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# Three-Dimensional Paracontact Walker Structures

G. CALVARUSO (*)

**Abstract.** – *We investigate paracontact metric three-manifolds equipped with an associated Walker metric. Some interesting paracontact metric properties are studied for the paracontact Walker structures introduced in* [10], *also clarifying their relationships with some curvature properties. Moreover, improving the result on* [4] *on locally symmetric Walker three-manifolds, we show that homogeneity conditions give some obstructions to the existence of compatible paracontact structures on a Walker three-manifold.*

## 1. – Introduction

In perfect correspondence with almost contact and complex structures, almost paracontact structures were introduced in [12], as a natural odd-dimensional analogue of almost paracomplex structures ([13], [11]). Since then, paracontact and almost paracontact metric manifolds have been investigated by several authors, although most of the results focused on the very special case of paraSasakian manifolds. Recently, the remarkable paper [17] started a systematic study of paracontact metric manifolds, describing the Levi-Civita connection, the curvature and a canonical connection (analogue to the Tanaka-Webster connection of the contact metric case) of a paracontact metric manifold. The technical apparatus introduced in [17] is essential for further investigations of paracontact metric geometry.

In dimension three, a metric $g$ compatible with a paracontact structure $(\varphi, \xi, \eta)$ has signature $(2, 1)$, that is, $g$ is Lorentzian. In [4], the present author classified homogeneous paracontact metric three-manifolds, that is, three-dimensional Lorentzian manifolds $(M, g)$, admitting a transitive group of isometries which leaves invariant the paracontact form $\eta$. Three of the four canonical forms of unimodular Lorentzian Lie groups [15] and two of the three canonical forms of the non-unimodular ones [9], do admit a left-invariant paracontact metric structure [4, Theorem 1.1]. On the other hand, a (locally) symmetric paracontact metric three-space is either flat or of con-

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stant sectional curvature $-1$. In particular, unless it is flat, a locally symmetric Walker metric can not be associated to any paracontact structure [4, Theorem 3.4].

*Walker metrics* are a peculiar class of pseudo-Riemannian metrics, which is responsible for many of most relevant differences between Riemannian and pseudo-Riemannian geometries. A Riemannian manifold $(M, g)$ admitting a parallel line field is locally reducible. This property remains true for a pseudo-Riemannian manifold admitting a parallel non degenerate line field, that is, generated by an either spacelike or timelike vector field. However, in the pseudo-Riemannian framework, a peculiar phenomenon arises: it can exist a parallel degenerate line field.

Walker three-manifolds are three-dimensional Lorentzian manifolds admitting a parallel degenerate line field. The systematic study of Walker three-manifolds has been undertaken in the basic paper [8], describing their Levi-Civita connection and curvature and deducing several geometric consequences. These manifolds are described in terms of a suitable system of local coordinates $(t, x, y)$ and form a large class, depending on an arbitrary three-variables function $f(t, x, y)$.

The purpose of this paper is to investigate three-dimensional *paracontact Walker structures*, that is, paracontact structures admitting a Walker metric as an associated metric. As proved in [10], under the assumption $(f_{tx}/f_{tt})_t \neq 0$, a Walker three-manifold $(M, g_f)$ admits a suitable contact 1-form $\eta$, which determines a paracontact metric structure $(\varphi, \xi, \eta, g_f)$, having the Walker metric $g_f$ as an associated metric. It was also shown that, considered the product manifold $M \times \mathbb{R}$ equipped with the metric $g_0 = e^{2t} g'$ conformal to the product metric $g'$, the almost paracomplex structure

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi X + a\xi, \eta(X)\frac{d}{dt}\right)$$

becomes almost para-Kähler.

In this paper, after reporting in Section 2 some basic information about paracontact metric manifolds, in Section 3 we will describe in detail the paracontact metric structure introduced in [10] and characterize some interesting paracontact metric properties of such structure, as being paraSasakian or $\eta$-Einstein. In particular, we shall prove that this structure is $\eta$-Einstein if and only if $(M, g_f)$ is *semi-symmetric* [5]. This result shows that the non-existence of paracontact locally symmetric Walker three-manifolds (except for the trivial flat case) does not extend to semi-symmetric spaces. As symmetric spaces are homogeneous, it is also a natural question to ask whether there exist paracontact (locally) homogeneous Walker three-manifolds. In Section 4, we shall provide a negative answer to the question above.

## 2. – Preliminaries

We briefly report the definition and basic properties of almost paracontact metric structures, referring to [12] and [17] for more details and several interesting examples. An *almost paracontact structure* on a $(2n+1)$-dimensional smooth manifold $M$ is a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1,1)$-tensor, $\xi$ a global vector field and $\eta$ a 1-form, such that

$$(2.1)\qquad \begin{array}{lll} (i) & \varphi(\xi) = 0, & \eta \circ \varphi = 0, \\ (ii) & \eta(\xi) = 1, & \varphi^2 = Id - \eta \otimes \xi \end{array}$$

and the restriction $J$ of $\varphi$ on the horizontal distribution Ker$\eta$ is an almost paracomplex structure ([13], [11]), that is, $J^2 = Id$ and the $\pm 1$-eigenspaces of $J$ have the same dimension $n$. A pseudo-Riemannian metric $g$ on $M$ is said to be *compatible* with the almost paracontact structure $(\varphi, \xi, \eta)$ if and only if

$$(2.2)\qquad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

Remark that, by (2.1) and (2.2), $\eta(X) = g(\xi, X)$ for any compatible metric. Any almost paracontact structure admits a compatible metric. Moreover, compatible metrics necessarily have signature $(n+1, n)$ [17].

An almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ has a structure group $\mathbb{U}(n, \mathbb{R}) \times Id$, where $\mathbb{U}(n, \mathbb{R})$ is the para-unitary group isomorvarphic to $\mathbb{GL}(n, \mathbb{R})$. Similarly to the almost contact metric case, each almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ admits a local orthonormal basis adapted to its almost paracontact metric structure, that is, of the form $\{\xi, e_i, \varphi e_i\}$, called a *$\varphi$-basis* [17].

Given an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$, consider the product manifold $M \times \mathbb{R}$. A vector field on $M \times \mathbb{R}$ is denoted by $\left(X, f\frac{d}{dt}\right)$, where $X$ is tangent to $M$, $t$ is the coordinate on $\mathbb{R}$ and $f$ is a $C^\infty$ function. Then,

$$\tilde{J} = \left(X, f\frac{d}{dt}\right) = \left(\varphi X + f\xi, \eta(X)\frac{d}{dt}\right)$$

defines an almost paracomplex structure on $M \times \mathbb{R}$. The almost paracontact structure $(\varphi, \xi, \eta)$ is said to be *normal* if and only if $\tilde{J}$ is integrable.

If a compatible metric $g$ on an almost paracontact manifold $(M, \varphi, \xi, \eta)$ satisfies

$$(2.3)\qquad g(X, \varphi Y) = (d\eta)(X, Y),$$

then the manifold $(M, \eta, g)$ (or $(M, \varphi, \xi, \eta, g)$) is called a *paracontact metric manifold* and $g$ the *associated metric*.

Let now $(M, \eta, g)$ be a paracontact metric manifold. By $\nabla$ and $R$ we shall denote the Levi-Civita connection and the curvature tensor of $M$, respectively, the latter taken with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ (note that this convention is opposite to the one used in [17]). Taking into account (2.1) and (2.3), tensors

$$h = \frac{1}{2}\mathcal{L}_\xi\varphi, \qquad \ell = R(\xi,\cdot)\xi, \tag{2.4}$$

$\mathcal{L}$ being the Lie derivative, are defined on $(M,\eta,g)$ and play an important role in describing its geometry. In particular, as proved in [17], $h$ is self-adjoint, $h\varphi = -\varphi h$ and $h\xi = \operatorname{tr} h = 0$. Moreover, the covariant derivative and the curvature satisfy the following properties:

$$\nabla_\xi\xi = 0, \qquad \nabla_\xi\varphi = 0, \tag{2.5}$$

$$\nabla_\xi X = -\varphi X + \varphi h X, \tag{2.6}$$

$$(\nabla_\xi h)X = -\varphi X + h^2\varphi X - \varphi\ell X, \tag{2.7}$$

$$\ell X + \varphi\ell\varphi X = 2h^2 X - 2\varphi^2 X. \tag{2.8}$$

We recall the following.

DEFINITION 2.1. – *A paracontact metric manifold* $(M,\eta,g)$ *is said to be*

(i) paraSasakian *if it is normal, that is, equivalently,*

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.9}$$

(ii) $K$-paracontact *if* $h = 0$*, that is, equivalently,* $\xi$ *is a Killing vector field.*

We explicitly remark that, contrarily to the contact Riemannian case, $|h|^2 = 0$ is a necessary *but not sufficient* condition in order to have a $K$-paracontact manifold, since in pseudo-Riemannian settings it holds whenever $hX$ is light-like for any tangent vector $X$. Moreover, paraSasakian manifolds are $K$-paracontact [17, Theorem 2.8]. Although the converse does not hold in general, a three-dimensional $K$-paracontact metric manifold $(M,\eta,g)$ is paraSasakian [4, Theorem 2.2].

## 3. – Paracontact Walker three-manifolds

In dimension three, a Walker manifold is a Lorentzian manifold $M$ admitting a parallel degenerate line field. It admits local coordinates $(t,x,y)$, such that with respect to the local frame field $\{\partial_t, \partial_x, \partial_y\}$ the Lorentzian metric tensor is expressed by

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix}, \tag{3.1}$$

for some function $f(t,x,y)$, where $\varepsilon = \pm 1$. The parallel degenerate line field is spanned by $\partial_t$. The special case when $U = \partial_t$ is a parallel null vector (*strictly Walker manifold*) is characterized by the fact that $f$ does not depend on the variable $t$, that is, $f_t = 0$ [16].

As shown in [8], with respect to the coordinate basis $\{\partial_t, \partial_x, \partial_y\}$, the Levi-Civita connection and curvature of $(M, g_f)$ are completely determined by the following possibly non-vanishing components:

$$\nabla_{\partial_t}\partial_y = \frac{1}{2} f_t\partial_t, \quad \nabla_{\partial_x}\partial_y = \frac{1}{2} f_x\partial_t, \quad \nabla_{\partial_y}\partial_y = \frac{1}{2}(ff_t + f_y)\partial_t - \frac{1}{2\varepsilon} f_x\partial_x - \frac{1}{2} f_t\partial_y, \tag{3.2}$$

and

$$\begin{aligned}
R(\partial_t, \partial_y)\partial_t &= -\frac{1}{2} f_{tt}\partial_t,\\
R(\partial_t, \partial_y)\partial_x &= -\frac{1}{2} f_{tx}\partial_t,\\
R(\partial_t, \partial_y)\partial_y &= -\frac{1}{2} ff_{tt}\partial_t + \frac{1}{2\varepsilon} f_{tx}\partial_x + \frac{1}{2} f_{tt}\partial_y,\\
R(\partial_x, \partial_y)\partial_t &= -\frac{1}{2} f_{tx}\partial_t,\\
R(\partial_x, \partial_y)\partial_x &= -\frac{1}{2} f_{xx}\partial_t,\\
R(\partial_x, \partial_y)\partial_y &= -\frac{1}{2} ff_{tx}\partial_t + \frac{1}{2\varepsilon} f_{xx}\partial_x + \frac{1}{2} f_{tx}\partial_y.
\end{aligned} \tag{3.3}$$

It is well known that the curvature of a three-dimensional Lorentzian manifold is completely determined by its Ricci tensor $\varrho$. Following [8], we have that, with respect to the coordinate basis $\{\partial_t, \partial_x, \partial_y\}$, the Ricci tensor $\varrho$ and the Ricci operator $Q$ of any metric (3.1) is determined by

$$\varrho = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{tt} \\ 0 & 0 & \frac{1}{2} f_{tx} \\ \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & \frac{1}{2}(\varepsilon ff_{tt} - f_{xx}) \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & -\frac{\varepsilon}{2} f_{xx} \\ 0 & 0 & \frac{\varepsilon}{2} f_{tx} \\ 0 & 0 & \frac{1}{2} f_{tt} \end{pmatrix} \tag{3.4}$$

and the Ricci eigenvalues are given by

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = \frac{1}{2} f_{tt}. \tag{3.5}$$

Several curvature properties of a Walker three-manifold $(M, g_f)$ were already investigated in [8].

Let now $(M, g_f)$ be an arbitrary Walker three-manifold. In what follows, we shall assume that the scalar curvature $Sc = f_{tt} \neq 0$ at any point of $(M, g_f)$. We explicitly remark that this assumption excludes the case of a strictly Walker three-manifold.

Following [10], we have that, whenever $f_{tt} \neq 0$, vector field

$$N := -\frac{f_{tx}}{f_{tt}} \partial_t + \partial_x$$

is an eigenvector for the distinguished Ricci eigenvalue $\lambda_1 = 0$. The one-form dual to $N$ is then given by

$$\tilde{\eta} := \varepsilon dx - \frac{f_{tx}}{f_{tt}} d_y \tag{3.6}$$

and we easily find

$$d\tilde{\eta} = -\left(\frac{f_{tx}}{f_{tt}}\right)_t dt \wedge d_y - \left(\frac{f_{tx}}{f_{tt}}\right)_x dx \wedge d_y, \qquad \tilde{\eta} \wedge d\tilde{\eta} = \varepsilon \left(\frac{f_{tx}}{f_{tt}}\right)_t dt \wedge dx \wedge d_y.$$

Therefore, we have that

$$\left(\frac{f_{tx}}{f_{tt}}\right)_t \neq 0 \tag{3.7}$$

is a necessary and sufficient condition for $\tilde{\eta}$ to be a contact form, defined in a natural way from the Ricci operator of $(M, g_f)$.

Whenever (3.7) holds, the contact one-form $\tilde{\eta}$ uniquely determines the corresponding characteristic vector field $\tilde{\xi}$ by conditions $\tilde{\eta}(\tilde{\xi}) = 1$ and $i_{\tilde{\xi}} d\tilde{\eta} = 0$. Writing $\tilde{\xi} = \bar{a}\partial_t + \bar{b}\partial_x + \bar{c}\partial_y$, for three smooth functions $\bar{a}, \bar{b}, \bar{c}$, from $\tilde{\eta}(\tilde{\xi}) = 1$ we get

$$\varepsilon \bar{b} - F\bar{c} = 1, \tag{3.8}$$

where we put $F := (f_{tx}/f_{tt})$. Next, as $i_{\tilde{\xi}} d\tilde{\eta} = 0$, we have

$$0 = d\tilde{\eta}(\xi, \partial_t) = -\bar{c}F_t.$$

By (3.7), $F_t \neq 0$ and so, $\bar{c} = 0$. Hence, (3.8) yields $\bar{b} = \varepsilon$ and so, $\tilde{\xi} = \bar{a}\partial_t + \varepsilon\partial_x$.

From $i_{\tilde{\xi}} d\tilde{\eta} = 0$ we now find that $d\tilde{\eta}(\tilde{\xi}, \partial_t) = d\tilde{\eta}(\tilde{\xi}, \partial_x) = 0$ hold identically, while

$$0 = d\tilde{\eta}(\tilde{\xi}, \partial_y) = -F_t\,\bar{a} - \varepsilon F_x,$$

which, taking into account (3.7), yields $\bar{a} = -\varepsilon(F_x/F_t)$. Therefore,

$$\tilde{\xi} = \varepsilon\left(-\frac{F_x}{F_t}\partial_t + \partial_x\right). \tag{3.9}$$

Clearly, in general $\tilde{\xi} \neq N$. However, $\tilde{\xi} = N$ holds if and only if $\varepsilon = 1$ and

$$F_x = FF_t. \tag{3.10}$$

Next, we focus on paracontact structures detemined by the one-form $\tilde{\eta}$ and having the Walker metric $g_f$ described in (3.1) as an associated metric.

We first remark that if $\tilde{\eta}$ and $g_f$ are respectively the contact form and the metric of a paracontact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$, then from $1 = \tilde{\eta}(\tilde{\xi}) = g_f(\tilde{\xi}, \tilde{\xi})$, we have at once $\varepsilon = 1$. Thus, the assumption $\varepsilon = 1$ made in [10] is indeed the only possible case, and from now on we shall take $\varepsilon = 1$ in equation (3.1) and derived formulae.

The tensor $\tilde{\varphi}$ is now completely determined by equations (2.1)-(2.3). In order to describe $\tilde{\varphi}$, we first remark that, with respect to the coordinate basis $\{\partial_t, \partial_x, \partial_y\}$, the 2-form $d\tilde{\eta}$ is given by

$$d\tilde{\eta} = \begin{pmatrix} 0 & 0 & -F_t \\ 0 & 0 & -F_x \\ F_t & F_x & 0 \end{pmatrix}.$$

It is easily seen that, because of (2.3), $\tilde{\varphi}$ with respect to $\{\partial_t, \partial_x, \partial_y\}$ must be written as

$$\tilde{\varphi} = \begin{pmatrix} a & c & fa \\ b & 0 & fb - c \\ 0 & -b & -a \end{pmatrix},$$

for three smooth functions $a, b, c$. By (2.1), $\tilde{\varphi}(\tilde{\xi}) = 0$, from which we get $b = 0$ and $c = -(F_x/F_t)a$. Finally, applying $\tilde{\varphi}^2 = Id - \tilde{\eta} \otimes \tilde{\xi}$ to $\partial_t$, $\partial_x$ and $\partial_y$, we find

$$a^2 = 1, \qquad -F\frac{F_x}{F_t} = -\left(\frac{F_x}{F_t}\right)^2, \qquad F = a^2\frac{F_x}{F_t},$$

which is equivalent to $a^2 = 1$ and $F = (F_x/F_t)$. Note that this last condition is exactly (3.10). Without loss of generality, we assume $a = 1$. So, with respect to $\{\partial_t, \partial_x, \partial_y\}$, tensor $\tilde{\varphi}$ is given by

$$\tilde{\varphi} = \begin{pmatrix} 1 & F & f \\ 0 & 0 & -F \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.11}$$

Thus, we obtained the following (see also [10]).

THEOREM 3.1. – *Let $g_f$ denote an arbitrary three-dimensional Walker metric, as described in* (3.1).

(a) *The* 1*-form* $\tilde{\eta}$*, naturally defined starting from the Ricci operator of* $g_f$*, exists if and only if* $F_t \neq 0$*, where we put* $F = f_{tx}/f_{tt}$*. In this case, the corresponding characteristic vector field is given by*

$$\tilde{\xi} = \varepsilon\left(-\frac{F_x}{F_t}\partial_t + \partial_x\right).$$

(b) $\tilde{\eta}$ *and* $g_f$ *determine a paracontact metric structure if and only if* $\varepsilon = 1$ *and* $FF_t = F_x$*. In this case,* $\tilde{\xi} = N$ *is an eigenvector for the distinguished Ricci eigenvalue* $\lambda_1 = 0$ *and* $\tilde{\varphi}$ *is given by* (3.11).

In the remaining part of this paper, by $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ we shall denote any paracontact Walker structure, described by equations (3.1), (3.6), $\tilde{\xi} = N$ and (3.11). Such a paracontact metric structure exists whenever $\varepsilon = 1$ and equations (3.7), (3.10) hold. We shall now consider some curvature properties for the paracontact Walker structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$.

We first determine tensor $h = (1/2)\mathcal{L}_{\tilde{\xi}}\tilde{\varphi}$. From equations (3.2), we deduce $[\tilde{\xi}, \partial_t]$, $[\tilde{\xi}, \partial_x]$ and $[\tilde{\xi}, \partial_y]$ and then calculate $h$ with respect to the coordinate basis $\{\partial_t, \partial_x, \partial_y\}$. We find

$$h(\partial_t) = 0, \qquad h(\partial_x) = 0, \qquad h(\partial_y) = \frac{1}{2}\{fF_t - f_tF + f_x - 2F_y - FF_x\}\partial_t. \tag{3.12}$$

We recall that given a paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, the Ricci curvature in the direction of $\xi$ is given by $\varrho(\xi, \xi) = -2n + |h|^2$. In particular, on a $K$-paracontact manifold we necessarily have $\varrho(\xi, \xi) = 2n$. Now, for any paracontact Walker structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$, the characteristic vector field $\tilde{\xi}$ is a Ricci eigenvector associated to the distinguished Ricci eigenvalue $\lambda_1 = 0$. Therefore, $\varrho(\tilde{\xi}, \tilde{\xi}) = 0$ and so, we have at once the following.

PROPOSITION 3.2. – *Paracontact Walker structures* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ *are never* $K$*-paracontact (equivalently, paraSasakian).*

Note that, by (3.12), $h^2 = 0$ (that is, $h$ is two-step nilpotent), although $h \neq 0$ by Proposition 3.2. It is well known that a contact metric manifold is $K$-contact if and only if $h^2 = 0$. Recently, this result has been extended to pseudo-Riemannian settings [7]. Henceforth, paracontact Walker structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ show that this contact metric and pseudo-metric results do not extend to paracontact metric structures.

As already remarked in [8], a three-dimensional Walker metric $g_f$ is Einstein if and only if it is flat. This rigidity result makes interesting to investigate some generalizations of the Einstein condition for paracontact Walker structures $(\varphi, \xi, \eta, g_f)$.

Following [17], a paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be *$\eta$-Einstein* is there exist two smooth functions $a, b : M \to R$, such that

$$\varrho = ag + b\eta \otimes \eta. \tag{3.13}$$

Condition (3.13) is tensorial and coincides with the analogue condition defining $\eta$-Einstein contact metric manifolds. As proved in [1], a three-dimensional contact metric manifold is $\eta$-Einstein if and only if $\varphi$ and the Ricci operator $Q$ commute. Moreover, both conditions are equivalent to the fact that $\xi$ belongs to the $\kappa$-nullity distribution. We recall that, more in general, $\xi$ belongs to the *$(\kappa, \mu)$-nullity distribution* if

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{3.14}$$

for all tangent vector fields $X, Y$, and $\xi$ is said to belong to the $\kappa$-nullity distribution when (3.14) holds with $\mu = 0$. A (para)contact metric manifold is said to be a $(\kappa, \mu)$-space if its characteristic vector field belongs to the $(\kappa, \mu)$-nullity distribution.

We can now calculate conditions above for paracontact Walker structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$. We obtain the following.

THEOREM 3.3. – *For an arbitrary paracontact Walker structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$, the following properties are equivalent:*

(i) *$(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ is $\eta$-Einstein;*
(ii) *$Q\varphi = \varphi Q$;*
(iii) *$\tilde{\xi}$ belongs to the $\kappa$-nullity distribution. In this case, $\kappa = 0$;*
(iv) *the defining function $f$ of the Walker metric $g_f$ satisfies $f_{tt} f_{xx} = f_{tx}^2$;*
(v) *$g_f$ is a semi-symmetric Walker metric.*

PROOF. – The equivalence of properties (iv) and (v) has been proved in [5].

(i) $\Leftrightarrow$ (iv): using equations (3.1) and (3.4), a straightfoward calculation gives that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ is $\eta$-Einstein if and only if $f_{tt} f_{xx} = f_{tx}^2$. Notice that in this case, $a = \frac{1}{2} f_{tt}$ and $b = -\frac{1}{2} f_{tt}$, which are not constant because of (3.7).

(ii) $\Leftrightarrow$ (iv): it easily follows from (3.4) and (3.11).

(iii) $\Leftrightarrow$ (iv): using (3.3) and the definition of $\tilde{\eta}$, we can apply condition (3.14) with $\mu = 0$ to the pairs $(\partial_t, \partial_x)$, $(\partial_t, \partial_y)$ and $(\partial_x, \partial_y)$, and we find that $\tilde{\xi}$ belongs to the $k$-nullity distribution if and only if $k = 0$ and $f_{tt} f_{xx} = f_{tx}^2$ □

As proved in [4], three-dimensional locally symmetric Walker metrics cannot be associated to any paracontact structure. Theorem 3.3 shows that this rigidity result does not extend to semi-symmetric Walker metrics, which can be associated to paracontact structures and also play a special role in terms of paracontact geometry.

A three-dimensional *contact* metric manifold satisfying $Q\varphi = \varphi Q$ is either Sasakian, flat or of constant $\xi$-sectional curvature $\kappa < 1$ and constant $\varphi$-sectional curvature $-\kappa$ [1, Theorem 3.3]. Here, $\kappa$ is the real constant such that $\xi$ belongs to the $\kappa$-nullity distribution. By Theorem 3.3 above, such a classification result cannot be adapted to paracontact settings, because paracontact Walker structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ do not satisfy any of these conditions.

With regard to condition (3.14), we get the following.

THEOREM 3.4. – *Any paracontact Walker manifold $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ is a $(\kappa, \mu)$-space, where*

$$\kappa = 0, \qquad \mu = \frac{f_{tt} f_{xx} - f_{tx}^2}{f_{tt}\left(fF_t - f_t F + f_x - 2F_y - FF_x\right)}.$$

PROOF. – Because of its tensorial character, condition (3.14) is fulfilled if and only if it holds when $(X, Y) = (\partial_t, \partial_x)$, $(\partial_t, \partial_y)$ or $(\partial_x, \partial_y)$. From (3.3) and the definiton of $\tilde{\xi}$, we have $R(\partial_t, \partial_x)\tilde{\xi} = 0$. Hence, taking into account (3.12) and the definition of $\tilde{\eta}$, (3.14) yields

$$\kappa\, \partial_t = \kappa(\eta(\partial_x)\partial_t - \eta(\partial_t)\partial_x) + \nu(\eta(\partial_x)h\partial_t - \eta(\partial_t)h\partial_x) = 0,$$

that is, $\kappa = 0$. Taking into account $\kappa = 0$, condition (3.14) gives an identity when $(X, Y) = (\partial_t, \partial_y)$, while for $(X, Y) = (\partial_x, \partial_y)$ it gives

$$-\frac{f_{tx}^2}{f_{tt}} + f_{xx} = \mu\left(fF_t - f_t F + f_x - 2F_y - FF_x\right),$$

which ends the proof, since (3.12) and Proposition 3.2 imply that $fF_t - f_t F + f_x - 2F_y - FF_x \neq 0$ □

REMARK 3.5 (On contact Walker structures). – Contact pseudo-metric structures $(\eta, g)$, where $\eta$ is a contact 1-form and the associated metric $g$ is allowed to be pseudo-Riemannian, are a natural generalization of contact metric structures [7]. A pseudo-Riemannian metric $g$ is said to be *compatible* with an almost contact structure $(\varphi, \xi, \eta)$ if

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{3.15}$$

where $\varepsilon = \pm 1$. Remark that $\eta(X) = \varepsilon g(\xi, X)$ for any compatible metric. In particular, $g(\xi, \xi) = \varepsilon$ and so, the characteristic vector field $\xi$ is either space-like or time-like, but cannot be light-like. If the compatible pseudo-Riemannian metric $g$ satisfies (2.3), then $(\varphi, \xi, \eta, g)$ is called a *contact pseudo-metric structure*.

As a three-dimensional Walker metric is Lorentzian, it could be the associated metric of contact pseudo-metric structure. However, we have the following.

PROPOSITION 3.6. – *For a three-dimensional Walker metric $g_f$, the contact form $\tilde{\eta}$ naturally defined from the Ricci operator never defines a contact Lorentzian structure.*

PROOF. – Suppose that $g_f$ and the contact form $\tilde{\eta}$ define a contact Lorentzian structure. Following the argument used to introduce the paracontact Walker structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$, we find that with respect to $\{\partial_t, \partial_x, \partial_y\}$, because of (2.3) and $\varphi(\tilde{\xi}) = 0$, tensor $\varphi$ of such a contact Lorentzian structure can be written as

$$\varphi = \begin{pmatrix} a & -\dfrac{F_x}{F_t}a & fa \\ 0 & 0 & -\varepsilon \dfrac{F_x}{F_t}a \\ 0 & 0 & -a \end{pmatrix},$$

for a smooth function $a$. But the tensor $\varphi$ of a contact structure must satisfy $\varphi^2 = -Id + \tilde{\eta} \otimes \tilde{\xi}$ and this yields $a^2 = -1$, which cannot occur. $\square$

## 4. – On homogeneous paracontact Walker three-manifolds

A paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is (locally) homogeneous if it admits a transitive (pseudo-)group of (local) isometries leaving invariant the contact form $\eta$ and so, the whole paracontact structure $(\varphi, \xi, \eta)$.

A three-dimensional (simply connected, complete) homogeneous Lorentzian manifold $(M, g)$ is either symmetric, or it is a Lie group and $g$ is left-invariant [2]. Consequently, a locally homogeneous Lorentzian three-manifold is either locally symmetric, or locally isometric to a Lie group equipped with a left-invariant Lorentzian metric.

As proved in [4], there exist no locally symmetric paracontact Walker three-manifolds $(M, \eta, g_f)$, except for the trivial case when $g_f$ is flat. In fact, by [2], a locally symmetric Walker three-manifold either is reducible or it admits a parallel null vector field. However, a reducible symmetric paracontact metric three-space is necessarily flat [4, Proposition 3.3]. A stronger rigidity result holds for strictly Walker three-manifolds: if a Lorentzian three-manifold $(M, g_f)$ with a parallel null vector field admits a paracontact metric structure $(\varphi, \xi, \eta, g)$ satisfying $\nabla_\xi h = 0$ (in particular, a locally symmetric paracontact metric structure), then $(M, g_f)$ is flat [4, Theorem 3.4].

We shall now improve the rigidity result about paracontact Walker three-manifolds, proving the following.

THEOREM 4.1. – *A (locally) homogeneous paracontact Walker three-manifold $(M, g_f)$ is necessarily flat.*

In order to prove Theorem 4.1, we start from the classification of homogeneous paracontact metric three-manifolds.

THEOREM 4.2 [4]. – *A simply connected complete homogeneous paracontact metric three-manifold is isometric to a Lie group $G$ with a left-invariant paracontact metric structure $(\varphi, \xi, \eta, g)$. More precisely, one of the following cases occurs:*

(i) *If $G$ is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with $e_3$ time-like, such that the Lie algebra of $G$ is one of the following:*

(1) $\mathfrak{g}_2 : \ [e_1, e_2] = \gamma e_2 - \beta e_3, \ [e_1, e_3] = -\beta e_2 + \gamma e_3, \ [e_2, e_3] = 2e_1$, *with* $\gamma \neq 0$.

*Then, $G$ is either the identity component of $O(1,2)$ or $\widetilde{SL}(2, \mathbb{R})$.*

(2) $\mathfrak{g}_3 : \ [e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = 2e_1.$

*In this case, $G$ is*

(2a) *the identity component of $O(1,2)$ or $\widetilde{SL}(2, \mathbb{R})$ if $\beta, \gamma > 0$ or $\beta, \gamma < 0$;*
(2b) $\widetilde{E}(2)$ *if $\beta > 0 = \gamma$ or $\beta = 0 > \gamma$;*
(2c) $E(1,1)$ *if $\beta < 0 = \gamma$ or $\beta = 0 < \gamma$;*
(2d) *either $SO(3)$ or $SU(2)$ if $\beta > 0$ and $\gamma < 0$;*
(2e) *the Heisenberg group $H_3$ if $\beta = \gamma = 0$.*
(3) $\mathfrak{g}_4 : [e_1, e_2] = -e_2 + (2\varepsilon - \beta) e_3, \ [e_1, e_3] = -\beta e_2 + e_3, \ [e_2, e_3] = 2e_1$, *with* $\varepsilon = \pm 1$.

*In this case, $G$ is*

(3a) *the identity component of $O(1,2)$ or $\widetilde{SL}(2, \mathbb{R})$ if $\beta \neq \varepsilon$;*
(3b) $\widetilde{E}(2)$ *if $\beta = \varepsilon = 1$;*
(3c) $E(1,1)$ *if $\beta = \varepsilon = -1$.*

(ii) *if $G$ is non-unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with $e_3$ time-like, such that the Lie algebra of $G$ is one of the following:*

(4) $\mathfrak{g}_5, \mathfrak{g}_6 : \ [e_1, e_2] = [e_1, e_3] = 0, \ [e_2, e_3] = 2e_1 + \delta e_2$, *with* $\delta \neq 0$.
(5) $\mathfrak{g}_7 : [e_1, e_2] = -[e_1, e_3] = -\beta (e_2 + e_3), \ [e_2, e_3] = 2e_1 + \delta (e_2 + e_3)$, *with* $\delta \neq 0$.

Notations $\mathfrak{g}_2 - \mathfrak{g}_7$ for Lie algebras listed in Theorem 4.2 refer to the classification of all three-dimensional Lorentzian Lie groups, obtained in [2].

In order to show that there exist no locally homogeneous paracontact Walker three-manifolds which are not flat, we must prove that above Lie groups do not admit a parallel degenerate line field, unless they are flat. Indeed, we can exclude most of the cases above by investigating the admissible forms of the Ricci operator. Because of the symmetries of the curvature tensor, $\varrho$ is symmetric and so, the *Ricci operator* $Q$, defined by $g(QX,Y)=\varrho(X,Y)$, is self-adjoint [14]. Hence, in the Riemannian case, there always exists an orthonormal basis diagonalizing $Q$, while in the Lorentzian case four different cases can occur [14], known as *Segre types*. The possible cases are the following:

1. *Segre type* $\{11,1\}$: the Ricci operator itself is symmetric and so, diagonalizable. The comma separates the spacelike and timelike eigenvectors. In the degenerate case, at least two of the Ricci eigenvalues coincide.

2. *Segre type* $\{1z\bar{z}\}$: the Ricci operator has one real and two complex conjugate eigenvalues.

3. *Segre type* $\{21\}$: the Ricci operator has two real eigenvalues (coinciding in the degenerate case), one of which has multiplicity two and each associated to a one-dimensional eigenspace.

4. *Segre type* $\{3\}$: the Ricci operator has three equal eigenvalues, associated to a one-dimensional eigenspace.

The Ricci operator of three-dimensional Lorentzian Lie groups was determined in [3]. Then, in [6], the author and O. Kowalski determined which Segre types, and under which restrictions, occur for the Ricci operator of any locally homogeneous Lorentzian 3-manifold. For the Lorentzian paracontact Lie algebras listed in Theorem 4.2, the results of [3] and [6] easily imply the following classification.

(1): by [6, Theorem 3.2], $Q$ is of Segre type $\{1z\bar{z}\}$ if $\beta\neq 1$ and of degenerate Segre type $\{11,1\}$ with real eigenvalues $\lambda_1<0$ and $\lambda_2=\lambda_3=0$ if $\beta=1$.

(2): $Q$ is of Segre type $\{11,1\}$. By [6, Theorem 3.3], the Ricci eigenvalues $\lambda_1,\lambda_2,\lambda_3$ satisfy either $\lambda_1\lambda_2\lambda_3<0$, or at least two of $\lambda_i$ are zero.

(3): $Q$ is either of Segre type $\{21\}$ with Ricci eigenvalues $\lambda_1=-2$, $\lambda_2=\lambda_3=2(1+\eta-\beta)$, or of degenerate Segre type $\{11,1\}$, with Ricci eigenvalues $\lambda_1\leq 0, \lambda_2=\lambda_3=0$ [6, Theorem 3.4].

(4): Theorems 3.5 and 3.6 of [6] yield that unless $\lambda_1=\lambda_2=\lambda_3\neq 0$, $Q$ is of nondegenerate Segre type $\{11,1\}$.

On the other hand, the following result holds.

THEOREM 4.3 [5]. – *At a given point $p$ of a Walker three-manifold $(M,g_f)$, the Ricci operator $Q_p$ is never of Segre type $\{1z\bar{z}\}$ and is*

- *of Segre type $\{21\}$ if and only if either $f_{tt}(p)=f_{tx}(p)=0\neq f_{xx}(p)$ (degenerate case), or $(f_{tx}^2-f_{tt}f_{xx})(p)\neq 0\neq f_{tt}(p)$ (nondegenerate case).*
- *of Segre type $\{3\}$ if and only if $f_{tx}(p)\neq 0=f_{tt}(p)$.*

- *of Segre type* $\{11,1\}$ *if and only if either* $(f_{tx}^2 - f_{tt}f_{xx})(p) = 0 \neq f_{tt}(p)$ *(degenerate case), or* $f_{tt}(p) = f_{tx}(p) = f_{xx}(p) = 0$ *(degenerate and flat case).*

Note that for a homogeneous manifold, the Ricci operator has the same Segre type at each point and constant Ricci eigenvalues. Now, none of the cases (1)-(4) listed above is compatible with the existence of a Walker metric, as they contradict either (3.5) or Theorem 4.3.

Thus, we are left with the case (5) of the classification given in Theorem 4.2. We exclude this last remaining case (and so, we complete the proof of Theorem 4.1) by proving the following.

PROPOSITION 4.4. – *A three-dimensional non-unimodular Lie group $G$, admitting a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with $e_3$ time-like, and Lie brackets described by*

$$[e_1, e_2] = -[e_1, e_3] = -\beta(e_2 + e_3), \quad [e_2, e_3] = 2e_1 + \delta(e_2 + e_3), \quad \textit{with } \delta \neq 0,$$

*does not admit any parallel degenerate line field.*

PROOF. – Suppose that there exist three smooth functions $a, b, c : G \to \mathbb{R}$, such that $X = ae_1 + be_2 + ce_3$ is a null vector field generating a parallel degenerate line field, that is, satisfies

$$\nabla X = \omega \otimes X, \tag{4.1}$$

for a suitable 1-form $\omega = \mu_1 e^1 + \mu_2 e^2 + \mu_3 e^3$, where $\{e^i\}$ is the dual basis of $\{e_i\}$ and $\mu_1, \mu_2, \mu_3$ are smooth functions. From the form of the Lie algebra of $G$, a straightforward calculation yields (see also [3])

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = \beta e_2 + (\beta+1)e_3, & \nabla_{e_3} e_1 = -(\beta-1)e_2 - \beta e_3, \\ \nabla_{e_1} e_2 = e_3, & \nabla_{e_2} e_2 = -\beta e_1 + \delta e_3, & \nabla_{e_3} e_2 = (\beta-1)e_1 - \delta e_3, \\ \nabla_{e_1} e_3 = e_2, & \nabla_{e_2} e_3 = (\beta+1)e_1 + \delta e_2, & \nabla_{e_3} e_3 = -\beta e_1 - \delta e_2. \end{array} \tag{4.2}$$

We then calculate condition (4.1) with respect to the basis $\{e_i\}$ and find that (4.1) is equivalent to the following system of partial differential equations:

(4.3)
$$\begin{cases} e_1(a) = \mu_1 a, & e_1(b) = \mu_1 b - c, & e_1(c) = \mu_1 c - b, \\ e_2(a) = \mu_2 a + \beta b - (\beta+1)c, & e_2(b) = \mu_2 b - \beta a - \delta c, & e_2(c) = \mu_2 c - (\beta+1)a - \delta b, \\ e_3(a) = \mu_3 a - (\beta-1)b + \beta c, & e_2(b) = \mu_3 b + (\beta-1)a + \delta c, & e_3(c) = \mu_3 c + \beta a + \delta b. \end{cases}$$

For all indices $i, j$, we can now calculate $[e_i, e_j](a), [e_i, e_j](b), [e_i, e_j](c)$ both using (4.2) and (4.3). In particular, comparing the expressions of

$[e_1, e_2](a), [e_1, e_2](b)$ and $[e_1, e_2](c)$, we get

$$\begin{cases} (e_1(\mu_2) - e_2(\mu_1))a + \beta(\mu_2 + \mu_3)a = -(1+2\beta)b + 2\beta c, \\ (e_1(\mu_2) - e_2(\mu_1))b + \beta(\mu_2 + \mu_3)b = (1+2\beta)a, \\ (e_1(\mu_2) - e_2(\mu_1))c + \beta(\mu_2 + \mu_3)c = 2\beta a. \end{cases} \tag{4.4}$$

Since $X$ is a null vector field, $0 = \|X\|^2 = a^2 + b^2 - c^2$. In particular, $c \neq 0$. So, we can calculate $e_1(\mu_2) - e_2(\mu_1)$ from the last equation in (4.4) and replace in the first two equations. In this way, we get

$$\begin{cases} 2\beta ab = (1+2\beta)ac, \\ 2\beta a^2 = -(1+2\beta)bc + 2\beta c^2, \end{cases}$$

that is, as $a^2 = b^2 - c^2$,

$$\begin{cases} 2\beta ab = (1+2\beta)ac, \\ 2\beta b^2 = (1+2\beta)bc. \end{cases} \tag{4.5}$$

Solutions of system (4.5) are either $a = b = 0$, or $2\beta b = (1+2\beta)c$. If $a = b = 0$, then $X = 0$. So, we must exclude this solution and necessarily have $2\beta b = (1+2\beta)c$. But then, as $c \neq 0$, we have $\beta \neq 0$ and so,

$$b = \frac{1+2\beta}{2\beta} c.$$

Hence, $e_i(b) = ((1+2\beta)/2\beta)e_i(c)$ for all indices $i = 1, 2, 3$. By (4.3), we now find

$$\mu_1 \frac{1+2\beta}{2\beta} c - c = e_1(b) = \frac{1+2\beta}{2\beta}\left(\mu_1 c - \frac{1+2\beta}{2\beta} c\right),$$

which, since $c \neq 0$, yields $\beta = -1/4$. Therefore, $b = -c$, which also implies $a = 0$. Again by (4.3), we now have

$$0 = e_2(a) = -\beta b - (\beta+1)c = -(1+2\beta)c = -\frac{1}{2}c,$$

that is, $c = 0$, which can not occur and this ends the proof. □

## 5. – Final remarks and conclusions

We proved that there are not locally homogeneous paracontact Walker structures in dimension three. In particular, the paracontact Walker structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ studied in Section 3 can not be locally homogeneous. Indeed, a stronger statement holds: *a paracontact Walker structure* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g_f)$ *has never constant Ricci eigenvalues. In particular, it is never curvature homogeneous*. In fact, by (3.5), if the Ricci eigenvalues are constant, then $f_{tt}$ is constant. But this contradicts condition (3.7), which is necessary to the existence of such a paracontact Walker structure.

The results of [4], [10] and Sections 3,4 above show that there exist no three-dimensional paracontact *strictly* Walker structures $(\varphi, \xi, \eta, g_f)$ satisfying any of the following conditions:

- $\nabla_\xi h = 0$ (in particular, locally symmetric);
- the contact form is metrically equivalent to the unit eigenvector for the distinguished Ricci eigenvalue $\lambda_1 = 0$;
- locally homogeneous.

These rigidity results lead to the following natural

QUESTION. – *Do there exist three-dimensional paracontact strictly Walker structures?*

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