure ξ is invariant under an isometry φ if $\xi \simeq \varphi^*\xi$, i.e. if there exists a (non-canonical) lift $\widetilde{d\varphi}$ completing the following diagram:

(1) 

If such a $\widetilde{d\varphi}$ exists, there are only two possibilities, linked by an exchange of the two sheets: in fact, $\widetilde{d\varphi}$ is a lifting of the map $\varphi^*\xi$ to a $2:1$ covering of the codomain ([6] prop. 1.34 pag. 62). Calling γ the sheet exchange, the two possible lifts are $\widetilde{d\varphi}$ and $\widetilde{d\varphi} \circ \gamma$. Then $\widetilde{d\varphi} \circ \gamma = \gamma \circ \widetilde{d\varphi}$ since, if $\widetilde{d\varphi}(p_x) = q_{\varphi(x)}$, then the only possibility is that $\widetilde{d\varphi}(\gamma(p_x)) = \gamma(q_{\varphi(x)})$ in order to cover $d\varphi$.

2.2 – *Double covering of a non-orientable manifold*

As is well-known, every non-orientable manifold X has an orientable double-cover $\tilde{X}$ with an orientation-reversing involution τ such that $X \simeq \tilde{X} / \tau$. For $\pi : \tilde{X} \to X$ the projection, the kernel of $\pi^* : H^1(X, \mathbb{Z}_2) \to H^1(\tilde{X}, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$, and it is generated by $w_1(X)$. This is a consequence of the following exact sequence in cohomology [9]:

$$\cdots \longrightarrow H^{i-1}(X, \mathbb{Z}_2) \stackrel{\cup w_1(X)}{\longrightarrow} H^i(X, \mathbb{Z}_2) \stackrel{\pi^*}{\longrightarrow} H^i(\tilde{X}, \mathbb{Z}_2) \longrightarrow H^i(X, \mathbb{Z}_2) \longrightarrow \cdots \ .$$

For $i = 1$, since $H^0(X, \mathbb{Z}_2) = \mathbb{Z}_2$ we have that $\mathrm{Im}(\cup w_1(X)) = w_1(X)$, thus by exactness $\mathrm{Ker}\,\pi^* = \{0, w_1(X)\} \simeq \mathbb{Z}_2$. This is what we expected: since the double covering is orientable, the pull-back π^* must kill $w_1(X)$.

In the sequel we will also need another general result, that the reader can easily verify.

LEMMA 2.1. – *For $\pi : \tilde{X} \to X$ the projection and $p : TX \to X$, $\tilde{p} : T\tilde{X} \to \tilde{X}$ the tangent bundles, there is the canonical bundle isomorphism:*

$$\begin{aligned} \varphi : T\tilde{X} &\stackrel{\simeq}{\longrightarrow} \pi^* TX \\ \varphi(v) &= (d\pi(v), \tilde{p}(v)) \ . \end{aligned}$$

Similarly, for the orthogonal frame bundles with respect to a metric g on X and its pull-back $\pi^* g$ on $\tilde{X}$, there is the canonical isomorphism:

$$\begin{aligned} \varphi_O : P_O\tilde{X} &\stackrel{\simeq}{\longrightarrow} \pi^* P_O X \\ \varphi_O(x) &= (d\pi(x), \tilde{p}(x)) \ . \end{aligned}$$

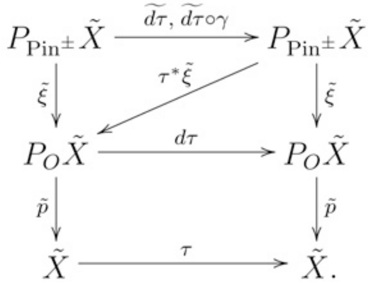
2.3 – *Pinors on the double covering*

We now want to compare pinors on a non-orientable manifold X and pinors on its double cover $\tilde{X}$ which are τ-invariant. We start with the following simple lemma:

LEMMA 2.2. – *If X admits a pin$^+$-structure or a pin$^-$-structure then $\tilde{X}$ is spin.*

PROOF. – By lemma 2.1 we have that $w_2(\tilde{X}) = \pi^* w_2(X)$. Since $w_1(X) \in \mathrm{Ker}\, \pi^*$, we obtain $\pi^* w_2(X) = \pi^*(w_2(X) + w_1(X) \cup w_1(X))$, thus if there is a pin$^\pm$ structure we get $w_2(\tilde{X}) = 0$. □

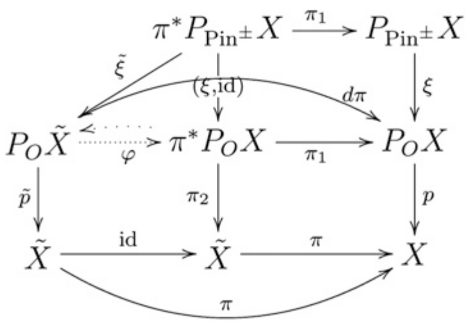
Let us suppose that a pin$^\pm$ structure on $\tilde{X}$ is τ-invariant. Thus we have two possible lifts of $d\tau$ in diagram (1):



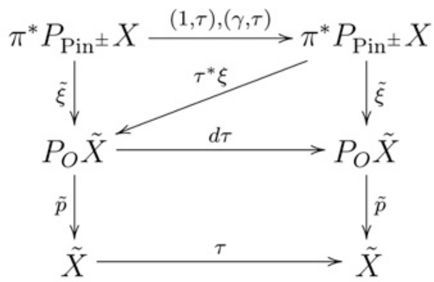
Since $\widetilde{d\tau} \circ \gamma = \gamma \circ \widetilde{d\tau}$, it follows that $(\widetilde{d\tau} \circ \gamma)^2 = \widetilde{d\tau}^2$, and the latter can be only id or γ, since it is an auto-equivalence of $\tilde{\xi}$ which covers $d\tau^2 = \mathrm{id}$.

We show that the pull-back of a pin$^\pm$ structure on X is a pin$^\pm$ structure on $\tilde{X}$ which is τ-invariant and such that $\widetilde{d\tau}^2 = \mathrm{id}$. Let us consider the following diagram:

(2)



where $\tilde{\xi}$ defines the pull-back on $\tilde{X}$ of the pin structure ξ of X, for φ defined in lemma 2.1. Thus we consider as the total space of the bundle exactly $\pi^* P_{\mathrm{Pin}^\pm} X$. It is now easy to see that $\tilde{\xi}$ is τ-invariant via the two possible equivalences:



where γ is the exchange of sheets of $P_{\mathrm{Pin}^\pm} X$ with respect to $P_O X$, while τ is the exchange of sheets of $\tilde{X}$ with respect to X. In fact, by diagram (2) we have $\tilde{\xi}(p', \tilde{x}) = \varphi^{-1} \circ (\xi, \mathrm{id})(p', \tilde{x}) = \varphi^{-1}(p, \tilde{x}) = d\pi_{\tilde{x}}^{-1}(p)$ where $\pi_{\tilde{x}}$ is π restricted to a neighborhood of $\tilde{x}$ on which it is a diffeomorphism. Therefore, for $\varepsilon = 1, \gamma$:

$$d\tau \circ \tilde{\xi}(p', \tilde{x}) = d\tau(d\pi_{\tilde{x}}^{-1}(p)) = d(\tau \circ \pi_{\tilde{x}}^{-1})(p) = d(\pi_{\tau(\tilde{x})}^{-1})(p)$$

$$\tilde{\xi} \circ (\varepsilon, \tau)(p', \tilde{x}) = \tilde{\xi}(\varepsilon(p'), \tau(\tilde{x})) = d(\pi_{\tau(\tilde{x})}^{-1})(p)$$

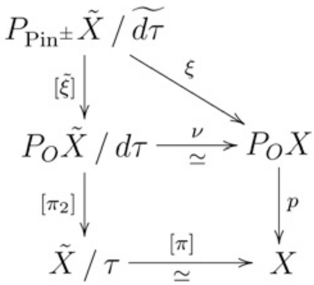
so that the diagram commutes. In particular, we see that the two possible isomorphisms $\widetilde{d\tau} = (1, \tau), (\gamma, \tau)$ have the property that $\widetilde{d\tau}^2 = 1$. We have thus constructed a function:

$$\Phi : \{\mathrm{pin}^\pm \text{ structures on } X\} \longrightarrow \{\mathrm{pin}^\pm \text{ structures on } \tilde{X} \tau\text{-invariant with } \widetilde{d\tau}^2 = 1\}.$$

We now show that Φ is surjective, i.e. that a τ-invariant pin$^\pm$ structure $\tilde{\xi}$ on $\tilde{X}$ satisfying $\widetilde{d\tau}^2 = 1$ is the pull-back of a pin$^\pm$ structure on X. The latter is:

$$\xi : P_{\mathrm{Pin}^\pm} \tilde{X} \,/\, \widetilde{d\tau} \longrightarrow P_O \tilde{X} \,/\, d\tau \simeq P_O X\ .$$

In more detail:



where $\nu([p]) = d\pi(p)$. From $\widetilde{d\tau}^2 = 1$ we get that the quotient is a 2-covering of $P_O X$, otherwise we would obtain a 1-covering, i.e. a bundle isomorphism, since $\gamma = \widetilde{d\tau}^2$ would identify also the two points of the same fiber. To see that $\tilde{\xi} \simeq \pi^*\xi$, we use the equivalence:

$$\begin{array}{ccc} P_{\mathrm{Pin}^\pm}\tilde{X} & \xrightarrow[\simeq]{\ \mu\ } & \pi^*\big(P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau}\big) \\ & {\scriptstyle\tilde{\xi}}\searrow \qquad \swarrow{\scriptstyle\varphi^{-1}\circ(\xi,\mathrm{id})} & \\ & P_O\tilde{X} & \end{array} \tag{3}$$

for $\mu(\tilde{p}_{\tilde{x}}) = ([\tilde{p}_{\tilde{x}}], \tilde{x})$. The inverse of μ is given by $\mu^{-1}([\tilde{p}_{\tilde{x}}], \tilde{x}) = \tilde{p}_{\tilde{x}}$ or equivalently $\mu^{-1}([\tilde{p}_{\tilde{x}}], \tau(\tilde{x})) = \widetilde{d\tau}(\tilde{p}_{\tilde{x}})$. The diagram is commutative: $\varphi^{-1} \circ (\xi, \mathrm{id}) \circ \mu(\tilde{p}_{\tilde{x}}) = \varphi^{-1} \circ (\xi, \mathrm{id})([\tilde{p}_{\tilde{x}}], \tilde{x}) = \varphi^{-1}((\nu \circ [\xi])([\tilde{p}_{\tilde{x}}]), \tilde{x}) = \varphi^{-1}(d\pi(\tilde{\xi}(\tilde{p}_{\tilde{x}})), \tilde{x}) = \tilde{\xi}(\tilde{p}_{\tilde{x}})$.

It is easy to show that Φ commutes via π^* with the actions of $H^1(X, \mathbb{Z}_2)$ and $H^1(\tilde{X}, \mathbb{Z}_2)$. In fact, for $\xi : P_{\mathrm{Pin}^\pm}X \to P_O X$ a pin$^\pm$ structure, up to isomorphism we can view $\Phi(\xi)$ as $\pi^*\xi : \pi^*P_{\mathrm{Pin}^\pm}X \to \pi^*P_O X$. We fix a Čech class $[\omega] \in \check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$ for a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ of X. If the transition function of $P_{\mathrm{Pin}^\pm}X$ are $s_{\alpha\beta}$ and we fix a representative ω, then the new transition functions are $s_{\alpha\beta} \cdot \omega_{\alpha\beta}$. On the two components of $\pi^{-1}U_{\alpha\beta}$, the transition functions were both $s_{\alpha\beta}$ and they become both $s_{\alpha\beta} \cdot \omega_{\alpha\beta}$, i.e., ω acts on the transition functions of $\pi^*\xi$ exactly as $\pi^*\omega$. Since $[\pi^*\omega] = \pi^*[\omega]$, we get the claim. Thus we have a diagram:

$$\begin{array}{ccc} \{\mathrm{Pin}^\pm\text{-structures on } X\} & \xrightarrow{\ \Phi\ } & \{\mathrm{Pin}^\pm\text{-structures on } \tilde{X}\ \tau\text{-invariant with } \widetilde{d\tau}^2 = 1\} \\ \circlearrowleft & & \circlearrowleft \\ \check{H}^1(X, \mathbb{Z}_2) & \xrightarrow{\ \pi^*\ } & \check{H}^1(\tilde{X}, \mathbb{Z}_2). \end{array}$$

This implies in particular that $\Phi^{-1}(\tilde{\xi})$ is made by two inequivalent pin$^\pm$ structures, obtainable from each other via the action of $w_1(X) \in \mathrm{Ker}\,\pi^*$. We will now show that the two inequivalent counter images can be recovered as $P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau}$ and $P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, (\widetilde{d\tau} \circ \gamma)$, by proving that these two quotients are inequivalent. In fact, let us suppose that there exists an equivalence:

$$\begin{array}{ccc} P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau} & \xrightarrow{\ \rho\ } & P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, (\widetilde{d\tau} \circ \gamma) \\ & {\scriptstyle\xi}\searrow \qquad \swarrow{\scriptstyle\xi'} & \\ & P_O X & \end{array}$$

then it lifts to an equivalence of the pull-backs:

$$\begin{array}{ccccc} & & P_O\tilde{X} & & \\ & \nearrow^{\tilde{\xi}} & & \nwarrow^{\tilde{\xi}'} & \\ \pi^*(P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau}) & & \xrightarrow{\quad\tilde{\rho}\quad} & & \pi^*(P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, (\widetilde{d\tau}\circ\gamma)) \\ \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_1} \\ P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau} & & \xrightarrow{\quad\rho\quad} & & P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, (\widetilde{d\tau}\circ\gamma) \\ & \searrow_{\xi} & & \swarrow_{\xi'} & \\ & & P_O X & & \end{array}$$

but, being both the pull-backs equivalent to $P_{\mathrm{Pin}^\pm}\tilde{X}$ via (3), the only two possibilities for $\tilde{\rho}$ are the following:

$$\begin{array}{ccc} P_{\mathrm{Pin}^\pm}\tilde{X} & \xrightarrow{\quad \mathrm{id},\gamma \quad} & P_{\mathrm{Pin}^\pm}\tilde{X} \\ {\scriptstyle \mu}\downarrow\uparrow & & \uparrow\downarrow{\scriptstyle \mu'} \\ \pi^*(P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, \widetilde{d\tau}) & \xrightarrow{\tilde{\rho}} & \pi^*(P_{\mathrm{Pin}^\pm}\tilde{X} \,/\, (\widetilde{d\tau}\circ\gamma)). \end{array}$$

Let us show that none of the two can be a lift of ρ. In fact, if it were so, they would be of the form:

$$\tilde{\rho}([p], \tilde{x}) = (\rho[p], \tilde{x}) \tag{4}$$

while:

$$([p_{\tilde{x}}], \tilde{x}) \xrightarrow{\quad \mu^{-1} \quad} p_{\tilde{x}} \xrightarrow{\quad \mathrm{id} \quad} p_{\tilde{x}} \xrightarrow{\quad \mu' \quad} ([p_{\tilde{x}}], \tilde{x})$$

$$([p_{\tilde{x}}], \tau(\tilde{x})) \xrightarrow{\mu^{-1}} \widetilde{d\tau}(p_{\tilde{x}}) \xrightarrow{\mathrm{id}} \widetilde{d\tau}(p_{\tilde{x}}) \xrightarrow{\mu'} ([\widetilde{d\tau}(p_{\tilde{x}})], \tau(\tilde{x}))$$

and in the codomain $[p_{\tilde{x}}] \neq [\widetilde{d\tau}(p_{\tilde{x}})]$ since the class is taken with respect to $\widetilde{d\tau}\circ\gamma$, thus (4) is inconsistent. The same would happen choosing γ instead of the identity. Thus $\tilde{\rho}$ lifts only the auto-equivalences of each of the two quotients, not an equivalence between them.

Now that we have seen the relationship between $\mathrm{pin}^\pm$ structures on X and the corresponding ones on $\tilde{X}$, we analyze such a relationship at the level of pinors (i.e. sections of the associated vector bundles). Let us start from X and pull-back a pin$^{\pm}$ structure as in the following diagram:

$$\begin{array}{ccc} \pi^* P_{\mathrm{Pin}^{\pm}} X & \xrightarrow{\pi_1} & P_{\mathrm{Pin}^{\pm}} X \\ {\scriptstyle \tilde{\xi}}\downarrow & & \downarrow{\scriptstyle \xi} \\ P_O \tilde{X} & \xrightarrow{d\pi} & P_O X. \end{array}$$

For the associated bundles of pinors, we have that $(\pi^* P_{\mathrm{Pin}^{\pm}} X) \times_\rho \mathbb{C}^{2^n} \simeq \pi^*(P_{\mathrm{Pin}^{\pm}} X \times_\rho \mathbb{C}^{2^n})$ canonically. Thus, given on X a pinor $s \in \Gamma(P_{\mathrm{Pin}^{\pm}} X \times_\rho \mathbb{C}^{2^n})$, we can naturally consider on $\tilde{X}$ its pull-back $\pi^* s \in \Gamma((\pi^* P_{\mathrm{Pin}^{\pm}} X) \times_\rho \mathbb{C}^{2^n})$. The natural equivalence between $\tilde{\xi}$ and $\tau^* \tilde{\xi}$ is given by $\widetilde{d\tau}(p, \tilde{x}) = (p, \tau(\tilde{x}))$, and, if we extend it to the associated vector bundles, we have that a section $s' \in \Gamma((\pi^* P_{\mathrm{Pin}^{\pm}} X) \times_\rho \mathbb{C}^{2^n})$ is the pull-back of a section on X if and only if $\widetilde{d\tau}(s') = s'$.

Vice versa, let us start from $\tilde{X}$. We fix a pin$^{\pm}$ structure $\tilde{\xi}$ such that $\tilde{\xi} \simeq \tau^* \tilde{\xi}$ with $\widetilde{d\tau}^2 = 1$. Then there are two natural vector space isomorphisms:

- $\widetilde{d\tau}$-invariant sections of the associated bundle correspond to sections of the pin$^{\pm}$ structure $P_{\mathrm{Pin}^{\pm}} \tilde{X} \,/\, \widetilde{d\tau}$ on X;
- $(\widetilde{d\tau} \circ \gamma)$-invariant sections of the associated bundle correspond to sections of the pin$^{\pm}$ structure $P_{\mathrm{Pin}^{\pm}} \tilde{X} \,/\, (\widetilde{d\tau} \circ \gamma)$ on X.

In particular, the two conditions of invariance for sections are equivalent to $s_x = \widetilde{d\tau}(s_{\tau(x)})$ in the first case, and $s_x = -\widetilde{d\tau}(s_{\tau(x)})$ in the second case, since the action of γ corresponds to the multiplication by $-1 \in \mathrm{Pin}^{\pm}(n)$. We remark that if we want to describe invariance of pinors under general isometries, we must take into account the sign ambiguity due to the *projective* action of isometries on pinors and spinors [4]. In the present case, since we distinguish $\widetilde{d\tau}$ and $\widetilde{d\tau} \circ \gamma$ on the basis of the associated quotient on X, we fix this ambiguity. In this way we compensate the lack of injectivity of the map Φ between pin$^{\pm}$-structures on X and τ-invariant pin$^{\pm}$-structures on $\tilde{X}$, restoring the injectivity on pinors as sections of the associated vector bundles.

3. – Surfaces

We show the explicit examples of pin structures on surfaces (cfr. [3, 5]).

3.1 – *Invariant structures on the torus*

The torus has trivial tangent bundle $T^2 \times \mathbb{R}^2 \simeq S^1 \times S^1 \times \mathbb{R}^2$. The four inequivalent Spin or pin$^{\pm}$ structures can be all obtained from the trivial principal bundle $S^1 \times S^1 \times \mathrm{Spin}(2)$ or $S^1 \times S^1 \times \mathrm{Pin}^{\pm}(2)$ in the following way:

$$\begin{array}{cccc} (\theta,\varphi,p') & (\theta,\varphi,p') & (\theta,\varphi,p') & (\theta,\varphi,p') \\ \Big\downarrow \tilde{\xi}_0 & \Big\downarrow \tilde{\xi}_1 & \Big\downarrow \tilde{\xi}_2 & \Big\downarrow \tilde{\xi}_3 \\ (\theta,\varphi,p) & (\theta,\varphi,R_\theta\cdot p) & (\theta,\varphi,R_\varphi\cdot p) & (\theta,\varphi,R_\varphi R_\theta\cdot p) \end{array}$$

where R_x is the rotation by the angle x. To see that, e.g., the first two are not equivalent, we notice that we would need a map:

$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{\quad\rho\quad} & (\theta,\varphi,\tilde{R}_{-\theta}p') \\ & {\scriptstyle\tilde{\xi}_0}\searrow \quad \swarrow{\scriptstyle\tilde{\xi}_1} & \\ & (\theta,\varphi,p) & \end{array}$$

for $\tilde{R}_{-\theta}$ a lift of $R_{-\theta}$ to Spin or $\mathrm{Pin}^{\pm}$. But in this way ρ is not well defined, since for θ and $\theta+2\pi$ we get two lifts differing by -1.

We now see that all these $\mathrm{pin}^{\pm}$ structures are τ-invariant, where τ is the involution giving the Klein bottle, namely $\tau(\theta,\varphi)=(\theta+\pi,-\varphi)$. On the tangent frame bundle we have the action $d\tau(\theta,\varphi,p)=(\theta+\pi,-\varphi,j_2p)$ where j_2 is the reflection along $e_2^{\perp}$, i.e. $(x,y)\to(x,-y)$. The equivalence between $\tilde{\xi}_0$ and $\tau^*\tilde{\xi}_0$ is given by the following diagram:

$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{\widetilde{d\tau}} & (\theta+\pi,-\varphi,e_2\cdot p') \\ \tilde{\xi}_0\Big\downarrow & \swarrow{\scriptstyle\tau^*\tilde{\xi}_0} & \Big\downarrow\tilde{\xi}_0 \\ (\theta,\varphi,p) & \xrightarrow{d\tau} & (\theta+\pi,-\varphi,j_2p) \end{array}$$

or equivalently by $\widetilde{d\tau}\circ\gamma$ which can be obtained by choosing $-e_2$. Here we see that for the Pin^+-structure, since $e_2^2=1$, we get $\widetilde{d\tau}^2=1$, while for the Pin^--structure we get $\widetilde{d\tau}^2=-1$. Thus, only the Pin^+-structure is the pull-back of a Pin^+-structure of K^2. For $\tilde{\xi}_1$:

$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{\widetilde{d\tau}} & (\theta+\pi,-\varphi,\tilde{R}_{-\theta-\pi}e_2\tilde{R}_\theta p') \\ \tilde{\xi}_1\Big\downarrow & \swarrow{\scriptstyle\tau^*\tilde{\xi}_1} & \Big\downarrow\tilde{\xi}_1 \\ (\theta,\varphi,R_\theta p) & \xrightarrow{d\tau} & (\theta+\pi,-\varphi,j_2R_\theta p) \end{array}$$

and $\widetilde{d\tau}$ is well-defined since with the shift $\theta\to\theta+2\pi$ we get a minus sign in both lifts of the rotations. Then $\widetilde{d\tau}^2=\tilde{R}_{-(\theta+\pi)-\pi}e_2\tilde{R}_{\theta+\pi}\tilde{R}_{-\theta-\pi}e_2\tilde{R}_\theta=\tilde{R}_{-2\pi}e_2^2=-e_2^2$, thus we get opposite results with respect to $\tilde{\xi}_0$. For $\tilde{\xi}_2$:

$$
\begin{array}{ccc}
(\theta, \varphi, p') & \xrightarrow{\widetilde{d\tau}} & (\theta + \pi, -\varphi, \tilde{R}_\varphi e_2 \tilde{R}_\varphi p') \\
\Big\downarrow \tilde{\xi}_2 & \overset{\tau^*\tilde{\xi}_2}{\swarrow} & \Big\downarrow \tilde{\xi}_2 \\
(\theta, \varphi, R_\varphi p) & \xrightarrow{d\tau} & (\theta + \pi, -\varphi, j_2 R_\varphi p)
\end{array}
$$

and $\widetilde{d\tau}$ is well-defined since with the shift $\theta \to \theta + 2\pi$ we get a minus sign in both lifts of the rotations. Then $\widetilde{d\tau}^2 = \tilde{R}_{-\varphi} e_2 \tilde{R}_{-\varphi} \tilde{R}_\varphi e_2 \tilde{R}_\varphi = e_2^2$, thus we get the same results of $\tilde{\xi}_0$. It is clear that $\tilde{\xi}_3$ behaves as $\tilde{\xi}_1$.

3.2 – *Invariant structures on the sphere*

We think of the sphere S^2 as the Riemann sphere $\mathbb{CP}^1$, with two charts $U_0 = \mathbb{CP}^1 \setminus \{N\}$ and $U_1 = \mathbb{CP}^1 \setminus \{S\}$ and transition function $g_{01}(z) = -\frac{1}{z}$. The antipodal involution τ is specified each of the two charts [1] by $\tau(z) = -\frac{1}{\bar{z}}$. We compute its Jacobian to find the action $d\tau$ on the tangent bundle. In real coordinates:

$$\tau(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

so that the Jacobian becomes:

$$J\tau(x, y) = \frac{1}{x^2 + y^2} \begin{bmatrix} x^2 - y^2 & 2xy \\ 2xy & y^2 - x^2 \end{bmatrix}$$

which, on the equator $|z| = 1$ becomes the orthogonal matrix:

$$J\tau(\cos\theta, \sin\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \qquad (5)$$

We now consider the sphere as the union of the two halves glued on the equator, so that we restrict both the charts U_0 and U_1 to the disc $|z| \leq 1$, and we glue them via g_{01}. Now we consider the trivial spin structure for each of the two discs and we glue via a lift of dg_{01} on $|z| = 1$. On the equator of both charts the transformation (5) is a reflection with respect to the real line generated by $(\cos\theta, \sin\theta)$, i.e. by $(-\sin\theta, \cos\theta)^\perp$. Thus, if we consider the point $(\cos\theta, \sin\theta) \in \mathbb{C} \simeq U_0$, we get $\tau(\cos\theta, \sin\theta) = -(\cos\theta, \sin\theta)$ and $d\tau_{(\cos\theta, \sin\theta)}$ acts on the tangent bundle as a rotation of π along the equator composed with a reflection of the orthogonal

(1) Note that τ commutes with g_{01}, that's why the expression is the same in both charts.

direction. Hence its possible lifts to a $\mathrm{Pin}^{\pm}$-principal bundle are:

$$\widetilde{d\tau}(\theta, p) = (\pi + \theta, \pm(- \sin\theta e_1 + \cos\theta e_2) \cdot p) \ .$$

Then $\widetilde{d\tau}^2$ is given by $(- \sin(\theta+\pi)e_1 + \cos(\theta+\pi)e_2)(- \sin\theta e_1 + \cos\theta e_2) = (\sin\theta e_1 - \cos\theta e_2)(- \sin\theta e_1 + \cos\theta e_2) = -\sin^2\theta e_1^2 - \cos^2\theta e_2^2$. Thus we see that $\widetilde{d\tau}^2 = 1$ if and only if $e_1^2 = e_2^2 = -1$, namely if the structure is Pin^-: this shows that $\mathbb{RP}^2 \simeq S^2 / \tau$ has two pin^- structures, lifting to the one of the sphere, but no pin^+ structures (compare with [8]).

4. – Manifolds with boundary

We now want to give the analogous description in the case of unorientable manifolds *with boundary*. We start with a brief review of the case of orientable manifolds with boundary.

4.1 – *Orientable manifolds with boundary*

Let X be an orientable manifold of dimension $2n$ with boundary ∂X, and let us consider its double X^d obtained considering two disjoint copies of X and identifying the corresponding boundary points. We mark one of the two copies considering an embedding $i : X \to X^d$. In this way, an orientation of X^d induces an orientation of X and the opposite one on the other copy. We have a natural orientation-reversing involution τ identifying corresponding points of the two copies, which is *not* a double covering since the boundary points are fixed.

We consider on X couples (ξ, θ) where $\theta : \xi|_{\partial X} \to \xi|_{\partial X}$ is an automorphism. Then we can glue two copies of ξ to a structure $\tilde{\xi}$ on X^d, which we call $\xi \cup_\theta \xi$, such that $\tilde{\xi}|_X := \xi$ and $\tilde{\xi}|_{X^d \setminus \mathrm{Int}(X)} := \tau^*\xi$, gluing on ∂X via the isomorphism θ. For every connected component $Y \subset \partial X$, since $\theta|_Y$ lifts the identity of the tangent bundle of Y, it must be the identity or γ. There is an equivalence of categories:

$$\left\{ \begin{array}{c} (\xi, \theta) \ : \ \xi \ \mathrm{pin}^{\pm} \text{ structure on } X \\ \theta : \xi|_{\partial X} \xrightarrow{\simeq} \xi|_{\partial X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \tilde{\xi} \ : \ \tilde{\xi} \ \mathrm{pin}^{\pm} \text{ structure on } X^d \text{ s.t.} \\ \exists\, \widetilde{d\tau} : \tilde{\xi}|_X \xrightarrow{\simeq} \tau^*(\tilde{\xi}|_{X^d \setminus \mathrm{Int}(X)}) \end{array} \right\}. \tag{6}$$

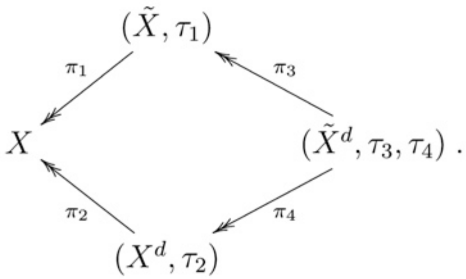
In fact, we have already shown how to pass from the l.h.s. to the r.h.s.; vice versa, if we have a $\mathrm{pin}^{\pm}$ structure $\tilde{\xi}$ on X^d, then $\tilde{\xi}$ is equivalent to $\tilde{\xi}|_X \cup_{\mathrm{id}} \tau^*(\tilde{\xi}|_{X^d \setminus \mathrm{Int}(X)})$. If there exists an isomorphism $\widetilde{d\tau} : \tilde{\xi}|_X \xrightarrow{\simeq} \tau^*(\tilde{\xi}|_{X^d \setminus \mathrm{Int}(X)})$, we restrict the latter to $\widetilde{d\tau}|_{\partial X} : \tilde{\xi}|_{\partial X} \xrightarrow{\simeq} \tilde{\xi}|_{\partial X}$ and if we apply $\widetilde{d\tau}^{-1}$ to the second component, we obtain $\tilde{\xi}|_X \cup_{\widetilde{d\tau}|_{\partial X}^{-1}} \tilde{\xi}|_X$ (we can freely choose $\widetilde{d\tau}$ or $\widetilde{d\tau} \circ \gamma$ since they differ by an overall sign, thus we can suppose $\widetilde{d\tau}^{-1} = \widetilde{d\tau}$). On sections, we just ask $s_x = \widetilde{d\tau}(s_{\tau(x)})$.

We briefly explain why to consider couples (ξ, θ) as above. With spinors, one usually considers on X one spin structure for the positive chirality, and one for the negative chirality, with the condition that they must glue at the boundary [2]. With pinors, which are extendable to the non-orientable case, there are no distinctions between chiralites: in particular, if we consider a pin structure on an orientable manifold and, fixing an orientation, we restrict it to a spin structure, we get the same spin structure for positive and negative spinors. That's why we fix only one pin structure ξ and an automorphism of it on the boundary.

4.2 – *Unorientable manifolds with boundary*

Let X be an unorientable manifold with boundary. Then we can consider the diagram:

(7)



We remark that we have immersions $X \subset X^d$ and $\tilde{X} \subset \tilde{X}^d$, while π_1 and π_4 are double coverings. In particular τ_2 and τ_3 have fixed points while τ_1 and τ_4 do not. By the definition of $\tilde{X}$ and X^d with the relevant involutions we easily get the following properties:

- $\pi_1 \circ \pi_3 = \pi_2 \circ \pi_4$;
- $\pi_4|_{\tilde{X}} = \pi_1$ and $\tau_4|_{\tilde{X}} = \tau_1$;
- $\tau_3 \circ \tau_4 = \tau_4 \circ \tau_3$.

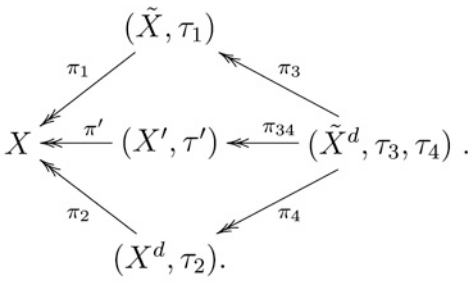
We want to show that there is a surjective map:

$$\left\{ \begin{array}{c} (\xi, \theta) : \xi \text{ pin}^{\pm} \text{ structure on } X \\ \theta : \xi|_{\partial X} \xrightarrow{\simeq} \xi|_{\partial X} \end{array} \right\} \to \left\{ \begin{array}{c} \xi' : \xi' \text{ pin}^{\pm} \text{ structure on } \tilde{X}^d \text{ s.t.} \\ \exists \widetilde{d\tau_3} : \xi'|_{\tilde{X}} \xrightarrow{\simeq} (\tau_3)_* (\xi'|_{\tilde{X}^d \setminus \mathrm{Int}(\tilde{X})}) \\ \exists \widetilde{d\tau_4} : \xi' \xrightarrow{\simeq} (\tau_4)_* \xi' \text{ with } \widetilde{d\tau_4}^2 = 1 \end{array} \right\} \tag{8}$$

which is not injective, but such that the counterimage of each element of the codomain contains 2 elements. As for the open oriented case, we must fix a couple (ξ, θ) with ξ a pin$^{\pm}$ structure on X and $\theta : \xi|_{\partial X} \to \xi|_{\partial X}$ an automorphism. To establish a correspondence with pin$^{\pm}$ structures on $\tilde{X}^d$, we can follow the upper or the lower paths of diagram (7). If we follow the lower path, we consider $\xi^{d(\theta)} := \xi \cup_\theta \xi$ on X^d, then we pull it back to $\pi_4^*(\xi^{d(\theta)})$. Otherwise, following the upper path, we first pull back ξ to $\pi_1^*\xi$ as in the closed case, so that $\xi|_{\partial X}$ pulls-back to $(\pi_1^*\xi)|_{\partial\tilde{X}}$ and the morphism θ pulls back to a morphism $\pi_1^*\theta : (\pi_1^*\xi)|_{\partial\tilde{X}} \to (\pi_1^*\xi)|_{\partial\tilde{X}}$. Then we double $\pi_1^*\xi$ on $\tilde{X}^d$ putting it on both copies of $\tilde{X}$ and using $\pi_1^*\theta$ as the isomorphism on $\partial\tilde{X}$, i.e. we consider $\pi_1^*\xi \cup_{\pi_1^*\theta} \pi_1^*\xi$, which we call $(\pi_1^*\xi)^{d(\pi_1^*\theta)}$. The two results are the same, in fact $(\pi_4^*(\xi^{d(\theta)}))|_{\tilde{X}} = (\pi_4|_{\tilde{X}})^*(\xi^{d(\theta)}|_X) = \pi_1^*(\xi) = (\pi_1^*\xi)^{d(\pi_1^*\theta)}|_{\tilde{X}}$, and the same for the other half of $\tilde{X}^d$ and for the isomorphism θ. Considering sections of the associated vector bundles of pinors, since under pull-back of pin$^\pm$ structures we pull-back also sections and under doubling we ask invariance of the sections, we obtain sections $s \in \Gamma(P_{\mathrm{Pin}^\pm,(\pi_1^*\xi)^{d(\pi_1^*\theta)}}(\tilde{X}^d) \times_\rho \mathbb{C}^{2^n})$, such that $s_x = s_{\tau_3(x)} = s_{\tau_4(x)} = s_{\tau_3\tau_4(x)}$. Here we do not have $\widetilde{d\tau_3}$ and $\widetilde{d\tau_4}$ since we are working with explicit pull-backs.

Vice versa, if we are given a pin$^\pm$ structure ξ' on $\tilde{X}^d$, such that there exists $\widetilde{d\tau_3} : \xi'|_{\tilde{X}} \simeq (\tau_3)_*(\xi'|_{\tilde{X}^d\setminus\mathrm{Int}(\tilde{X})})$ and $\widetilde{d\tau_4} : \xi' \xrightarrow{\simeq} (\tau_4)_*\xi'$ with $\widetilde{d\tau_4}^2 = 1$, then we can find a pin$^\pm$ structure on X such that $\xi' \simeq (\pi_1^*\xi)^{d(\theta)}$. We can find it using the two paths of the diagram. If we follow the upper path, we consider the couple $(\xi'|_{\tilde{X}}, \mathrm{id})$ where $\mathrm{id} : \xi'|_{\partial\tilde{X}} \to \xi'|_{\partial\tilde{X}}$ is the restriction of $\widetilde{d\tau_3}$. Then, since $\tau_4|_{\tilde{X}} = \tau_1$, if follows that $\xi'|_{\tilde{X}}$ is τ_1-invariant with $\widetilde{d\tau_1}^2 = 1$, thus we can consider $\xi = (\xi'|_{\tilde{X}}) \,/\, \widetilde{d\tau_1}$ as in the closed case. For sections on $\tilde{X}^d$, we must ask $s_x = \widetilde{d\tau_4}(s_{\tau_4(x)}) = \widetilde{d\tau_3}(s_{\tau_3(x)})$. If we follow the lower path of the diagram, we first quotient by $\widetilde{d\tau_4}$ and then we use the projection of $\widetilde{d\tau_3}$ to X^d by π_4.

We can also consider the orientation-*preserving* involution $\tau_{34} = \tau_3 \circ \tau_4$ on $\tilde{X}^d$. The space $X' = \tilde{X}^d \,/\, \tau_{34}$ is an oriented and closed manifold with an orientation-reversing involution τ' such that $X' \,/\, \tau' \simeq X$. In fact, τ_{34} has no fixed points: $\tau_{34}(x) = x$ is equivalent to $\tau_3(x) = \tau_4(x)$, but if $x \notin \partial\tilde{X} \subset \tilde{X}^d$, then τ_3 maps it to a point of the other copy of $\tilde{X}$, while τ_4 exchanges the sheets of the covering of the same copy of X; instead, if $x \in \partial\tilde{X} \subset \tilde{X}^d$, then $\tau_3(x) = x$ while τ_4 has no fixed points. Therefore $\tau_3(x) = \tau_4(x)$ is impossible. Hence $X' = \tilde{X}^d \,/\, \tau_{34}$ is a smooth closed orientable manifold double-covered by $\tilde{X}^d$. Then τ_3 and τ_4 projects at the quotient to the same involution τ'. We can thus complete the diagram:



The previous picture is analogous to considering a pin$^{\pm}$ structure ξ' on X' which is τ'-invariant with $\widetilde{d\tau'}^2 = 1$. Via π_{34} we pull-back it to a structure on $\tilde{X}^d$ satisfying the previous requirements.

4.3 – *Moebius strip*

We now study as an example pinors on the Moebius strip. In this case diagram (7) becomes (calling Cyl the finite cylinder or annulus and M^2 the Moebius strip):

$$\begin{array}{ccccc} & & (\mathrm{Cyl}, \tau_1) & & \\ & \swarrow^{\pi_1} & & \nwarrow^{\pi_3} & \\ M^2 & \xleftarrow{\pi'} & (T^2, \tau') & \xleftarrow{\pi_{34}} & (T^2, \tau_3, \tau_4)\ . \\ & \nwarrow_{\pi_2} & & \swarrow_{\pi_4} & \\ & & (K^2, \tau_2). & & \end{array}$$

with the involutions we now describe. We represent all the four surfaces involved as the square $[0, 2\pi] \times [0, 2\pi]$ with suitable identifications on the edges. In particular, for M^2 we identify $(0, y) \sim (2\pi, 2\pi - y)$, for Cyl $(0, y) \sim (2\pi, y)$, for T^2 $(0, y) \sim (2\pi, y)$ and $(x, 0) \sim (x, 2\pi)$, and for K^2 $(0, y) \sim (2\pi, y)$ and $(x, 0) \sim (2\pi - x, 2\pi)$. When two edges are identified with the same direction (*and only in this case*), we think of the orthogonal coordinate as a 2π-periodical coordinate $\mathbb{R} / 2\pi\mathbb{Z}$. With these conventions a possible choice of involutions is:

$$\begin{array}{ll} \tau_1(x, y) = (x + \pi, 2\pi - y) & \tau_2(x, y) = (y - x, y) \\ \tau_3(x, y) = (- x, y) & \tau_4(x, y) = (\pi - x, y + \pi)\ . \end{array}$$

We now analyze pin$^{\pm}$ structures on T^2. We can prove as before that they are all τ_4-invariant. In the (θ, φ)-coordinates τ_4 becomes $\tau_4(\theta, \varphi) = (- \theta + \pi, \varphi + \pi)$. Then:

$$\begin{array}{ccc} (\theta, \varphi, p') & \xrightarrow{\widetilde{d\tau_4}} & (-\theta + \pi, \varphi + \pi, e_1 \cdot p') \\ \downarrow \tilde{\xi}_0 & \swarrow_{(\tau_4)^* \tilde{\xi}_0} & \downarrow \tilde{\xi}_0 \\ (\theta, \varphi, p) & \xrightarrow{d\tau_4} & (-\theta + \pi, \varphi + \pi, j_1 p) \end{array}$$

$$\begin{array}{ccc} (\theta, \varphi, p') & \xrightarrow{\widetilde{d\tau_4}} & (-\theta + \pi, \varphi + \pi, \tilde{R}_{\theta - \pi} \cdot e_1 \cdot \tilde{R}_{\theta} \cdot p') \\ \downarrow \tilde{\xi}_1 & \swarrow_{(\tau_4)^* \tilde{\xi}_1} & \downarrow \tilde{\xi}_1 \\ (\theta, \varphi, R_{\theta} \cdot p) & \xrightarrow{d\tau_4} & (-\theta + \pi, \varphi + \pi, j_1 \cdot R_{\theta} \cdot p) \end{array}$$

$$\begin{array}{ccc} (\theta, \varphi, p') & \xrightarrow{\widetilde{d\tau_4}} & (-\theta + \pi, \varphi + \pi, \tilde{R}_{-\varphi - \pi} \cdot e_1 \cdot \tilde{R}_{\varphi} \cdot p') \\ \downarrow \tilde{\xi}_2 & \swarrow_{(\tau_4)^* \tilde{\xi}_2} & \downarrow \tilde{\xi}_2 \\ (\theta, \varphi, R_{\varphi} \cdot p) & \xrightarrow{d\tau_4} & (-\theta + \pi, \varphi + \pi, j_1 \cdot R_{\varphi} \cdot p) \end{array}$$

so that $\widetilde{d\tau_4}^2$ becomes respectively:

- e_1^2 for ξ_0;
- $\tilde{R}_{-\theta} \cdot e_1 \cdot \tilde{R}_{-\theta+\pi} \cdot \tilde{R}_{\theta-\pi} \cdot e_1 \cdot \tilde{R}_\theta = e_1^2$ for ξ_1;
- $\tilde{R}_{-\varphi-2\pi} \cdot e_1 \cdot \tilde{R}_{\varphi+\pi} \cdot \tilde{R}_{\theta-\pi} \cdot e_1 \cdot \tilde{R}_\theta = -e_1^2$ for ξ_2.

The structures ξ_1 and ξ_2 have opposite behavior with respect to the involution previously considered, since in this case the variable changing sign is x and not y.

We now analyze the situation for τ_3. First of all we can show that all the four structures satisfy $\xi_i|_{\mathrm{Cyl}} \simeq (\tau_3)_*(\xi_i|_{T^2\setminus\mathrm{Int(Cyl)}})$ exactly in the same way as for τ_4, and in this case we do not have to require that the isomorphism squares to 1. Actually, we will now prove that ξ_0 and ξ_1 (and similarly ξ_2 and ξ_3) restrict to equivalent structures on Cyl, but they differ by the isomorphism θ at the boundary. In fact, the equivalence:

(9)
$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{\quad\rho\quad} & (\theta,\varphi,\tilde{R}_{-\theta}p') \\ & {\scriptstyle\xi_0}\searrow \quad \swarrow{\scriptstyle\xi_1} & \\ & (\theta,\varphi,p) & \end{array}$$

is not well defined on T^2 since $\tilde{R}_\theta = -\tilde{R}_{\theta+2\pi}$, but if we restrict θ to the interval $[0,\pi]$, corresponding to the cylinder, there is no ambiguity left. This reasoning does not work between ξ_0 and ξ_2 since the interval of φ is not halved. For ξ_0 we have the diagram:

$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{(\widetilde{d\tau_3})_0} & (-\theta,\varphi,e_1\cdot p') \\ {\scriptstyle\tilde{\xi}_0}\downarrow & \swarrow{\scriptstyle(\tau_3)^*\tilde{\xi}_0} & \downarrow{\scriptstyle\tilde{\xi}_0} \\ (\theta,\varphi,p) & \xrightarrow{d\tau_3} & (-\theta,\varphi,j_1p) \end{array}$$

while for ξ_1:

$$\begin{array}{ccc} (\theta,\varphi,p') & \xrightarrow{(\widetilde{d\tau_3})_1} & (-\theta,\varphi,\tilde{R}_\theta e_1\tilde{R}_\theta p') \\ {\scriptstyle\tilde{\xi}_1}\downarrow & \swarrow{\scriptstyle(\tau_3)^*\tilde{\xi}_1} & \downarrow{\scriptstyle\tilde{\xi}_1} \\ (\theta,\varphi,R_\theta p) & \xrightarrow{d\tau_3} & (-\theta,\varphi,j_1R_\theta p) \end{array}$$

and we can show that the two couples $(\xi_0, (\widetilde{d\tau_3})_0|_{\partial X})$ and $(\xi_1, (\widetilde{d\tau_3})_1|_{\partial X})$ are not equivalent. In fact, they are equivalent to the triples $(\xi_0, (\tau_3)_*\xi_0, \mathrm{id})$ and $(\xi_1, (\tau_3)_*\xi_1, \mathrm{id})$ via the equivalences ρ of diagram (9) and $(\tau_3)_*\rho$ of the following diagram:

$$
\begin{array}{ccc}
(-\theta, \varphi, e_1 p') & \xrightarrow{(\tau_3)_* \rho} & (-\theta, \varphi, \tilde{R}_\theta e_1 p') \\
& {\scriptstyle (\tau_3)_* \xi_0} \searrow \quad \swarrow {\scriptstyle (\tau_3)_* \xi_1} & \\
& (\theta, \varphi, p). &
\end{array}
\tag{10}
$$

Comparing (9) and (10) we can see that the diagram:

$$
\begin{array}{ccc}
(\tau_3)_* \xi_0|_{\partial X} & \xrightarrow{(\tau_3)_* \rho|_{\partial X}} & (\tau_3)_* \xi_1|_{\partial X} \\
\uparrow {\scriptstyle \mathrm{id}} & & \uparrow {\scriptstyle \mathrm{id}} \\
\xi_0|_{\partial X} & \xrightarrow{\rho|_{\partial X}} & \xi_1|_{\partial X}
\end{array}
$$

does *not* commute or anti-commute. In fact, for $\theta = 0$ we get $\rho(0, \varphi, q') = (\tau_3)_* \rho(0, \varphi, q')$ while for $\theta = \pi$ we get $\rho(\pi, \varphi, q') = -(\tau_3)_* \rho(\pi, \varphi, q')$ since $\tilde{R}_{-\pi} = -\tilde{R}_\pi$. The diagram would not commute either by choosing $\rho \circ \gamma$ or $(\tau_3)_* \rho \circ \gamma$ or both.

Some comments about the behavior of $\xi_0, \xi_1, (\tau_3)_* \xi_0, (\tau_3)_* \xi_0$ at the boundary are needed in order to avoid possible confusion. We deal with the spin structures for simplicity, then the reader can see what happens for pin structures considering basis with the opposite orientation. If we embed $P_{\mathrm{SO}}(\partial \mathrm{Cyl}) \subset P_{\mathrm{SO}}(T^2)$ via the outward orthogonal normal unit vector, it follows that $\{e_1, e_2\}$ is the only orthonormal oriented basis [2] at a boundary point $(0, \varphi)$ with e_1 outward, while for (π, φ) the only embedded basis is $\{-e_1, -e_2\}$. Thus, since the principal bundle $P_{\mathrm{SO}}(T^2)$ is the bundle of isomorphisms from the trivial bundle $T^2 \times \mathbb{R}^2$ to the tangent bundle $T(T^2)$, which is also trivial, it follows that the embedded basis for $\theta = 0$ corresponds to $(0, \varphi, \mathrm{id}) \in S^1 \times S^1 \times \mathrm{SO}(2)$, while the embedded basis for $\theta = \pi$ corresponds to $(\pi, \varphi, -\mathrm{id}) \in S^1 \times S^1 \times \mathrm{SO}(2)$. Thus, is we consider the Spin-bundles $P_{\mathrm{Spin}}(\partial \mathrm{Cyl}) \subset P_{\mathrm{Spin}}(T^2)$ we have that the lifts of the embedded basis are:

	$\theta = 0$	$\theta = \pi$
$\tilde{\xi}_0$-lift:	$(0, \varphi, \pm 1)$	$(\pi, \varphi, \pm e_1 e_2)$
$\tilde{\xi}_1$-lift:	$(0, \varphi, \pm 1)$	$(\pi, \varphi, \pm 1)$
$(\tau_3)^* \tilde{\xi}_0$-lift:	$(0, \varphi, \pm e_1)$	$(\pi, \varphi, \pm (e_1)^2 e_2)$
$(\tau_3)^* \tilde{\xi}_1$-lift:	$(0, \varphi, \pm e_1)$	$(\pi, \varphi, \pm e_1)$

([2]) Since the boundary has dimension 1 there is only one oriented orthonormal basis.

It may seem strange that at the boundary, whose tangent space is generated only by e_2, also the outward vector e_1 is involved, but that's due to the fact that on π there is a -1 to lift due to the orientation and for all the structures different from $\tilde{\tilde{\xi}}_0$ there is a twist in the projection of the third factor $\mathrm{Spin}(2) \to \mathrm{SO}(2)$ which makes e_1 enter in the lifting. The isomorphisms of spin structures we dealt with until now are then at the boundary:

$$\begin{aligned} &\rho(0,\varphi,\pm 1) = (0,\varphi,\pm 1) && \rho(\pi,\varphi,\pm e_1 e_2) = (\pi,\varphi,\mp 1) \\ &(\widetilde{d\tau_3})_0(0,\varphi,\pm 1) = (0,\varphi,\pm e_1) && (\widetilde{d\tau_3})_0(\pi,\varphi,\pm e_1 e_2) = (\pi,\varphi,\pm (e_1)^2 e_2) \\ &(\widetilde{d\tau_3})_1(0,\varphi,\pm 1) = (0,\varphi,\pm e_1) && (\widetilde{d\tau_3})_1(\pi,\varphi,\pm 1) = (\pi,\varphi,\pm e_1)\,. \end{aligned}$$

Summarizing, there are two pin$^{\pm}$-structures on the torus which lift a pin$^{\pm}$-structure (ξ,θ) on a Moebius strip. We consider the pin^{+} ones, i.e. $\tilde{\tilde{\xi}}_0$ and $\tilde{\tilde{\xi}}_1$. We know that on the torus $\tilde{\tilde{\xi}}_0 \cup_{\mathrm{id}} (\tau_3)^* \tilde{\tilde{\xi}}_0 \simeq \tilde{\tilde{\xi}}_0 \cup_{(\widetilde{d\tau_3})_0} \tilde{\tilde{\xi}}_0$ and the same for $\tilde{\tilde{\xi}}_1$. Some representatives of the two equivalence classes are then:

$$\begin{aligned} &\text{Class of } \tilde{\tilde{\xi}}_0 \text{ on } T^2: \quad \tilde{\tilde{\xi}}_0 \cup_{(\widetilde{d\tau_3})_0} \tilde{\tilde{\xi}}_0 \;\simeq\; \tilde{\tilde{\xi}}_0 \cup_{\mathrm{id}} (\tau_3)^* \tilde{\tilde{\xi}}_0 \;\simeq\; \tilde{\tilde{\xi}}_1 \cup_{\mathrm{id}} \tilde{\tilde{\xi}}_1 \\ &\text{Class of } \tilde{\tilde{\xi}}_1 \text{ on } T^2: \quad \tilde{\tilde{\xi}}_1 \cup_{(\widetilde{d\tau_3})_1} \tilde{\tilde{\xi}}_1 \;\simeq\; \tilde{\tilde{\xi}}_1 \cup_{\mathrm{id}} (\tau_3)^* \tilde{\tilde{\xi}}_1 \;\simeq\; \tilde{\tilde{\xi}}_0 \cup_{\mathrm{id}} \tilde{\tilde{\xi}}_0\,. \end{aligned}$$

The situation is analogous for the pin^{-} case. In particular, there are 4 inequivalent pin^{+} structures and 4 inequivalent pin^{-} structures on the Moebius strip. It is easy to find the invariance conditions for pinors as sections of the associated vector bundles. Moreover, all this picture is equivalent to considering (T^2,τ'); we leave the details to the reader.

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