Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **5** (2012), n.2, p. 357–368.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_2_357_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012),357-368

Finite Energy Solutions of Nonlinear Dirichlet Problems with Discontinuous Coefficients

LUCIO BOCCARDO

A Enrico Magenes, uno di coloro che hanno dato l'anima per darci una patria libera.
Al Professor Magenes, uno dei padri della matematica italiana del dopo-guerra.
A Enrico, che era più forte di me anche nei 100 piani.

Abstract. – *This paper dedicated to the memory of Enrico Magenes, concerning a nonlinear Dirichlet problem, follows the previous one* ([1]) *dedicated to the memory of Guido Stampacchia, concerning a similar linear problem (see* [14]*).*

1. – Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$; let $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that, for some $0 < \alpha \le \beta$,

$$\alpha|\xi|^2 \le M(x)\,\xi\,\xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N\,; \tag{1}$$

let E and f be functions such that

$$E \in (L^N(\Omega))^N, \quad f \in L^{\frac{2N}{N+2}}(\Omega). \tag{2}$$

Under these assumptions, existence and uniqueness of the weak solution $u \in W_0^{1,2}(\Omega)$ of the linear Dirichlet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

is studied in [17] by Guido Stampacchia (with slightly stronger assumptions), in [1], where are studied also the cases $f \in L^m(\Omega)$, $m \ge 1$, with solutions of finite or infinite energy, and in [16].

In this paper, we consider a nonlinear version of the boundary value problem (3) whose simplest example is

$$\begin{cases} -\operatorname{div}(b(x)|\nabla u|^{p-2}\nabla u) = -\operatorname{div}(|u|^{p-2}u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\alpha \le b(x) \le \beta$, for some $0<\alpha \le \beta$ and

$$1<p<N. \tag{5}$$

Here (in the above model case (4) and in the general case (7) below), as in the linear cases studied in [17] and [1] (see also [6], [15], [18]), the main difficulty is due to the noncoercivity on $W_0^{1,p}(\Omega)$ of the differential operator.

Now let us define the differential operator

$$A(v) = -\mathrm{div}\,(a(x,\nabla v))$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that the following holds (for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$ and η in $\mathbb{R}^N$, with $\xi \ne \eta$):

$$\begin{cases} a(x,\xi)\,\xi \ge \alpha\,|\xi|^p\,, \\ |a(x,\xi)| \le \beta\,|\xi|^{p-1}\,, \\ \big(a(x,\xi) - a(x,\eta)\big)\,(\xi-\eta) > 0\,, \end{cases} \tag{6}$$

where α, β are strictly positive constants.

Thanks to (6), A is a monotone and coercive differential operator acting between $W_0^{1,p}(\Omega)$ and its dual; hence, it is surjective (see [10], [11], [13]).

In this paper, we study existence and uniqueness of weak solutions of the following nonlinear boundary problem

$$\begin{cases} A(u) = -\,\mathrm{div}(g(u)\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

under the assumptions

$$E \in \big(L^{\frac{N}{p-1}}(\Omega)\big)^N, \tag{8}$$

$$f \in L^m(\Omega),\ m \ge (p^*)', \tag{9}$$

$$g(s) \text{ is a real continuous function such that } |g(s)| \le \gamma |s|^{p-1}, \tag{10}$$

for some $\gamma > 0$.

To this aim, let us consider the following approximate Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega) : -\,\mathrm{div}\,(a(x,\nabla u_n)) = -\,\mathrm{div}\left(\frac{g(u_n)}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + \frac{f(x)}{1+\frac{1}{n}|f(x)|}. \tag{11}$$

Note that a weak solution u_n of (11) exists thanks to Schauder fixed point Theorem. Moreover, since for every fixed n the function

$$\frac{g(u_n)}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}$$

belongs to $(L^\infty(\Omega))^N$, every u_n is bounded thanks to Stampacchia's boundedness theorem (see [17]).

2. – Basic estimates

Even if in this paper we assume $E \in \left(L^{\frac{N}{p-1}}(\Omega)\right)^N$, in this section we will only need that $E \in (L^{p'}(\Omega))^N$.

LEMMA 2.1. – *Assume* (5), (6), (10), $E \in (L^{p'}(\Omega))^N$ *and* $f \in L^1(\Omega)$. *Then the solutions* u_n *of* (11) *satisfy*

$$L\left[\int_\Omega |\log(1+|u_n|)|^{p^*}\right]^{\frac{p}{p^*}} \le \int_\Omega |E|^{p'} + \int_\Omega |f|, \tag{12}$$

where $L = L(\alpha, p, \gamma)$ *is a strictly positive constant.*

PROOF. – Take $\dfrac{1}{(p-1)}\left[1 - \dfrac{1}{(1+|u_n|)^{p-1}}\right]\text{sign}(u_n)$ as test function in (11). We have, using (6) and (10) and since $\dfrac{|u_n|}{1+|u_n|} \le 1$ we have

$$\alpha \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \le \gamma \int_\Omega \frac{|E|\,|\nabla u_n|}{(1+|u_n|)} + \frac{1}{(p-1)} \int_\Omega |f|,$$

so that (thanks to Young inequality),

$$C_1 \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} \le \int_\Omega |E|^{p'} + \int_\Omega |f|,$$

here, as in all the paper, we denote by C_i strictly positive constants independent of n. Note that $p' < \dfrac{N}{p-1}$ (since $p < N$) which implies

$$L\left[\int_\Omega |\log(1+|u_n|)|^{p^*}\right]^{\frac{p}{p^*}} \le C_2 \int_\Omega |\nabla \log(1+|u_n|)|^p \le \int_\Omega |E|^{p'} + \int_\Omega |f|,$$

which is (12). □

REMARK 2.2. – Remark that for every $\sigma > 0$, it is possible to choose k_σ such that

$$\text{meas}\left\{x \in \Omega : |u_n(x)| > k\right\}^{\frac{p}{p^*}} \leq \sigma, \quad \forall\, k > k_\sigma,$$

thanks to the estimate (12), which implies also

$$\text{meas}\left\{x \in \Omega : |u_n(x)| > k\right\}^{\frac{p}{p^*}} \leq \frac{1}{L\,|\log(1+k)|^p} \int_\Omega [|E|^{p'} + |f|]. \tag{13}$$

□

We recall the definitions of $T_k(s)$ and $G_k(s)$, for s and k in $\mathbb{R}$, with $k \geq 0$: $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$.

LEMMA 2.3. – *Assume* (5), (6), (10), $E \in (L^{p'}(\Omega))^N$ *and* $f \in L^1(\Omega)$. *Then, for every* $k \in \mathbb{R}^+$, *the sequence* $T_k(u_n)$ *is bounded in* $W_0^{1,p}(\Omega)$. *More precisely we have*

$$\Lambda \int_\Omega |\nabla T_k(u_n)|^p \leq k^p \int_\Omega |E|^{p'} + k \int_\Omega |f|, \tag{14}$$

where $\Lambda = \Lambda(\alpha, p, \gamma)$ *is a strictly positive constant.*

PROOF. – Using $T_k(u_n)$ as test function in (11) and using (6) and (10), we get

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \leq \gamma k^{p-1} \int_\Omega |E|\,|\nabla T_k(u_n)| + k \int_\Omega |f|.$$

Then Young inequality implies the estimate (14). □

3. – Existence of weak solutions

LEMMA 3.1. – *Assume* (5), (6), (8), (9), (10). *Then there exists* k_0 *and* $\Gamma(k, E, f, \alpha, p, \gamma)$ *such that, for every* $k > k_0$,

$$\|G_k(u_n)\|_{W_0^{1,p}(\Omega)} \leq \Gamma(k, E, f, \alpha, p, \gamma), \text{ for every } k > k_0. \tag{15}$$

PROOF. – Define

$$A_n(k) = \{x \in \Omega : k \leq |u_n(x)|\}.$$

The use of $G_k(u_n)$ as test function in (11), with Young, Hölder and Sobolev inequalities imply that

$$C_{\alpha,p}\int_{\Omega} |\nabla G_k(u_n)|^p \leq$$

$$\gamma\int_{\Omega} |G_k(u_n)|^{p-1}|E||\nabla G_k(u_n)| + \gamma k^{p-1}\int_{\Omega} |E|\,|\nabla G_k(u_n)| + \int_{\Omega} |G_k(u_n)|\,|f|$$

$$\leq C_1\gamma\left[\int_{A_n(k)} |E|^{\frac{N}{p-1}}\right]^{1-\frac{1}{p}-\frac{p-1}{p^*}} \int_{\Omega} |\nabla G_k(u_n)|^p$$

$$+\gamma k^{p-1}\left[\int_{A_n(k)} |E|^{p'}\right]^{\frac{1}{p'}}\left[\int_{\Omega} |\nabla G_k(u_n)|^p\right]^{\frac{1}{p}} + C_1\left[\int_{\Omega} |\nabla G_k(u_n)|^p\right]^{\frac{1}{p}}\left[\int_{A_n(k)} |f|^{(p^*)'}\right]^{\frac{1}{(p^*)'}}$$

Then

$$\left\{C_{\alpha,p} - C_1\gamma\left[\int_{A_n(k)} |E|^{\frac{N}{p-1}}\right]^{\frac{p-1}{N}}\right\}\left[\int_{\Omega} |\nabla G_k(u_n)|^p\right]^{1-\frac{1}{p}}$$

$$\leq \gamma k^{p-1}\left[\int_{A_n(k)} |E|^{p'}\right]^{\frac{1}{p'}} + C_1\left[\int_{A_n(k)} |f|^{(p^*)'}\right]^{\frac{1}{(p^*)'}}$$

Now Remark 2.2 implies that there exists k_0, such that

$$C_{\alpha,p} - C_1\gamma\left[\int_{A_n(k)} |E|^{\frac{N}{p-1}}\right]^{\frac{p-1}{N}} \geq \frac{C_{\alpha,p}}{2}, \quad k \geq k_0.$$

Thus we have, if $k \geq k_0$,

$$\frac{C_{\alpha,p}}{2}\left[\int_{\Omega} |\nabla G_k(u_n)|^p\right]^{1-\frac{1}{p}} \leq \gamma k^{p-1}\left[\int_{\Omega} |E|^{p'}\right]^{\frac{1}{p'}} + C_1\left[\int_{\Omega} |f|^{(p^*)'}\right]^{\frac{1}{(p^*)'}},$$

that is (15). □

COROLLARY 3.2. – *Assume* (5), (6), (8), (9), (10). *Then the sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

PROOF. – The estimates (14) and (15) imply that, if $k \geq k_0$ (k_0 of Lemma 3.1),

$$\int_{\Omega} |\nabla u_n|^p \leq M(\alpha, p, E, f, \gamma),$$

where

$$M(\alpha,p,E,f)=\frac{k^p}{\Lambda}\int_\Omega |E|^{p'}+\frac{k}{\Lambda}\int_\Omega |f|+\Gamma.$$

□

This Corollary ensures the existence of a subsequence (not relabelled) and a function u in $W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \text{ converges weakly to } u \text{ in } W_0^{1,p}(\Omega), \\ u_n(x) \text{ converges a.e. to } u(x). \end{cases} \tag{16}$$

In some sense, the next lemma improves Lemma 3.1.

LEMMA 3.3. – *Assume* (5), (6), (8), (9), (10). *Then, for every* $k > k_0$,

$$\tilde{\Gamma}\left[\int_\Omega |\nabla G_k(u_n)|^p\right]^{1-\frac{1}{p}} \leq \left[\int_{A_n(k)} |E|^{\frac{N}{p-1}}\right]^{\frac{p-1}{N}} + \left[\int_{A_n(k)} |f|^{(p^*)'}\right]^{\frac{1}{(p^*)'}} \tag{17}$$

where $\tilde{\Gamma}=\tilde{\Gamma}(\alpha,p,\gamma,E,f)$ *is a strictly positive constant.*

PROOF. – The use of $G_k(u_n)$ as test function in (11), and Hölder and Sobolev inequalities imply that (thanks to Corollary 3.2)

$$\alpha\int_\Omega |\nabla G_k(u_n)|^p \leq \gamma\int_\Omega |u_n|^{p-1}|E||\nabla G_k(u_n)| + \int_\Omega |G_k(u_n)|\,|f|$$

$$\leq C_M\left[\int_{A_n(k)} |E|^{\frac{N}{p-1}}\right]^{1-\frac{1}{p}-\frac{p-1}{p^*}}\left[\int_\Omega |\nabla G_k(u_n)|^p\right]^{\frac{1}{p}} + C_1\left[\int_\Omega |\nabla G_k(u_n)|^p\right]^{\frac{1}{p}}\left[\int_{A_n(k)} |f|^{(p^*)'}\right]^{\frac{1}{(p^*)'}},$$

which implies the inequality (17). □

COROLLARY 3.4. – *Thanks to the absolute continuity of the integral and Corollary* 3.2, *we can say that*

$$\lim_{k\to\infty}\int_\Omega |\nabla G_k(u_n)|^p = 0, \quad \textit{uniformly with respect to } n. \tag{18}$$

□

LEMMA 3.5. –

$$u_n \textit{ converges strongly to } u \textit{ in } W_0^{1,p}(\Omega). \tag{19}$$

PROOF. – In the first step of the proof, we show that, for every $k > 0$, we have.

$$\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\,\nabla[T_k(u_n) - T_k(u)] \to 0. \tag{20}$$

Note that

$$-\operatorname{div}(a(x,\nabla u)) = -\operatorname{div}(a(x,\nabla T_k(u_n))) - \operatorname{div}(a(x,\nabla G_k(u_n))).$$

Moreover it results

$$-\operatorname{div}(a(x,\nabla T_k(u_n))) - \operatorname{div}(a(x,\nabla G_k(u_n)))$$

$$= -\operatorname{div}\left(\frac{g(u_n)\,\chi_{\{|u_n|\le k\}}}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}\right) - \operatorname{div}\left(\frac{g(u_n)\,\chi_{\{|u_n|>k\}}}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}\right) + f_n(x).$$

Note that the contribution of terms of the type $a(x,\nabla T_k(v))\nabla G_k(v)$ is zero. Then the use of $[T_k(u_n) - T_k(u)]$ as test function implies

$$\int_\Omega a(x,\nabla T_k(u_n))\nabla[T_k(u_n)-T_k(u)] - \int_\Omega a(x,\nabla u_n)\nabla T_k(u)\chi_{\{|u_n|>k\}}$$

$$= \int_\Omega \frac{g(u_n)\,\chi_{\{|u_n|\le k\}}}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla[T_k(u_n)-T_k(u)]$$

$$-\int_\Omega \frac{g(u_n),\chi_{\{|u_n|>k\}}}{1+\frac{1}{n}|u_n|^{p-1}}\,\frac{E(x)}{1+\frac{1}{n}|E(x)|}\nabla T_k(u) + \int_\Omega f_n(x)[T_k(u_n)-T_k(u)].$$

Now note that, for almost every $k>0$,

$$\begin{cases} a(x,\nabla u_n) \text{ converges weakly to } Y(x) \text{ in } (L^{p'}(\Omega))^N, \\ \nabla T_k(u)\chi_{\{|u_n|>k\}} \text{ converges strongly to } \nabla T_k(u)\chi_{\{|u|>k\}} = 0 \text{ in } (L^p(\Omega))^N, \end{cases}$$

and

$$\begin{cases} \dfrac{g(u_n)\,\chi_{\{|u_n|\le k\}}}{1+\frac{1}{n}|u_n|^{p-1}}\,\dfrac{E(x)}{1+\frac{1}{n}|E(x)|} \text{ converges strongly in } (L^{p'}(\Omega))^N, \\ \nabla[T_k(u_n)-T_k(u)] \text{ converges weakly to } 0 \text{ in } (L^p(\Omega))^N. \end{cases}$$

Thus we can prove the convergence (20), which implies that

$$T_k(u_n) \text{ converges strongly to } T_k(u) \text{ in } W_0^{1,p}(\Omega), \tag{21}$$

thanks to the assumptions and to a result in [11] and [9] (see also [8]).

Since $u_n = G_k(u_n) + T_k(u_n)$, in order to prove that u_n converges strongly to u in $W_0^{1,p}(\Omega)$, we only need to put together (17) and (21). □

THEOREM 3.6. – *Assume* (5), (6), (8), (9), (10). *Then there exists* $u \in W_0^{1,p}(\Omega)$ *weak solution of* (7); *that is*

$$\int_\Omega a(x, \nabla u)\nabla v = \int_\Omega g(u)E(x)\nabla v + \int_\Omega f\, v, \qquad \forall\ v \in W_0^{1,p}(\Omega).$$

PROOF. – Since the sequence $\{u_n\}$ converges strongly to u (see (19)) in $W_0^{1,p}(\Omega)$, it is possible to pass to the limit, as n tends to infinity, in the weak formulation of (11). Therefore u is a weak solution of (7). □

COROLLARY 3.7. – *If* $f(x) \geq 0$ *then* $u(x) \geq 0$.

PROOF. – Use $T_h(u^-)$ as test function in (7). Thus we have

$$\int_\Omega a(x, -\nabla T_h(u^-))\nabla T_h(u^-) = \int_\Omega g(u)E(x)\nabla T_h(u^-) + \int_\Omega f\, T_h(u^-),$$

which implies

$$\alpha \int_\Omega |\nabla T_h(u^-)|^p \leq \int_\Omega |g(u)|\,|E|\,|\nabla T_h(u^-)| - \int_\Omega f\, T_h(u^-) \leq \int_\Omega |g(u)|\,|E|\,|\nabla T_h(u^-)|.$$

Let $0 < h < \delta$. Then the inequalities

$$C_1 \left[\int_\Omega |T_h(u^-)|^p\right]^{\frac{1}{p'}} \leq \alpha \left[\int_\Omega |\nabla T_h(u^-)|^p\right]^{\frac{1}{p'}} \leq \gamma\, h^{p-1} \left[\int_{-h<u<0} |E|^{p'}\right]^{\frac{1}{p'}}$$

imply

$$C_1\, h^{\frac{p}{p'}} \text{ meas}\,\{u < -\delta\} \leq \gamma\, h^{p-1} \left[\int_{-h<u<0} |E|^{p'}\right]^{\frac{1}{p'}},$$

that is

$$C_1 \text{ meas}\,\{u < -\delta\} \leq \gamma \left[\int_{-h<u<0} |E|^{\frac{p}{p-1}}\right]^{\frac{1}{p'}}.$$

Since $|E| \in L^{\frac{N}{p-1}}(\Omega)$, the right hand side goes to 0, as $h \to 0$. Thus meas $\{u < -\delta\} = 0$, for every $\delta > 0$. □

4. – Uniqueness of weak solutions

Note that Corollary 3.2 implies the uniqueness of the weak solution, if $f = 0$. The uniqueness in the general case is more difficult.

We are able to prove the following partial (because of the assumption (24) below, see also [7]) result.

THEOREM 4.1. – *Assume* (8), (9), (10) *and consider the boundary value problem*

$$\begin{cases} -\operatorname{div}(b(x)|\nabla u|^{p-2}\nabla u) = -\operatorname{div}(g(u)E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where

$$\alpha \le b(x) \le \beta, \quad \text{for some } 0 < \alpha \le \beta.$$

Moreover we assume also that $g(s)$ *is a* C^1 *increasing function such that*

$$|g'(s)| \le \mu |s|^{p-1} + \mu, \tag{23}$$

for some $\mu > 0$, *and*

$$1 < p \le 2. \tag{24}$$

Then the weak solution of (22) *is unique.*

PROOF. – For simplicity we will consider positive solutions (see Corollary 3.7); thus let $u, w \ge 0$ be solutions of (22) and use $T_h(u-w)^+$ as test function. Then we have

$$\int_\Omega b(x)[|\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w]\nabla T_h(u-w)^+$$
$$= \int_{\{0<u-w<h\}} (g(u)-g(w))E(x)\nabla T_h(u-w)^+.$$

Now we use the following coercivity inequality (see also [12]). Let $1<p\le 2$. There exists $H_p > 0$ such that, for everey $\eta, \xi \in \mathbb{R}^N$

$$H_p \frac{|\eta-\xi|^2}{[|\eta|+|\xi|]^{2-p}} \le [|\eta|^{p-2}\eta - |\xi|^{p-2}\xi][\eta-\xi],$$

so that

$$\alpha H_p \int_\Omega \frac{|\nabla T_h(u-w)^+|^2}{(|\nabla u|+|\nabla w|)^{2-p}} \le \int_{\{0<u-w<h\}} |g(u)-g(w)|\,|E|\,|\nabla T_h(u-w)^+| \tag{25}$$
$$= \int_{\{0<u-w<h\}} (g(u)-g(w))|E||\nabla T_h(u-w)^+|$$
$$\le \int_{\{0<u-w<h\}} (g(w+h)-g(w))|E||\nabla T_h(u-w)^+|.$$

We use the following inequality in (25)

$$\int_\Omega |\nabla T_h(u-w)^+|^p = \int_{\{0<u-w<h\}} \frac{|\nabla T_h(u-w)^+|^p}{(|\nabla u|+|\nabla w|)^{\frac{(2-p)p}{2}}}(|\nabla u|+|\nabla w|)^{\frac{(2-p)p}{2}}$$

$$\leq \left[\int_\Omega \frac{|\nabla T_h(u-w)^+|^2}{(|\nabla u|+|\nabla w|)^{2-p}}\right]^{\frac{p}{2}} \left[\int_{\{0<u-w<h\}} (|\nabla u|+|\nabla w|)^p\right]^{\frac{2-p}{2}}$$

so that it results

$$(26)\quad C_2\int_\Omega |\nabla T_h(u-w)^+|^p$$

$$\leq \left[\int_{\{0<u-w<h\}} (g(w+h)-g(w))|E||\nabla T_h(u-w)^+|\right]^{\frac{p}{2}} \left[\int_{\{0<u-w<h\}} (|\nabla u|+|\nabla w|)^p\right]^{\frac{2-p}{2}}$$

$$\leq C_E\left[\int_{\{0<u-w<h\}} (g(w+h)-g(w))^{\frac{p^*}{p-1}}\right]^{\frac{p(p-1)}{2p^*}} \left[\int_\Omega |\nabla T_h(u-w)^+|^p\right]^{\frac{1}{2}} \left[\int_{\{0<u-w<h\}} (|\nabla u|+|\nabla w|)^p\right]^{\frac{2-p}{2}}.$$

Let $0<h<\delta$. The Hölder and Poincaré inequalities with (26) yield

$$C_3\, h^{\frac{p}{2}} \operatorname{meas}\{\delta<u-w\}^{\frac{1}{2}} \leq \left[\int_\Omega |T_h(u-w)^+|^p\right]^{\frac{1}{2}} \leq \left[\int_\Omega |\nabla T_h(u-w)^+|^p\right]^{\frac{1}{2}}$$

$$\leq C_4\left[\int_{\{0<u-w<h\}} (g(w+h)-g(w))^{\frac{p^*}{p-1}}\right]^{\frac{p(p-1)}{2p^*}} \left[\int_{\{0<u-w<h\}} (|\nabla u|+|\nabla w|)^p\right]^{\frac{2-p}{2}},$$

which implies

$$C_5 \operatorname{meas}\{\delta<u-w\}^{\frac{1}{2}} \leq \left[\int_\Omega \left(\frac{g(w+h)-g(w)}{h}\right)^{\frac{p^*}{p-1}}\right]^{\frac{p(p-1)}{2p^*}} \left[\int_{\{0<u-w<h\}} (|\nabla u|+|\nabla w|)^p\right]^{\frac{2-p}{2}}.$$

On the right hand side note that the first integral converges, as $h\to 0$, to

$$\int_\Omega g'(w)^{\frac{p^*}{p-1}}$$

which is finite, because of (23); on the other hand the second integral converges to zero since

$$\bigcap_{h>0}\{0<u(x)-w(x)<h\} = \{0<u(x)-w(x)\leq 0\} = \emptyset\ ,$$

and the continuity of the measure with respect to intersection then implies that

$$\text{meas}(\{0 < u(x) - w(x) < h\}) \to 0, \text{ as } h \to 0.$$

Thus $\text{meas}\{\delta < u(x) - w(x)\} = 0$ for any $\delta > 0$, that is $u(x) = w(x)$ a.e. in Ω. □

REMARK 4.2. – Note that, unfortunately, the simple case $g(t) = |t|^{p-2}t$ satisfies assumption (10), but it does not satisfy assumption (23), since $1 < p \leq 2$.

5. – Summability and boundedness

In the spirit of [17], if the summability assumption of the right hand side f is stronger that (9), it is possible to prove a stronger summability result on the weak solutions of (7). Following [1], [4] and [5] it is possible to prove the following theorem.

THEOREM 5.1. – *Assume* (5), (6), (8), (10) *and if* $f \in L^m(\Omega)$, $\dfrac{pN}{pN - N + p} < m < \dfrac{N}{p}$, *then there exists a weak solution* u *of* (3), *which belongs to* $W_0^{1,p}(\Omega) \cap L^{\frac{(pm)^*}{p'}}(\Omega)$.

Moreover, if $f \in L^m(\Omega)$, $m > \dfrac{N}{p}$, *and* $E \in (L^r(\Omega))^N$, $r > \dfrac{N}{p-1}$, *then there exists a bounded weak solution* u *of* (3).

REMARK 5.2. – The previous techniques can be adapted easily to differential problems with more difficult assumptions of coercivity (see [2], [3]), with respect to (6)-1, like

$$a(x, s, \xi)\,\xi \geq \alpha\,(1 + |s|)^{\gamma} |\xi|^p$$

or

$$a(x, s, \xi)\,\xi \geq \alpha\,\frac{|\xi|^p}{(1 + |s|)^{\gamma}},$$

where $\gamma > 0$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

REFERENCES

[1] L. BOCCARDO, *Some developments on Dirichlet problems with discontinuous coefficients*, Boll. Unione Mat. Ital., **2** (2009), 285-297.

[2] L. BOCCARDO, *A remark on some nonlinear elliptic problems, 2001-Luminy Conference on Quasilinear Elliptic and Parabolic Equations and Systems*, Electron. J. Diff. Eqns., Conf., 8 (2002), 47-52.

[3] L. BOCCARDO - A. DALL'AGLIO - L. ORSINA, *Existence and regularity results for some*

elliptic equations with degenerate coercivity. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996). Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), suppl., 51-81.
[4] L. BOCCARDO - D. GIACHETTI, *Existence results via regularity for some nonlinear elliptic problems*, Comm. P.D.E., **14** (1989), 663-680.
[5] L. BOCCARDO - D. GIACHETTI, *A nonlinear interpolation result with application to the summability of minima of some integral functionals*, Discrete and Continuous Dynamical Systems, Series B, **11** (2009), 31-42.
[6] L. BOCCARDO - J. I. DIAZ - D. GIACHETTI - F. MURAT, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms*, J. Diff. Eq., **106** (1993), 215-237.
[7] L. BOCCARDO - T. GALLOUET - F. MURAT, *Unicité de la solution pour des equations elliptiques non linéaires*, C. R. Acad. Sc. Paris, **315** (1992), 1159-1164.
[8] L. BOCCARDO - F. MURAT, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal TMA, **19** (1992), 581-597.
[9] L. BOCCARDO - F. MURAT - J.-P. PUEL, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl., **152** (1988), 183-196.
[10] H. BREZIS, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier (Grenoble), **18** (1968), 115-175.
[11] F. E. BROWDER, *Existence theorems for nonlinear partial differential equations*, Proceedings of Symposia in Pure Mathematics, vol. XVI, American Mathematical Society, Providence (1970), 1-60.
[12] J. CASADO-DIAZ - F. MURAT - A. PORRETTA, *Uniqueness results for pseudomonotone problems with* $p > 2$, C. R. Math. Acad. Sci. Paris, **344** (2007), 487-492.
[13] J. LERAY - J. L. LIONS, *Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder*, Bull. Soc. Math. France, **93** (1965), 97-107.
[14] E. MAGENES - G. STAMPACCHIA, *I problemi al contorno per le equazioni differenziali di tipo ellittico*, Ann. Scuola Norm. Sup. Pisa, **12** (1958), 247-358.
[15] G. MOSCARIELLO, *Existence and uniquesness for elliptic equations with lower-order terms*, Adv. Calc. Var., **4** (2011), 421-444.
[16] S. PUGLISI, *Some regularity results for solutions to Dirichlet problems with discontinuous coefficients*, preprint.
[17] G. STAMPACCHIA, *Le probléme de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble), **15** (1965), 189-258.
[18] G. ZECCA, *Existence and uniqueness for nonlinear elliptic equations with lower-order terms*, Nonlinear Anal., **75** (2012), 899-912.

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Received November 3, 2011 and in revised form April 2, 2012