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Weak L^∞ and BMO in Metric Spaces

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Abstract. – *Bennett, DeVore and Sharpley introduced the space weak L^∞ in 1981 and studied its relationship with functions of bounded mean oscillation. Here we characterize the weak L^∞ in measure spaces without using the decreasing rearrangement of a function. Instead, we use exponential estimates for the distribution function. In addition, we consider a localized version of the characterization that leads to a new characterization of BMO.*

1. – Introduction

Bennett, DeVore and Sharpley introduced the space weak L^∞ in [3]. The definition of the weak L^∞ is based on decreasing rearrangements (see also the generalizations in [2], [10] and [12]). We give here more geometric characterizations of the weak L^∞ by analyzing the decay of the distribution functions. The main result in [3] states that the weak $L^\infty(Q)$, where Q is a Euclidean cube, is the rearrangement invariant hull of BMO (see also [4] and [8]). We show by an example that the weak L^∞ is not the rearrangement invariant hull of BMO in a general metric space.

We localize the geometric characterization of the weak L^∞ to obtain a new characterization of BMO, too. To show that our argument is based on a general principle, we study the characterizations in doubling measure spaces. Indeed, there has been a considerable interest in extending classical results in harmonic analysis to the metric setting, see e.g. [6] and [7]. The most important ingredient in our argument is a Calderón-Zygmund type covering lemma, which may be useful also elsewhere. There are several versions of this type of covering lemmas in the literature, see [6] and [9], but the version presented here seems not to follow immediately from any of them.

2. – A Characterisation of The Weak L^∞

Let (X, μ) be a measure space. The distribution function of a real-valued function f defined in X is the function $d_{f,\mu} : [0, \infty) \to [0, \infty]$ defined by the formula

$$d_{f,\mu}(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}).$$

If $d_{f,\mu}$ is invertible, its inverse function f^* is called the decreasing rearrangement of f. More generally we define the decreasing rearrangement of f to be the decreasing function $f^* : (0,\infty) \to [0,\infty]$ defined by

$$f^*(t) = \inf\{\lambda : d_{f,\mu}(\lambda) \leq t\}.$$

Here we use the convention $\inf \emptyset = \infty$. The decreasing rearrangement is unique, right-continuous, homogeneous, sublinear and satisfies $f^*(d_{f,\mu}(\lambda)) \leq \lambda$, whenever $d_{f,\mu}(\lambda)$ is finite, and $d_{f,\mu}(f^*(t)) \leq t$, whenever $f^*(t)$ is finite (c.f. [4]). The preceding inequalities can be taken as a definition of f^* as well (as in [13]). We say that f defined in (X,μ) and g defined in (Y,v) are equimeasurable functions if $d_{f,\mu} = d_{g,v}$. Observe that f and f^* are equimeasurable.

The following Cavalieri principle is useful for us: if f is μ-measurable, then for $0 < p < \infty$ we have

$$\int_X |f|^p d\mu = p \int_0^\infty \lambda^{p-1} d_{f,\mu}(\lambda) d\lambda.$$

We define the maximal function f^{**} of f^* by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

The space weak $L^\infty(X)$ or $L_w^\infty(X)$ is the collection of all f so that f^* is finite everywhere and

$$\sup_{t>0} (f^{**}(t) - f^*(t)) < \infty.$$

Observe that the weak $L^\infty(X)$ is rearrangement invariant and that it fails to be a vector space. The next theorem gives a characterization of the weak $L^\infty(X)$.

THEOREM 2.1. − *Let $f \in L_{loc}^1(X)$ so that $d_{f,\mu}$ is not identically infinite. Then the following conditions are equivalent:*

(i) *The rearrangement of f is finite for all $t > 0$ and there exists $M \geq 0$ independent of t for which*

$$f^{**}(t) - f^*(t) \leq M.$$

(ii) *There exist constants $\alpha \geq 0$ and $M \geq 0$ so that for all $\lambda > \alpha$, we have $d_{f,\mu}(\lambda) < \infty$ and*

$$\int_{\{x \in X : |f(x)| > \lambda\}} |f| d\mu \leq (\lambda + M) d_{f,\mu}(\lambda).$$

(iii) *There exist constants $\alpha \geq 0$ and $M \geq 0$ so that for all $\lambda > \alpha$, we have*

$d_{f,\mu}(\lambda) < \infty$ and

$$\int\limits_{\lambda}^{\infty} d_{f,\mu}(s)ds \leq Md_{f,\mu}(\lambda).$$

   (iv) *There exist constants $c_1, c_2 > 0$ so that*

$$d_{f,\mu}(\lambda_2) \leq c_1 d_{f,\mu}(\lambda_1) e^{c_2(\lambda_1 - \lambda_2)}$$

for all $\lambda_2 > \lambda_1 \geq 0$.

   PROOF. – Observe that, as a consequence of Cavalieri's principle, if $f^*(t)$ is finite, we have

$$\int\limits_{0}^{t} f^*(s)ds = \int\limits_{0}^{f^*(t)} t d\lambda + \int\limits_{f^*(t)}^{\infty} |\{s > 0 : f^*(s) > \lambda\}| d\lambda$$

$$= tf^*(t) + \int\limits_{f^*(t)}^{\infty} d_{f,\mu}(\lambda)d\lambda.$$

This implies that

(2.2) $$f^{**}(t) - f^*(t) = \frac{1}{t}\int\limits_{f^*(t)}^{\infty} d_{f,\mu}(\lambda)d\lambda.$$

   Let us assume the condition (i) and show that (ii) follows. For $f \in L_w^\infty(X)$ we define

$$\alpha = \lim_{t \to \infty} f^*(t).$$

The limit exists since f^* is decreasing and bounded below. Pick any $s > \alpha$. If $d_{f,\mu}(s) = 0$, there is nothing to prove. Hence, we may assume $d_{f,\mu}(s) > 0$. Using Cavalieri's principle, we get

(2.3) $$\int\limits_{\{x \in X : |f(x)| > s\}} |f| d\mu = \int\limits_{s}^{\infty} d_{f,\mu}(\lambda)d\lambda + s d_{f,\mu}(s).$$

Since $f^*(d_{f,\mu}(s)) \leq s$, by (2.2) we have

$$\int\limits_{\{x \in X : |f(x)| > s\}} |f| d\mu \leq \int\limits_{f^*(d_{f,\mu}(s))}^{\infty} d_{f,\mu}(\lambda)d\lambda + s d_{f,\mu}(s)$$

$$= \big(f^{**}(d_{f,\mu}(s)) - f^*(d_{f,\mu}(s)) + s\big) d_{f,\mu}(s)$$

$$\leq (M + s) d_{f,\mu}(s).$$

Assume then that the condition (ii) holds. Take any $t > 0$. Since

$$d_{f,\mu}(f^*(t)) \leq t,$$

by inequalities (2.2) and (2.3) we have

$$t(f^{**}(t) - f^*(t)) = \int_{f^*(t)}^{\infty} d_{f,\mu}(\lambda)d\lambda$$

$$= \int_{\{x \in X : |f(x)| > f^*(t)\}} |f|d\mu - f^*(t)d_{f,\mu}(f^*(t))$$

$$\leq M d_{f,\mu}(f^*(t)) \leq Mt.$$

This proves the equivalence of the first two conditions.

The conditions (ii) and (iii) are equivalent by the formula (2.3).

Assume then that the condition (iii) is valid. The condition (iv) is a consequence of a general real analysis principle. Indeed, given any decreasing function g that satisfies

$$\int_{s}^{\infty} g(t)dt \leq Mg(s)$$

for every $s > \alpha$, we have

$$g(s + t) \leq c_1 g(s)e^{-c_2 t/M}$$

for every $s > \alpha$ and $t > 0$ with $c_1 = 4$ and $c_2 = \log 2$. To see this, observe that since g is decreasing, the integral condition can be rewritten as

$$\sum_{i=1}^{\infty} g(s + iM) \leq g(s)$$

for all $s > \alpha$. Using the condition recursively, we have

$$g(s) \geq 2 \sum_{i=2}^{\infty} g(s + iM) \geq 2^{k-1} g(s + kM)$$

for all $s > \alpha$ and every positive integer k. Since there exists a smallest positive integer so that $t < kM$, we may apply the preceding inequality to obtain

(2.4) $$g(s + t) \leq g(s + (k-1)M) \leq 2^{2-k} g(s) \leq 4e^{-\log 2t/M} g(s).$$

The condition (iv) follows with g replaced by the distribution function of f.

Assume now the condition (iv). Then

$$\int\limits_{\{x\in X:|f(x)|>\lambda\}} |f|d\mu = \int\limits_{\lambda}^{\infty} d_{f,\mu}(s)ds + \lambda d_{f,\mu}(\lambda)$$

$$\leq \int\limits_{\lambda}^{\infty} c_1 d_{f,\mu}(\lambda)e^{c_2(\lambda-s)}ds + \lambda d_{f,\mu}(\lambda)$$

$$\leq \left(\frac{c_1}{c_2} + \lambda\right) d_{f,\mu}(\lambda).$$

Hence we have the second condition with $M = c_1/c_2$. $\qquad\square$

The previous theorem provides us with interesting knowledge on the behaviour of the functions in $L_w^\infty(X)$. Indeed, there are no big gaps in the distribution function of f. In terms of decreasing rearrangements we see that given f and M as in the second condition, f^* is continuous except for a countable set of points where the size of the jump is at most M. In addition, the mass is always concentrated on the low level sets.

COROLLARY 2.5. – *Let $f \in L_w^\infty(X)$. Suppose that $d_{f,\mu}(\lambda)$ is positive and finite. Then for any $\gamma > 2$ we have*

$$\frac{\mu(\{x \in X : \lambda < |f(x)| \leq \lambda + \gamma M\})}{\mu(\{x \in X : \lambda < |f(x)|\})} \geq (1 - 2^{2-\gamma}),$$

where M is the smallest constant satisfying the second condition in Theorem 2.1.

PROOF. – By Theorem 2.1(ii) we have

$$\int\limits_{\{x\in X:|f(x)|>\lambda\}} |f|d\mu \leq (\lambda + M)d_{f,\mu}(\lambda).$$

Inequality (2.4) in the proof of Theorem 2.1 implies

$$d_{f,\mu}(\lambda + \gamma M) \leq d_{f,\mu}(\lambda)2^{2-\gamma}$$

and the claim follows. $\qquad\square$

Sometimes it is possible to calculate precisely the integral average of a function over its level sets. In the following example we have a function which is extremal for the second condition of Theorem 2.1.

EXAMPLE 2.6. – Let $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = \log(|x|^{-n})\chi_{B(0,1)}$, when $x \neq 0$ and $f(0) = 0$. Then f belongs to $L_w^\infty(\mathbb{R}^n)$, since

$$\int\limits_{\{x \in \mathbb{R}^n : | f(x)| > \lambda\}} | f(y)|dy = (\lambda + 1)|\{x \in \mathbb{R}^n : | f(x)| > \lambda\}|$$

for all $\lambda \geq 0$.

3. – Functions of bounded mean tail oscillation

In this section we localize the definition of the weak $L^\infty(X)$. Let (X, d, μ) be a metric measure space. A ball with radius $r > 0$ and center x is denoted by

$$B(x, r) = \{y \in X : d(y, x) < r\}$$

Let f be a real-valued function defined in (X, d, μ). We write

$$f_B = \fint_B fd\mu = \frac{1}{\mu(B)} \int\limits_B fd\mu$$

for the mean value integral over the ball B. If $\mu(B) = 0$, then we set $f_B = 0$. A locally integrable function f is of bounded mean oscillation, if there exists $M \geq 0$ so that

$$\fint_B |f - f_B|d\mu \leq M$$

for every ball $B \subset X$ and we write $f \in BMO$. Similarly, if there exists $M \geq 0$ so that

$$\int\limits_{\{x \in B : |f(x) - f_B| > \lambda\}} |f - f_B|d\mu \leq (\lambda + M)\mu(\{x \in B : |f(x) - f_B| > \lambda\})$$

for all $\lambda > 0$ and for all balls $B \subset X$, we say that f is of bounded mean tail oscillation and write $f \in BMTO(X)$. Functions of bounded mean tail oscillation are of bounded mean oscillation and satisfy the John-Nirenberg inequality.

THEOREM 3.1. – *Let $f \in L_{loc}^1(X)$. Then the following conditions are equivalent:*

(i) *There exists $M \geq 0$ so that*

$$\int\limits_{\{x \in B : |f(x) - f_B| > \lambda\}} |f - f_B|d\mu$$
$$\leq (\lambda + M)\mu(\{x \in B : |f(x) - f_B| > \lambda\})$$

for all $\lambda \geq 0$ and for all balls $B \subset X$.

(ii) *There exist constants $c_1, c_2 > 0$ so that*

$$\mu(\{x \in B : |f(x) - f_B| > \lambda_2\})$$
$$\leq c_1 \mu(\{x \in B : |f(x) - f_B| > \lambda_1\}) e^{c_2(\lambda_2 - \lambda_1)}$$

for all $0 \leq \lambda_1 < \lambda_2$ and for all balls $B \subset X$.

In addition, if f is of bounded mean tail oscillation, then $f \in BMO(X)$ with

$$\sup_B \fint_B |f - f_B| d\mu \leq M$$

and f satisfies the John-Nirenberg inequality: there exist $c_1, c_2 > 0$ so that

$$\mu(\{x \in B : |f - f_B| > \lambda\}) \leq c_1 \mu(B) e^{-c_2 \frac{\lambda}{M}}$$

for all $\lambda > 0$.

PROOF. – Suppose that $f \in BMTO(X)$. Fix a ball B and consider the measure space $(B, \mu|_B)$. Then $f - f_B \in L_w^\infty(B, \mu|_B)$ with $\alpha = 0$ and we may apply Theorem 2.1. Hence, by inequality (2.4), we have

$$\mu(\{x \in B : |f(x) - f_B| > \lambda_2\}) \leq 4\mu(\{x \in B : |f(x) - f_B| > \lambda_1\})2^{(\lambda_1 - \lambda_2)/M}.$$

Since B is arbitrary, the second condition follows.

Assume then that f satisfies the second condition. Applying Theorem 2.1 to measure space $(B, \mu|_B)$ for every ball $B \subset X$ we get the first condition with $M = c_1/c_2$.

For the BMO condition we use the $BMTO$ condition with $\lambda = 0$. The John-Nirenberg inequality follows at once. $\qquad\square$

4. – BMO and the weak L^∞ for doubling measures

In this section the focus is in the connection between BMO and L_w^∞. We assume that the measure space is doubling which guarantees covering properties of the space. These can be used to prove that the space BMO is included in L_w^∞. We also characterize essentially bounded functions and functions of bounded mean oscillation with a condition similar to the $BMTO$ condition.

4.1 – *Doubling measures*

Let (X, d, μ) be a metric space endowed with a metric d and a Borel regular measure μ so that all open balls have positive and finite measure. During the rest of the current section we assume that the measure μ is doubling, i.e. there exists

a constant $c_\mu \geq 1$, called the doubling constant of μ, so that

$$\mu(B(x, 2r)) \leq c_\mu \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

The doubling condition implies a covering theorem. Indeed, given any collection of balls with uniformly bounded radius, there exists a pairwise disjoint, countable subcollection of balls, whose 5-dilates cover the union of the original collection. This theorem implies Lebesgue's differentiation theorem, which guarantees that any locally integrable function can be approximated at almost every point by integral averages of the function over a contracting sequence of balls. For the proofs we refer to [6] and [7].

4.2 – Essentially bounded functions

Let us study the localized weak L^∞. Here we show that essentially bounded functions can be characterized as functions satisfying the weak L^∞ condition in every ball of the space.

LEMMA 4.1. – *Let f be a measurable function in X. Then $f \in L^\infty(X)$ if and only if there exists $M \geq 0$ so that*

$$\int_{\{x \in B : |f(x)| > \lambda\}} |f| d\mu \leq (\lambda + M)\mu(\{x \in B : |f(x)| > \lambda\}),$$

for all balls $B \subset X$ and for all $\lambda \geq 0$.

PROOF. – Let $f \in L^\infty(X)$ and $B \subset X$ an arbitrary ball. Then

$$\int_{\{x \in B : |f(x)| > \lambda\}} |f| d\mu \leq \|f\|_{L^\infty(X)} \mu(\{x \in B : |f(x)| > \lambda\})$$

and the necessity of the condition follows.

For the sufficiency, let f be a function on X satisfying

$$\int_{\{x \in B : |f(x)| > \lambda\}} |f| d\mu \leq (\lambda + M)\mu(\{x \in B : |f(x)| > \lambda\}),$$

for all balls $B \subset X$ and for all $\lambda \geq 0$ with some $M \geq 0$. Define $A = \{x \in X : |f(x)| > 2M\}$. Suppose $\mu(A) > 0$. By Lebesgue's differentiation theorem, A has at least one density point, say $a \in A$. Hence, there exists a ball

$B(a,r) \subset X$ so that

$$\frac{\mu(B(a,r) \cap A)}{\mu(B(a,r))} > \frac{1}{2}.$$

On the other hand

$$2M\mu(B(a,r) \cap A) \leq \int\limits_{B(a,r) \cap A} |f| d\mu$$

$$\leq \int\limits_{\{x \in B(a,r): |f(x)| > 0\}} |f| d\mu$$

$$\leq (0 + M)\mu(B(a,r)).$$

This is a contradiction and hence the proof is complete. $\square$

4.3 – A covering lemma

Here we present a Calderón-Zygmund type covering lemma which is a generalization of the Lemma 3.2 in [3].

LEMMA 4.2. – *Let B_0 be a ball. Let $F \subset 3B_0$ be a measurable set and denote $E = F \cap B_0$. If $\mu(F) \leq \frac{1}{2}\mu(B_0)$, then there exists a disjoint family of balls contained in $3B_0$ so that*

$$(i) \quad \mu(5B_i \cap F) \leq \mu(5B_i \setminus F),$$

$$(ii) \quad \mu\left(E \setminus \bigcup_{i=1}^{\infty} 5B_i\right) = 0,$$

$$(iii) \quad \sum_{i=1}^{\infty} \mu(5B_i) \leq c\mu(F),$$

where c depends only on the doubling constant of the measure μ.

PROOF. – Let $x \in E$ be a density point of E. Since

$$\mu(F \cap B(x, 2r_0)) \leq \mu(F) \leq \frac{1}{2}\mu(B_0) \leq \frac{1}{2}\mu(B(x, 2r_0)),$$

we have

$$\mu(B(x, 2r_0) \cap F) \leq \mu(B(x, 2r_0) \setminus F)$$

and since x is a density point of E, there exists a greatest integer k so that $5B_x = B(x, 2^{1-k}r_0)$ satisfies

$$\mu(5B_x \cap F) \leq \mu(5B_x \setminus F).$$

By the maximality of k we also get

$$\mu(B(x, 2^{1-j}r_0) \cap F) > \mu(B(x, 2^{1-j}r_0) \setminus F)$$

for all $j > k$. By a covering theorem for the balls B_x, there exists a countable family of balls $\{B_i\}$ which satisfy the first condition of the lemma. Since almost every point is a density point the condition (ii) follows. For the condition (iii), we observe that

$$\sum_{i=1}^{\infty} \mu(5B_i) \leq c_\mu^3 \sum_{i=1}^{\infty} \mu\left(\frac{5}{8}B_i\right)$$

$$\leq 2c_\mu^3 \sum_{i=1}^{\infty} \mu\left(\frac{5}{8}B_i \cap F\right)$$

$$\leq 2c_\mu^3 \mu(F).$$

$\square$

4.4 – *Functions of bounded mean oscillation*

In doubling metric measure space it is possible to characterize the functions of bounded mean oscillation with a condition similar to that of bounded mean tail oscillation.

THEOREM 4.3. – *Let $f \in L^1_{loc}(X)$. Then $f \in BMO(X)$ if and only if there exists $M \geq 0$ so that*

$$\int\limits_{\{x \in B : |f(x) - f_B| > \lambda\}} |f - f_B| d\mu \leq (\lambda + M)\mu(\{x \in 3B : |f(x) - f_B| > \lambda\})$$

for all $\lambda \geq 0$ and for all balls $B \subset X$.

PROOF. – Assume $f \in BMO(X)$ and write $\|f\|_*$ for the *BMO*-norm of the function f. Fix a ball B. Given $\lambda > 0$ we write

$$E_\lambda = \{x \in B : |f(x) - f_B| > \lambda\}$$

and

$$F_\lambda = \{x \in 3B : |f(x) - f_B| > \lambda\}.$$

We have

$$\fint_{3B} |f - f_B| d\mu \leq \fint_{3B} |f - f_{3B}| d\mu + |f_B - f_{3B}| \leq (1 + c_\mu^2)\|f\|_*.$$

Hence,

$$\mu(F_\lambda) \le \int_{3B} \frac{|f - f_B|}{\lambda} d\mu \le \frac{1}{2}\mu(B)$$

whenever $\lambda \ge \lambda_0 = 2c_\mu^2(1 + c_\mu^2)\|f\|_*$. Fix any $\lambda > \lambda_0$. Now we apply Lemma 4.2 to the sets E_λ and F_λ. We have

$$\int_{E_\lambda} (|f - f_B| - \lambda)d\mu \le \sum_{i=1}^\infty \int_{5B_i \cap F_\lambda} (|f - f_B| - \lambda)d\mu$$

$$\le \sum_{i=1}^\infty \int_{5B_i \cap F_\lambda} |f - f_{5B_i}|d\mu$$

$$+ \sum_{i=1}^\infty \mu(5B_i \cap F_\lambda)(|(f - f_B)_{5B_i}| - \lambda).$$

Let P be the set of indices for which the last sum has a positive term. Then

$$\sum_{i=1}^\infty \mu(5B_i \cap F_\lambda)(|(f - f_B)_{5B_i}| - \lambda) \le \sum_{i \in P} \mu(5B_i \setminus F_\lambda)(|f_B - f_{5B_i}| - \lambda)$$

$$\le \sum_{i=1}^\infty \int_{5B_i \setminus F_\lambda} (|f_B - f_{5B_i}| - |f - f_B|)d\mu$$

$$\le \sum_{i=1}^\infty \int_{5B_i \setminus F_\lambda} |f - f_{5B_i}|d\mu.$$

This gives

$$\int_{E_\lambda} (|f - f_B| - \lambda)d\mu \le \sum_i \int_{5B_i} |f - f_{5B_i}|d\mu.$$

$$\le \|f\|_* \sum_i \mu(5B_i)$$

$$\le 2c\|f\|_* \mu(F_\lambda).$$

The last inequality follows from Lemma 4.2. For $\lambda < \lambda_0$ we have

$$\int_{E_\lambda} |f - f_B|d\mu = \int_{E_{\lambda_0}} |f - f_B|d\mu + \int_{E_\lambda \setminus E_{\lambda_0}} |f - f_B|d\mu$$

$$\le (\lambda_0 + c\|f\|_*)\mu(F_{\lambda_0}) + \lambda_0(\mu(E_\lambda) - \mu(E_{\lambda_0}))$$

$$\le (\lambda + 2\lambda_0 + c\|f\|_*)\mu(F_\lambda).$$

Now the theorem follows. $\qquad\square$

REMARK 4.4. — If we assume

$$\mu(F) \leq \frac{\mu(B_0)}{2c_\mu^k},$$

then we have

$$\bigcup 5B_i \subset (1 + 2^{-k})B_0$$

in the covering lemma. Consequently, we have another version of the Theorem 4.3. Indeed, let $\rho > 1$. Then $f \in BMO(X)$ if and only if there exists $M \geq 0$ so that

$$\int_{\{x \in B : |f(x) - f_B| > \lambda\}} |f - f_B| d\mu \leq (\lambda + M)\mu(\{x \in \rho B : |f(x) - f_B| > \lambda\}).$$

Rewriting the proof of the lemma with $\rho > 1$ instead of constant 3 shows that the bound M in the final estimate blows up when ρ approaches 1.

REMARK 4.5. — In some metric spaces the above result can be sharpened. Indeed, if every pair of points can be joined by a curve with a length as close to their distance as wished, a similar argument as above shows that $f \in BMO(X)$ if and only if there exists $M \geq 0$ so that

$$\int_{\{x \in B : |f(x) - f_B| > \lambda\}} |f - f_B| d\mu \leq (\lambda + M)\mu(\{x \in B : |f(x) - f_B| > \lambda\}).$$

In particular, this characterization is valid for doubling measures in $\mathbb{R}^n$.

4.5 — *The Weak L^∞ and BMO*

The following theorem establishes the connection between the weak $L^\infty(X)$ and $BMO(X)$. Briefly, every function of bounded mean oscillation with a finite distribution function belongs to the weak $L^\infty(X)$.

THEOREM 4.6. — *If $f \in BMO(X)$ and there exists $\alpha \geq 0$ so that $d_{f,\mu}(\lambda)$ is finite for all $\lambda > \alpha$, then there exists $M \geq 0$ so that*

$$\int_{\{x \in X : |f(x)| > \lambda\}} |f| d\mu \leq (\lambda + M)\mu(\{x \in X : |f(x)| > \lambda\})$$

for all $\lambda > \alpha$.

PROOF. — Let $f \in BMO(X)$. We may assume f is positive since $|f|$ is also in $BMO(X)$ with a norm at most twice that of f.

We split the proof in two cases according to the total mass of the space X. First we suppose $\mu(X) = \infty$. Let $\varepsilon > 0$ and consider $\lambda \geq \alpha + \varepsilon$, where $\alpha \geq 0$ is given by the assumption. Fix a point $x_0 \in X$ and define

$$E_k^\lambda = \{x \in B(x_0, k) : f(x) > \lambda\}$$

and $F_k^\lambda = E_{3k}^\lambda$. Since $d_{f,\mu}(\alpha + \varepsilon)$ is finite,

$$\lim_{k \to \infty} \frac{\mu(F_k^\lambda)}{\mu(B(x_0, k))} = 0$$

and consequently there exists an integer N_ε so that the above quotion is at most $1/2$ for any $\lambda \geq \alpha + \varepsilon$ whenever $k > N_\varepsilon$. Fix some $k > N_\varepsilon$, apply Lemma 4.2 to the set F_k^λ and obtain a collection of balls $\{B_i\}$. This implies

$$\int_{E_k^\lambda} (f - \lambda) d\mu \leq \sum_{i=1}^\infty \left(\int_{5B_i \cap F_k^\lambda} |f - f_{5B_i}| d\mu + \mu(5B_i \cap F_k^\lambda)(f_{5B_i} - \lambda) \right).$$

Similar to the proof of Theorem 4.3 we estimate the last term in the summand with

$$\mu(5B_i \cap F_k^\lambda)(f_{5B_i} - \lambda) \leq \int_{5B_i \setminus F_k^\lambda} |f - f_{5B_i}| d\mu$$

and hence

$$\int_{E_k^\lambda} (f - \lambda) d\mu \leq \sum_{i=1}^\infty \int_{5B_i} |f - f_{5B_i}| d\mu \leq c\|f\|_* \mu(F_k^\lambda).$$

By monotone convergence theorem the above inequality may be passed to the limit (w.r.t. k) and the theorem follows since the conclusion is independent of ε.

Let us then consider the case $\mu(X) < \infty$. Then α may be discarded since all sets in X have finite measure. Write

$$E^\lambda = \{x \in X : f(x) > \lambda\}.$$

Since f is finite μ almost everywhere, $\mu(E^\lambda)$ approaches zero as λ tends to infinity. Hence we may pick a λ_0 so that

$$\mu(E^\lambda) \leq \frac{1}{4} \mu(X)$$

for any $\lambda \geq \lambda_0$. Since $\mu(B(x_0, k)) \to \mu(X)$ as k tends to infinity, we have $N > 0$ so that

$$\frac{\mu(E_k^\lambda)}{\mu(B(x_0, k))} \leq \frac{1}{2}$$

for all $\lambda \geq \lambda_0$ and $k > N$. As above we apply Lemma 4.2 and monotone convergence theorem. We have

$$\int_{E^\lambda} (f - \lambda) \leq c\|f\|_* d_{f,\mu}(\lambda)$$

for all $\lambda \geq \lambda_0$. The case of small λ is treated as in the end of the proof of the Theorem 4.3. $\square$

The preceding theorem shows that any function of bounded mean oscillation belongs to the L_w^∞ as soon as the the measure space is doubling. The converse is not true in general since the local behaviour of the function in L_w^∞ is not controlled.

In a Euclidean cube, equipped with the Lebesgue measure, there is a converse result stating that every function in the weak L^∞ is equimeasurable to some function of bounded mean oscillation [3]. However, if the measure is changed, the result is no longer true as the next example shows.

LEMMA 4.7. – *There exists a doubling metric measure space (X, μ) and a function $f \in L_w^\infty(X)$ so that no function g defined on X, equimeasurable to f, belongs to BMO(X).*

PROOF. – We define $X \subset \mathbb{R}$ to be the countable collection of points

$$X = \{x_k\}_{k=0}^\infty = \{2^{-k}\}_{k=0}^\infty$$

with the Euclidean distance. We set

$$\mu(x_k) = 2^{-k}$$

for every $x_k \in X$. The metric measure space (X, μ) is doubling with a doubling constant $c_\mu = 4$ and $\mu(X) = 2$. Let us now define

$$f(x_k) = (-1)^k k.$$

and fix $\lambda > 0$. Then

$$\int_{\{x \in X: |f(x)| > \lambda\}} |f| d\mu \leq (\lambda + 2)\mu(\{x \in X : |f(x)| > \lambda\}).$$

and hence $f \in L^\infty(X)$. Nevertheless, f is not of bounded mean oscillation. This can be seen by considering the balls

$$B_k = B\left(x_{2k}, 5 \cdot 2^{-2k-3}\right)$$

since now

$$\fint_{B_k} |f - f_{B_k}| d\mu = \frac{16k + 10}{27}$$

which blows up as k grows. Observe that because of the special structure of the space, f is the only function defined on X which is equimeasurable to f. This proves the lemma. $\square$

REMARK 4.8. – The Hardy-Littlewood maximal operater is bounded in BMO for doubling metric measure spaces [1], for Euclidean case see [3] and [5]. The argument in [3] showing that maximal operator preserves the weak L^∞ in the Euclidean case, depend only on the weak and strong type estimates of the maximal operator. Since these estimates are available also in doubling metric measure spaces, we can conclude that the maximal operator is bounded in $L_w^\infty(X)$ as well.

5. – BMO and the weak L^∞ for non-doubling measures

In this section we show that the arguments of the previous section can be generalized to some non-doubling measures in Euclidean spaces. Indeed, we study positive Radon measures μ for which no hyperplane L, orthogonal to one of the coordinate axes, contains mass. These measures are not rare since for every nonnegative Radon measure for which $\mu(p) = 0$ at every point $p \in \mathbb{R}^n$, there exists an orthonormal system so that the above mentioned hyperplane condition is satisfied (for further details, see [11]). In this case Lemma 4.2 is replaced by the following result.

LEMMA 5.1. – *Let μ be a positive Radon measure in $\mathbb{R}^n$ such that for every hyperplane L, orthogonal to one of the coordinate axes, $\mu(L) = 0$. Let E be a subset of $\mathbb{R}^n$. Suppose E is contained in a cube Q_0 with sides parallel to the coordinate axes, and suppose that $\mu(E) \le \frac{1}{2}\mu(Q_0)$. Then there exists a sequence $\{Q_i\}$ of cubes with sides parallel to the coordinate axes and contained in Q_0 such that*

$$(i) \quad \mu(Q_i \cap E) \le \mu(Q_i \setminus E)$$

$$(ii) \quad \mu\left(E \setminus \bigcup_{i=1}^\infty Q_i\right) = 0,$$

$$(iii) \quad \sum_{i=1}^\infty \mu(Q_i) \le c(n)\mu(E),$$

with $c(n)$ depending only on the dimension of $\mathbb{R}^n$.

Following exactly the same line of arguments as in the proofs of Theorems 4.3 and 4.6 we obtain the following results.

THEOREM 5.2. — *Let μ be a positive Radon measure in $\mathbb{R}^n$ such that for every hyperplane L, orthogonal to the coordinate axes, $\mu(L) = 0$. Then $f \in BMO(\mu)$ if and only if there exists $M \geq 0$ such that*

$$\int\limits_{\{x \in Q : |f(x) - f_Q| > \lambda\}} |f - f_Q| d\mu \leq (\lambda + M)\mu(\{x \in Q : |f(x) - f_Q| > \lambda\})$$

for all cubes Q and $\lambda \geq 0$.

THEOREM 5.3. — *Let μ be a positive Radon measure in $\mathbb{R}^n$ such that for every hyperplane L, orthogonal to the coordinate axes, $\mu(L) = 0$. If $f \in BMO(\mu)$ and there exists $\alpha \geq 0$ so that $d_{f,\mu}(\lambda)$ is finite for all $\lambda > \alpha$, then $f \in L_w^\infty(\mu)$.*

Whether there exists a converse, i.e. if for every $f \in L_w^\infty(\mu)$ there exists an equimeasurable function of bounded mean oscillation, is not clear. Observe that the counterexample in the previous section was singular with respect to Lebesgue measure and does not satisfy the hyperplane condition.

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