Lucio Boccardo

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Finite Energy Solutions of Nonlinear Dirichlet Problems with Discontinuous Coefficients

LUCIO BOCCHARDI

A Enrico Magenes, uno di coloro che hanno dato l’anima per darci una patria libera.
Al Professor Magenes, uno dei padri della matematica italiana del dopo-guerra.
A Enrico, che era più forte di me anche nei 100 piani.

Abstract. – This paper dedicated to the memory of Enrico Magenes, concerning a nonlinear Dirichlet problem, follows the previous one ([1]) dedicated to the memory of Guido Stampacchia, concerning a similar linear problem (see [14]).

1. – Introduction

Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^N, N > 2 \); let \( M : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N^2} \), be a bounded and measurable matrix such that, for some \( 0 < \alpha \leq \beta \),

\[
\alpha |\xi|^2 \leq M(x) \xi \xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N;
\]

let \( E \) and \( f \) be functions such that

\[
E \in (L^N(\Omega))^N, \quad f \in L^{\frac{2N}{N+2}}(\Omega).
\]

Under these assumptions, existence and uniqueness of the weak solution \( u \in W^{1,2}_0(\Omega) \) of the linear Dirichlet problem

\[
\begin{cases}
-\text{div}(M(x)\nabla u) = -\text{div}(uE(x)) + f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

is studied in [17] by Guido Stampacchia (with slightly stronger assumptions), in [1], where are studied also the cases \( f \in L^m(\Omega), m \geq 1 \), with solutions of finite or infinite energy, and in [16].

In this paper, we consider a nonlinear version of the boundary value problem (3) whose simplest example is

\[
\begin{cases}
-\text{div}(b(x)|\nabla u|^{p-2}\nabla u) = - \text{div}(|u|^{p-2}uE(x)) + f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where \( z \leq b(x) \leq \beta \), for some \( 0 < z \leq \beta \) and
\[
(5) \quad 1 < p < N.
\]

Here (in the above model case (4) and in the general case (7) below), as in the linear cases studied in [17] and [1] (see also [6], [15], [18]), the main difficulty is due to the noncoercivity on \( W^{1,p}_0(\Omega) \) of the differential operator.

Now let us define the differential operator
\[
A(v) = -\text{div} (a(x, \nabla v))
\]
where \( a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) be a Carathéodory function such that the following holds (for almost every \( x \in \Omega \), for every \( \xi \in \mathbb{R}^N \) and \( \eta \in \mathbb{R}^N \), with \( \xi \neq \eta \):
\[
\begin{align*}
\alpha (x, \xi) \xi & \geq \alpha |\xi|^p, \\
|a(x, \xi)| & \leq \beta |\xi|^{p-1}, \\
(a(x, \xi) - a(x, \eta)) (\xi - \eta) & > 0,
\end{align*}
\]
where \( \alpha, \beta \) are strictly positive constants.

Thanks to (6), \( A \) is a monotone and coercive differential operator acting between \( W^{1,p}_0(\Omega) \) and its dual; hence, it is surjective (see [10], [11], [13]).

In this paper, we study existence and uniqueness of weak solutions of the following nonlinear boundary problem
\[
(7) \quad \begin{cases}
A(u) = -\text{div} (g(u) E(x)) + f(x) & \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]
under the assumptions
\[
(8) \quad E \in (L^\infty(\Omega))^N,
\]
\[
(9) \quad f \in L^m(\Omega), \ m \geq (p')',
\]
\[
(10) \quad g(s) \text{ is a real continuous function such that } |g(s)| \leq \gamma |s|^{p-1},
\]
for some \( \gamma > 0 \).

To this aim, let us consider the following approximate Dirichlet problems
\[
(11) \quad u_n \in W^{1,p}_0(\Omega): -\text{div} (a(x, \nabla u_n)) = -\text{div} \left( \frac{g(u_n)}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{E(x)}{1 + \frac{1}{n} |E(x)|} \right) + \frac{f(x)}{1 + \frac{1}{n} |f(x)|}.
\]

Note that a weak solution \( u_n \) of (11) exists thanks to Schauder fixed point Theorem. Moreover, since for every fixed \( n \) the function
\[
\frac{g(u_n)}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{E(x)}{1 + \frac{1}{n} |E(x)|}
\]
belongs to \((L^\infty(\Omega))^N\), every \(u_n\) is bounded thanks to Stampacchia’s boundedness theorem (see [17]).

2. – Basic estimates

Even if in this paper we assume \(E \in (L^\infty(\Omega))^N\), in this section we will only need that \(E \in (L^{p'}(\Omega))^N\).

**Lemma 2.1.** – Assume (5), (6), (10), \(E \in (L^{p'}(\Omega))^N\) and \(f \in L^1(\Omega)\). Then the solutions \(u_n\) of (11) satisfy

\[
L \left[ \int_\Omega \log^p (1 + |u_n|) \right] \leq \frac{1}{p'} \left[ \int_\Omega |E|^{p'} + \int_\Omega |f| \right],
\]

where \(L = L(x, p, \gamma)\) is a strictly positive constant.

**Proof.** – Take 

\[
\frac{1}{(p-1)} \left[ 1 - \frac{1}{1 + |u_n|^{p-1}} \right] \text{sign}(u_n)
\]

as test function in (11).

We have, using (6) and (10) and since \(\frac{|u_n|}{1 + |u_n|} \leq 1\) we have

\[
x \int_\Omega \frac{|
abla u_n|^p}{(1 + |u_n|)^p} \leq \frac{1}{p'} \int_\Omega |E| |
abla u_n| + \frac{1}{p-1} \int_\Omega |f|,
\]

so that (thanks to Young inequality),

\[
C_1 \int_\Omega \frac{|
abla u_n|^p}{(1 + |u_n|)^p} \leq \frac{1}{p'} \int_\Omega |E|^{p'} + \int_\Omega |f|,
\]

here, as in all the paper, we denote by \(C_i\) strictly positive constants independent of \(n\). Note that \(p' < \frac{N}{p-1}\) (since \(p < N\)) which implies

\[
L \left[ \int_\Omega \log^p (1 + |u_n|) \right] \leq C_2 \int_\Omega |
abla \log (1 + |u_n|)|^p \leq \frac{1}{p'} \left[ \int_\Omega |E|^{p'} + \int_\Omega |f| \right],
\]

which is (12).
Remark 2.2. – Remark that for every $\sigma > 0$, it is possible to choose $k_{\sigma}$ such that
\[ \text{meas}\left\{ x \in \Omega : |u_n(x)| > k \right\}^{\frac{1}{p}} \leq \sigma, \quad \forall k > k_{\sigma}, \]
thanks to the estimate (12), which implies also
\[ \text{meas}\left\{ x \in \Omega : |u_n(x)| > k \right\}^{\frac{1}{p}} \leq \frac{1}{L} \left| \log(1 + k) \right|^{\frac{1}{p}} \int_{\Omega} \left| [E]^{p'} + |f| \right|. \]

We recall the definitions of $T_k(s)$ and $G_k(s)$, for $s$ and $k$ in $\mathbb{R}$, with $k \geq 0$: $T_k(s) = \max (-k, \min (k, s))$ and $G_k(s) = s - T_k(s)$.

**Lemma 2.3.** – Assume (5), (6), (10), $E \in (L^p(\Omega))^N$ and $f \in L^1(\Omega)$. Then, for every $k \in \mathbb{R}^+$, the sequence $T_k(u_n)$ is bounded in $W_0^1,p(\Omega)$. More precisely we have
\[ A \int_{\Omega} |\nabla T_k(u_n)|^p \leq k^p \int_{\Omega} |E|^{p'} + k \int_{\Omega} |f|, \]
where $A = A(\alpha, p, \gamma)$ is a strictly positive constant.

**Proof.** – Using $T_k(u_n)$ as test function in (11) and using (6) and (10), we get
\[ \alpha \int_{\Omega} |\nabla T_k(u_n)|^p \leq \gamma k^{p-1} \int_{\Omega} |E| |\nabla T_k(u_n)| + k \int_{\Omega} |f|. \]
Then Young inequality implies the estimate (14).

3. – Existence of weak solutions

**Lemma 3.1.** – Assume (5), (6), (8), (9), (10). Then there exists $k_0$ and $\Gamma(k, E, f, \alpha, p, \gamma)$ such that, for every $k > k_0$,
\[ \|G_k(u_n)\|_{W_0^1,p(\Omega)} \leq \Gamma(k, E, f, \alpha, p, \gamma), \text{ for every } k > k_0. \]

**Proof.** – Define
\[ A_n(k) = \{ x \in \Omega : k \leq |u_n(x)| \}. \]
The use of $G_k(u_n)$ as test function in (11), with Young, Hölder and Sobolev
inequalities imply that
\[ C_{x,p} \int_\Omega |\nabla G_k(u_n)|^p \leq \]
\[ \gamma \int_\Omega |G_k(u_n)|^{p-1} |E| |\nabla G_k(u_n)| + \gamma k^{p-1} \int_\Omega |E| |\nabla G_k(u_n)| + \int_\Omega |G_k(u_n)||f| \]
\[ \leq C_1 \gamma \left( \int_{A_{n(k)}} |E|^{\frac{N}{p-1}} \right)^{-\frac{1}{p}} \int_\Omega |\nabla G_k(u_n)|^p \]
\[ + \gamma k^{p-1} \left[ \int_{A_{n(k)}} |E|^p \right]^{\frac{1}{p}} \left[ \int_\Omega |\nabla G_k(u_n)|^p \right]^{\frac{1}{p}} + C_1 \left[ \int_\Omega |\nabla G_k(u_n)|^p \right]^{\frac{1}{p}} \left[ \int_{A_{n(k)}} |f|^{(p')^r} \right]^{\frac{1}{(p')^r}} \]
Then
\[ \left\{ C_{x,p} - C_1 \gamma \left( \int_{A_{n(k)}} |E|^{\frac{N}{p-1}} \right)^{-\frac{1}{p}} \left[ \int_\Omega |\nabla G_k(u_n)|^p \right]^{-\frac{1}{p}} \right\} \leq \gamma k^{p-1} \left[ \int_{A_{n(k)}} |E|^p \right]^{\frac{1}{p}} + C_1 \left[ \int_{A_{n(k)}} |f|^{(p')^r} \right]^{\frac{1}{(p')^r}} \]
Now Remark 2.2 implies that there exists \( k_0 \), such that
\[ C_{x,p} - C_1 \gamma \left( \int_{A_{n(k)}} |E|^{\frac{N}{p-1}} \right)^{-\frac{1}{p}} \geq \frac{C_{x,p}}{2}, \quad k \geq k_0. \]
Thus we have, if \( k \geq k_0 \),
\[ \frac{C_{x,p}}{2} \left[ \int_\Omega |\nabla G_k(u_n)|^p \right]^{\frac{1}{p}} \leq \gamma k^{p-1} \left[ \int_\Omega |E|^p \right]^{\frac{1}{p}} + C_1 \left[ \int_\Omega |f|^{(p')^r} \right]^{\frac{1}{(p')^r}}, \]
that is (15). \( \square \)

**Corollary 3.2.** - Assume (5), (6), (8), (9), (10). Then the sequence \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \).

**Proof.** - The estimates (14) and (15) imply that, if \( k \geq k_0 \) (\( k_0 \) of Lemma 3.1),
\[ \int_\Omega |\nabla u_n|^p \leq M(x, p, E, f, \gamma), \]
where
\[
M(\alpha, p, E, f) = \frac{k^p}{A} \int_{\Omega} |E|^{\phi} + \frac{k}{A} \int_{\Omega} |f| + \Gamma.
\]

This Corollary ensures the existence of a subsequence (not relabelled) and a function \( u \) in \( W_0^{1,p}(\Omega) \) such that
\[
\begin{align*}
\{ u_n \text{ converges weakly to } u \text{ in } W_0^{1,p}(\Omega), \\
u_n(x) \text{ converges a.e. to } u(x).
\end{align*}
\]

(16)

In some sense, the next lemma improves Lemma 3.1.

**Lemma 3.3.** – Assume (5), (6), (8), (9), (10). Then, for every \( k > k_0 \),
\[
\bar{\Gamma} \left[ \int_{\Omega} |\nabla G_k(u_n)|^p \right]^{1-\frac{1}{p}} \leq \left[ \int_{A_n(k)} |E|^{\frac{p^\ast}{p-1}} \right]^{\frac{p^\ast}{p}} + \left[ \int_{A_n(k)} |f|^{(p')^\gamma} \right]^{\frac{1}{(p')^\gamma}}
\]

where \( \bar{\Gamma} = \Gamma(\alpha, p, \gamma, E, f) \) is a strictly positive constant.

**Proof.** – The use of \( G_k(u_n) \) as test function in (11), and Hölder and Sobolev inequalities imply that (thanks to Corollary 3.2)
\[
\int_{\Omega} |\nabla G_k(u_n)|^p \leq \int_{\Omega} |u_n|^{p-1} |E| |\nabla G_k(u_n)| + \int_{\Omega} |G_k(u_n)| |f|
\]
\[
\leq C_M \left[ \int_{A_n(k)} |E|^{\frac{p^\ast}{p-1}} \right]^{1-\frac{1}{p}} \left[ \int_{\Omega} |\nabla G_k(u_n)|^p \right]^{\frac{1}{p}} + C_1 \left[ \int_{\Omega} |\nabla G_k(u_n)|^p \right]^{\frac{1}{2}} \left[ \int_{A_n(k)} |f|^{(p')^\gamma} \right]^{\frac{1}{(p')^\gamma}},
\]

which implies the inequality (17).

**Corollary 3.4.** – Thanks to the absolute continuity of the integral and Corollary 3.2, we can say that
\[
\lim_{k \to \infty} \int_{\Omega} |\nabla G_k(u_n)|^p = 0, \quad \text{uniformly with respect to } n.
\]

(18)

**Lemma 3.5.** –
\[
u_n \text{ converges strongly to } u \text{ in } W_0^{1,p}(\Omega).
\]

**Proof.** – In the first step of the proof, we show that, for every \( k > 0 \), we have.
\[
\int_{\Omega} [a(\alpha, \nabla T_k(u_n)) - a(\alpha, \nabla T_k(u))] \nabla [T_k(u_n) - T_k(u)] \to 0.
\]

(20)
Note that  
\[- \text{div}(a(x, \nabla u)) = - \text{div}(a(x, \nabla T_k(u_n))) - \text{div}(a(x, \nabla G_k(u_n))).\]

Moreover it results  
\[- \text{div}(a(x, \nabla T_k(u_n))) - \text{div}(a(x, \nabla G_k(u_n)))
= - \text{div} \left( \frac{g(u_n) \chi_{\{|u_n| \leq k\}}}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{E(x)}{1 + \frac{1}{n} |E(x)|} \right) - \text{div} \left( \frac{g(u_n) \chi_{\{|u_n| > k\}}}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{E(x)}{1 + \frac{1}{n} |E(x)|} \right) + f_n(x).\]

Note that the contribution of terms of the type \(a(x, \nabla T_k(v))\nabla G_k(v)\) is zero. Then the use of \([T_k(u_n) - T_k(u)]\) as test function implies  
\[\int_{\Omega} a(x, \nabla T_k(u_n))[\nabla[T_k(u_n) - T_k(u)] - \int_{\Omega} a(x, \nabla u_n)\nabla T_k(u) \chi_{\{|u_n| > k\}} \]
\[= \int_{\Omega} g(u_n) \chi_{\{|u_n| \leq k\}} \frac{E(x)}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{1}{1 + \frac{1}{n} |E(x)|} \nabla[T_k(u_n) - T_k(u)]\]
\[- \int_{\Omega} g(u_n) \chi_{\{|u_n| > k\}} \frac{E(x)}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{1}{1 + \frac{1}{n} |E(x)|} \nabla T_k(u) + \int_{\Omega} f_n(x)[T_k(u_n) - T_k(u)].\]

Now note that, for almost every \(k > 0\),  
\[
\begin{align*}
\{ a(x, \nabla u_n) \} & \text{ converges weakly to } Y(x) \text{ in } (L^{p'}(\Omega))^N, \\
\nabla T_k(u) \chi_{\{|u_n| > k\}} & \text{ converges strongly to } \nabla T_k(u) \chi_{\{|u| > k\}} = 0 \text{ in } (L^p(\Omega))^N,
\end{align*}
\]
and  
\[
\begin{align*}
\{ g(u_n) \chi_{\{|u_n| \leq k\}} \frac{E(x)}{1 + \frac{1}{n} |u_n|^{p-1}} \frac{1}{1 + \frac{1}{n} |E(x)|} \} & \text{ converges strongly in } (L^{p'}(\Omega))^N, \\
\nabla[T_k(u_n) - T_k(u)] & \text{ converges weakly to } 0 \text{ in } (L^{p}(\Omega))^N.
\end{align*}
\]

Thus we can prove the convergence (20), which implies that  
\[T_k(u_n) \text{ converges strongly to } T_k(u) \text{ in } W^{1,p}_0(\Omega),\]

thanks to the assumptions and to a result in [11] and [9] (see also [8]).

Since \(u_n = G_k(u_n) + T_k(u_n)\), in order to prove that \(u_n\) converges strongly to \(u\) in \(W^{1,p}_0(\Omega)\), we only need to put together (17) and (21).
THEOREM 3.6. – Assume (5), (6), (8), (9), (10). Then there exists \( u \in W^{1,p}_0(\Omega) \)
weak solution of (7); that is
\[
\int_{\Omega} a(x, \nabla u) \nabla v = \int_{\Omega} g(u) E(x) \nabla v + \int_{\Omega} f(v), \quad \forall \ v \in W^{1,p}_0(\Omega).
\]

PROOF. – Since the sequence \( \{u_n\} \) converges strongly to \( u \) (see (19)) in
\( W^{1,p}_0(\Omega) \), it is possible to pass to the limit, as \( n \) tends to infinity, in the weak
formulation of (11). Therefore \( u \) is a weak solution of (7).

COROLLARY 3.7. – If \( f(x) \geq 0 \) then \( u(x) \geq 0 \).

PROOF. – Use \( T_h(u^-) \) as test function in (7). Thus we have
\[
\int_{\Omega} a(x, -\nabla T_h(u^-)) \nabla T_h(u^-) = \int_{\Omega} g(u) E(x) \nabla T_h(u^-) + \int_{\Omega} f T_h(u^-),
\]
which implies
\[
\alpha \int_{\Omega} |\nabla T_h(u^-)|^p \leq \int_{\Omega} |g(u)| |E| |\nabla T_h(u^-)| - \int_{\Omega} f T_h(u^-) \leq \int_{\Omega} |g(u)| |E| |\nabla T_h(u^-)|.
\]
Let \( 0 < h < \delta \). Then the inequalities
\[
C_1 \left[ \int_{\Omega} |T_h(u^-)|^p \right]^\frac{1}{p} \leq \alpha \left[ \int_{\Omega} |\nabla T_h(u^-)|^p \right]^\frac{1}{p} \leq \gamma h^{p-1} \left[ \int_{-h < u < 0} |E|^{\frac{p}{p-1}} \right]^\frac{1}{p}
\]
imply
\[
C_1 h^\frac{p}{p-1} \text{ meas } \{ u < -\delta \} \leq \gamma h^{p-1} \left[ \int_{-h < u < 0} |E|^{\frac{p}{p-1}} \right]^\frac{1}{p},
\]
that is
\[
C_1 \text{ meas } \{ u < -\delta \} \leq \gamma \left[ \int_{-h < u < 0} |E|^{\frac{p}{p-1}} \right]^\frac{1}{p}.
\]
Since \( |E| \in L^{\frac{p}{p-1}}(\Omega) \), the right hand side goes to 0, as \( h \to 0 \). Thus
meas \( \{ u < -\delta \} = 0 \), for every \( \delta > 0 \).

4. – Uniqueness of weak solutions

Note that Corollary 3.2 implies the uniqueness of the weak solution, if \( f = 0 \).
The uniqueness in the general case is more difficult.
We are able to prove the following partial (because of the assumption (24) below, see also [7]) result.

**Theorem 4.1.** — Assume (8), (9), (10) and consider the boundary value problem

\[
\begin{align*}
- \text{div} (b(x)|\nabla u|^{p-2} \nabla u) &= - \text{div} (g(u) E(x)) + f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where

\[\alpha \leq b(x) \leq \beta, \quad \text{for some } 0 < \alpha \leq \beta.\]

Moreover we assume also that \( g(s) \) is a \( C^1 \) increasing function such that

\[|g'(s)| \leq \mu |s|^{p-1} + \mu,\]

for some \( \mu > 0 \), and

\[1 < p \leq 2.\]

Then the weak solution of \( (22) \) is unique.

**Proof.** — For simplicity we will consider positive solutions (see Corollary 3.7); thus let \( u, w \geq 0 \) be solutions of \( (22) \) and use \( T_h(u - w)^+ \) as test function. Then we have

\[
\int_{\Omega} b(x)[|\nabla u|^{p-2} \nabla u - |\nabla w|^{p-2} \nabla w]|\nabla T_h(u - w)^+ \\
= \int_{\{0 < u - w < h\}} (g(u) - g(w)) E(x) |\nabla T_h(u - w)^+ |
\]

Now we use the following coercivity inequality (see also [12]). Let \( 1 < p \leq 2. \) There exists \( H_p > 0 \) such that, for every \( \eta, \zeta \in \mathbb{R}^N \)

\[
H_p \frac{||\eta - \zeta||^2}{||\eta|| + ||\zeta||}^{2-p} \leq [||\eta||^{p-2} \eta - ||\zeta||^{p-2} \zeta][\eta - \zeta],
\]

so that

\[
\begin{align*}
\alpha H_p \int_{\Omega} \frac{||\nabla T_h(u - w)^+||^2}{(|\nabla u| + |\nabla w|)^{2-p}} &\leq \int_{\{0 < u - w < h\}} (g(u) - g(w)) ||E|| |\nabla T_h(u - w)^+ | \\
&= \int_{\{0 < u - w < h\}} (g(u) - g(w)) ||E|| |\nabla T_h(u - w)^+ | \\
&\leq \int_{\{0 < u - w < h\}} (g(w + h) - g(w)) ||E|| |\nabla T_h(u - w)^+ |.
\end{align*}
\]
We use the following inequality in (25)

\[
\int_{\Omega} |\nabla T_h(u - w)^+|^p = \int_{\{0 < u - w < h\}} \frac{|\nabla T_h(u - w)^+|^p}{(|\nabla u| + |\nabla w|)^{2-\frac{p}{2}}} \leq \left( \int_{\{0 < u - w < h\}} \frac{|\nabla T_h(u - w)^+|^2}{(|\nabla u| + |\nabla w|)^{2-\frac{p}{2}}} \right)^{\frac{2}{p}} \left( \int_{\{0 < u - w < h\}} (|\nabla u| + |\nabla w|)^p \right)^{\frac{2-p}{p}}
\]

so that it results

\[
(26) \quad C_2 \int_{\Omega} |\nabla T_h(u - w)^+|^p \leq \left( \int_{\{0 < u - w < h\}} (g(w + h) - g(w)) |E||\nabla T_h(u - w)^+| \right)^{\frac{2}{p}} \left( \int_{\{0 < u - w < h\}} (|\nabla u| + |\nabla w|)^p \right)^{\frac{2-p}{p}} \leq C_E \left( \int_{\{0 < u - w < h\}} (g(w + h) - g(w)) \right)^{\frac{2(p-1)}{2p}} \left( \int_{\Omega} |\nabla T_h(u - w)^+|^p \right)^{\frac{2}{p}} \left( \int_{\{0 < u - w < h\}} (|\nabla u| + |\nabla w|)^p \right)^{\frac{2-p}{p}}.
\]

Let \(0 < h < \delta\). The Hölder and Poincaré inequalities with (26) yield

\[
C_3 \ h^{\frac{p}{2}} \text{ meas } \{\delta < u - w\} \leq \left( \int_{\Omega} |T_h(u - w)^+|^p \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} |\nabla T_h(u - w)^+|^p \right)^{\frac{1}{2}} \leq C_4 \left( \int_{\{0 < u - w < h\}} (g(w + h) - g(w)) \right)^{\frac{2(p-1)}{2p}} \left( \int_{\{0 < u - w < h\}} (|\nabla u| + |\nabla w|)^p \right)^{\frac{2-p}{p}}
\]

which implies

\[
C_5 \text{ meas } \{\delta < u - w\} \leq \left( \int_{\Omega} \left( \frac{g(w + h) - g(w)}{h} \right)^p \right)^{\frac{2(p-1)}{2p}} \left( \int_{\{0 < u - w < h\}} (|\nabla u| + |\nabla w|)^p \right)^{\frac{2-p}{p}}.
\]

On the right hand side note that the first integral converges, as \(h \to 0\), to

\[
\int_{\Omega} g^p(w)^{\frac{p}{p-1}}
\]

which is finite, because of (23); on the other hand the second integral converges to zero since

\[
\bigcap_{h>0} \{0 < u(x) - w(x) < h\} = \{0 < u(x) - w(x) \leq 0\} = \emptyset,
\]
and the continuity of the measure with respect to intersection then implies that
\[ \text{meas}\{0 < u(x) - w(x) < h\} \to 0, \text{ as } h \to 0. \]
Thus \( \delta < u(x) - w(x) = 0 \) for any \( \delta > 0 \), that is \( u(x) = w(x) \) a.e. in \( \Omega. \)

**Remark 4.2.** – Note that, unfortunately, the simple case \( g(t) = |t|^{p-2}t \) satisfies assumption (10), but it does not satisfy assumption (23), since \( 1 < p \leq 2 \).

5. – Summability and boundedness

In the spirit of [17], if the summability assumption of the right hand side \( f \) is stronger than (9), it is possible to prove a stronger summability result on the weak solutions of (7). Following [1], [4] and [5] it is possible to prove the following theorem.

**Theorem 5.1.** – Assume (5), (6), (8), (10) and if \( f \in L^m(\Omega), \frac{pN}{pN - N + p} < m < \frac{N}{p} \), then there exists a weak solution \( u \) of (3), which belongs to \( W_0^{1,p}(\Omega) \cap L^{\frac{pN}{p}}(\Omega) \).

Moreover, if \( f \in L^m(\Omega), m > \frac{N}{p} \), and \( E \in (L^r(\Omega))^N, r > \frac{N}{p - 1} \), then there exists a bounded weak solution \( u \) of (3).

**Remark 5.2.** – The previous techniques can be adapted easily to differential problems with more difficult assumptions of coercivity (see [2], [3]), with respect to (6)-1, like
\[ a(x, s, \xi) \xi \geq \alpha (1 + |s|)^\gamma |\xi|^p \]
or
\[ a(x, s, \xi) \xi \geq \alpha \frac{|\xi|^p}{(1 + |s|)^\gamma}, \]
where \( \gamma > 0 \) and \( a: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory function.

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Dipartimento di Matematica, Università di Roma I
Piazza A. Moro 2, 00185 Roma
E-mail: boccardo@mat.uniroma1.it

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