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The Atmospheric Equation of Water Vapor with Saturation

MICHELE COTI ZELATI - ROGER TEMAM

*This article is dedicated to the memory of Enrico Magenes
with deep respect for the man and the mathematician*

Abstract. – *We analyze the equation of water vapor content in the atmosphere taking into account the saturation phenomenon. This equation is considered alone or coupled with the equation describing the evolution of the temperature T. The concentration of water vapor q belongs to the interval* $[0,1]$ *and the saturation concentration* $q_s \in (0,1)$ *is the threshold after which the vapor condensates and becomes water (rain). The equation for q (as well as the coupled q-T system) thus accounts for possible change of phase.*

1. – Introduction

The equations describing the motion of the atmosphere, also called Primitive Equations (PEs), are the classical tools used in the study of climate and weather prediction when the hydrostatic assumption is enforced, see e.g. [7, 10, 11, 16]. To the best of our knowledge, the mathematical study of the PEs was initiated in [15]. The equations considered in [15] and in the previously quoted references are the equations of the general dynamics of the atmosphere, for the whole atmosphere or for midlatitude regions. When studying the climate dynamics around the equator, the humidity equation, which describes the ratio of vapor in the air, $q \in [0,1]$, becomes very important and it is necessary to account for the possible saturation of vapor leading to condensation and rain.

In this article we study the equation of concentration of vapor accounting for saturation. Note that the equation considered in [15] (see also [8, 9]) does not account for saturation and thus it is just a transport equation. The saturation concentration $q_s \in (0,1)$ usually depends on the temperature T, but for simplicity we take it constant; introducing its dependence on the temperature would only add minor technical difficulties.

In Section 2 we consider the equation for the specific humidity q, assuming that the temperature is given. We first study an approximated problem in Section 3. Then, passing to the limit, we derive the existence of solutions for the specific humidity equation. We also establish, using the maximum principle, that $q \in [0,1]$. Uniqueness is proven under an additional physically reasonable assumption. In Sections 4 and 5 we study the coupling of the specific humidity equation with the temperature equation, only emphasizing the new points. Note that uniqueness is not guaranteed in this case and that, in both cases, the fluid velocity field $\boldsymbol{u}$ is assumed to be given. The coupling with the fluid equations (PEs, as in [15]) raises additional difficulties which will be addressed elsewhere [2].

2. – The equation of specific humidity

In this section and the next one, we will concentrate our attention on the equation ruling the evolution of the specific humidity, establishing existence and uniqueness of weak solutions. In order to simplify the presentation, we will consider a slightly simpler form of the problem. We will assume that $\mathcal{M} \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\mathcal{M}$. Given $\mathcal{T}_\star > 0$ and a velocity vector field $\boldsymbol{u} : [0, \mathcal{T}_\star] \to \mathbb{R}^3$, we consider the equation

$$
\begin{cases}
\partial_t q - \Delta_3 q + \boldsymbol{u}(t) \cdot \nabla_3 q \in H(q - q_s) f(t), & \text{in } \mathcal{M}, \\
q = 0, \quad \text{on } \partial\mathcal{M}, \\
q(x, y, p, 0) = q_0(x, y, p), \quad (x, y, p) \in \mathcal{M},
\end{cases}
\tag{2.1}
$$

where f is a general time-dependent forcing term. In what follows, we will only keep track of the time-dependencies of f and $\boldsymbol{u}$. Here, we have adopted the (x, y, p)-coordinate system typical of the three-dimensional PEs of the atmosphere, for which $\Delta_3 = \partial_x^2 + \partial_y^2 + \partial_p^2$ and $\nabla_3 = (\partial_x, \partial_y, \partial_p)$. Lastly, the Heaviside multivalued function

$$
H(r) = \begin{cases}
0, & r < 0, \\
[0,1], & r = 0, \\
1, & r > 0,
\end{cases}
$$

depends on the saturation specific humidity q_s, which we impose to be a constant value.

2.1 – *Function spaces*

Let H be the real Hilbert space $L^2(\mathcal{M})$ with the usual scalar product $(\cdot, \cdot)$ and the induced norm $|\cdot|$. Setting $A = -\Delta_3$ with Dirichlet boundary conditions,

$$
A : \mathcal{D}(A) \to H \quad \text{with} \quad \mathcal{D}(A) = H_0^1(\mathcal{M}) \cap H^2(\mathcal{M}),
$$

we define the Hilbert space V to be

$$V = \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{M}),$$

denoting by $((v, w)) = (A^{1/2}v, A^{1/2}w)$ its scalar product and by $\|v\| = |A^{1/2}v|$ its norm. The dual space V^* of V is endowed with the dual norm $\|\cdot\|_*$, and the symbol $\langle\cdot,\cdot\rangle$ will indicate the duality pairing between V and V^*. If $\lambda_1 > 0$ is the first eigenvalue of A, then we have the well-known Poincaré inequality

$$\lambda_1|v|^2 \le \|v\|^2, \quad \forall v \in V.$$

Moreover, for $\ell \in (0,1)$, we define the scale of Hilbert spaces

$$H^\ell = \mathcal{D}(A^{\ell/2}), \qquad \langle v, w\rangle_\ell = (A^{\ell/2}v, A^{\ell/2}w), \qquad \|v\|_\ell = |A^{\ell/2}v|.$$

We denote by $\boldsymbol{L}^2(\mathcal{M}) = \{L^2(\mathcal{M})\}^3$ and $\boldsymbol{H}^1(\mathcal{M}) = \{H^1(\mathcal{M})\}^3$ the usual Lebesgue and Sobolev spaces of vector valued functions on $\mathcal{M}$. Setting

$$\mathfrak{D} = \big\{\boldsymbol{u} \in C_0^\infty(\mathcal{M}, \mathbb{R}^3) : \operatorname{div} \boldsymbol{u} = 0\big\},$$

we consider the usual Hilbert spaces associated with the Navier-Stokes equations [19],

$$\boldsymbol{H} = \text{closure of } \mathfrak{D} \text{ in } \boldsymbol{L}^2(\mathcal{M}),$$
$$\boldsymbol{V} = \text{closure of } \mathfrak{D} \text{ in } \boldsymbol{H}^1(\mathcal{M}),$$

which will serve us as the natural spaces for the vector field $\boldsymbol{u}$.

2.2 – *Notation*

For any function $v : \mathcal{M} \to \mathbb{R}$, we define

$$v^+ = v^+(x,y,p) = \max\{v(x,y,p), 0\} \quad \text{and} \quad v^- = v^-(x,y,p) = \max\{-v(x,y,p), 0\}$$

to be the positive and negative part functions, respectively. If $v \in H$ (resp. $v \in V$), then v^+ and v^- are in H (resp. in V), with $|v^+| \le |v|$ and $|v^-| \le |v|$ (resp. $\|v^+\| \le \|v\|$ and $\|v^-\| \le \|v\|$).

Given any normed space X other than the ones already defined, we denote by $\|\cdot\|_X$ its norm. Moreover, the Lebesgue measure of a set σ is indicated by the symbol $|\sigma|$.

Throughout the article, C and $\mathcal{Q}(\,\cdot\,)$ will refer to a *generic* positive constant and to a *generic* increasing positive function, whose values may change even in the same line of a certain equation. In the case of a specific constant or function, an index will be added (e.g. $C_1, \mathcal{Q}_1$), and the respective value will be explicitly computed.

2.3 – *The trilinear form b*

Given $\boldsymbol{u} \in \boldsymbol{V}$ and $q, q^\star \in V$ we set

$$b(\boldsymbol{u}, q, q^\star) = \int_{\mathcal{M}} (\boldsymbol{u} \cdot \nabla q) q^\star \mathrm{d}\mathcal{M}.$$

It is easy to check that b is well-defined for such $\boldsymbol{u}, q, q^\star$ and

$$|b(\boldsymbol{u}, q, q^\star)| \leq C \|\boldsymbol{u}\|_{\boldsymbol{V}} \|q\| \|q^\star\|, \qquad \forall q, q^\star \in V, \boldsymbol{u} \in \boldsymbol{V}, \tag{2.2}$$

so that b is a continuous trilinear form on $\boldsymbol{V} \times V \times V$, and we infer the existence of a bilinear continuous operator $B : \boldsymbol{V} \times V \to V^*$, sometimes called the *coupling operator*, defined by

$$\langle B(\boldsymbol{u}, q), q^\star \rangle = b(\boldsymbol{u}, q, q^\star), \qquad \forall q, q^\star \in V, \boldsymbol{u} \in \boldsymbol{V}.$$

Moreover, from the calculation

$$\int_{\mathcal{M}} (\boldsymbol{u} \cdot \nabla q) q \,\mathrm{d}\mathcal{M} = \frac{1}{2} \int_{\mathcal{M}} \boldsymbol{u} \cdot \nabla(q^2) \,\mathrm{d}\mathcal{M} = \frac{1}{2} \int_{\partial\mathcal{M}} q^2 \boldsymbol{u} \cdot \nu \,\mathrm{d}\partial\mathcal{M} - \frac{1}{2} \int_{\mathcal{M}} q^2 \mathrm{div}\, \boldsymbol{u} \,\mathrm{d}\mathcal{M} = 0,$$

we obtain the orthogonality property

$$b(\boldsymbol{u}, q, q) = 0, \qquad \forall q \in H^1(\mathcal{M}), \boldsymbol{u} \in \boldsymbol{V}. \tag{2.3}$$

Notice also that this implies the skew-symmetric property

$$b(\boldsymbol{u}, q, q^\star) + b(\boldsymbol{u}, q^\star, q) = 0, \qquad \forall q, q^\star \in H^1(\Omega), \boldsymbol{u} \in \boldsymbol{V}. \tag{2.4}$$

Finally, from the Sobolev embedding $H^{1/2} \subset L^3(\mathcal{M})$ in dimension three, along with the interpolation inequality

$$\|q^\star\|_{1/2} \leq |q^\star|^{1/2} \|q^\star\|^{1/2}, \qquad \forall q^\star \in V,$$

we obtain the useful estimates

$$|b(\boldsymbol{u}, q, q^\star)| \leq C \|\boldsymbol{u}\|_{\boldsymbol{V}} \|q\|^{1/2} |Aq|^{1/2} |q^\star|, \qquad \forall q \in \mathcal{D}(A), q^\star \in H, \boldsymbol{u} \in \boldsymbol{V}. \tag{2.5}$$

and

$$|b(\boldsymbol{u}, q, q^\star)| \leq C \|\boldsymbol{u}\|_{\boldsymbol{V}} |q|^{1/2} \|q\|^{1/2} \|q^\star\|, \qquad \forall q, q^\star \in V, \boldsymbol{u} \in \boldsymbol{V}. \tag{2.6}$$

REMARK 2.1. – Due to the structure of the physical problem, we will be interested in the case in which the vector field $\boldsymbol{u}$ is time-dependent. In this case, the trilinear form $b(\boldsymbol{u}(t), \cdot, \cdot)$ and the coupling operator $B(\boldsymbol{u}(t), \cdot)$ will be time-dependent as well. If we fix $\mathcal{T}_\star > 0$, from (2.6) we deduce that

$$\int_0^{\mathcal{T}_\star} \|B(\boldsymbol{u}(t), q(t))\|_*^2 \mathrm{d}t \leq C \int_0^{\mathcal{T}_\star} \|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2 |q(t)| \|q(t)\| \mathrm{d}t.$$

Since the function $t \mapsto B(\boldsymbol{u}(t), \cdot)$ is measurable, and, equivalently, so is $t \mapsto b(\boldsymbol{u}(t), \cdot, \cdot)$, we can conclude that

$$B(\boldsymbol{u}(\,\cdot\,), q(\,\cdot\,)) \in L^2(0, \mathcal{T}_\star; V^*)$$

whenever $\boldsymbol{u} \in L^4(0, \mathcal{T}_\star; \boldsymbol{V})$ and $q \in L^2(0, \mathcal{T}_\star; V) \cap L^\infty(0, \mathcal{T}_\star; H)$. Therefore, both the trilinear form and the coupling operator will be defined for a.e. $t \in [0, \mathcal{T}_\star]$.

2.4 – *The abstract problem*

We are now ready to rephrase problem (2.1) in an abstract way. Fix $\mathcal{T}_\star > 0$, $f \in L^2(0, \mathcal{T}_\star; H)$ and $q_s \in (0, 1)$. For $t \in (0, \mathcal{T}_\star]$, we examine the evolution equation in the unknown $q = q(t)$:

$$(\mathbf{P}) \qquad \partial_t q + Aq + B(\boldsymbol{u}(t), q) \in H(q - q_s) f(t),$$

with initial datum $q(0) = q_0 \in H$. We give the following definition of a weak solution.

DEFINITION 2.2. – *A function $q \in L^2(0, \mathcal{T}_\star; V) \cap C([0, \mathcal{T}_\star]; H)$ is a solution to (**P**) if $\partial_t q \in L^2(0, \mathcal{T}_\star; V^*)$ and*

$$(2.7) \quad \langle \partial_t q, q^\star \rangle + ((q, q^\star)) + b(\boldsymbol{u}(t), q, q^\star) = \langle h_q f(t), q^\star \rangle, \quad \text{a.e. } t \in (0, \mathcal{T}_\star], \ \forall q^\star \in V,$$

for some $h_q \in L^\infty(\mathcal{M} \times [0, \mathcal{T}_\star])$ which satisfies the variational inequality

$$(2.8) \quad ([q^\star - q_s]^+, 1) - ([q - q_s]^+, 1) \geq \langle h_q, q^\star - q \rangle, \quad \text{a.e. } t \in (0, \mathcal{T}_\star], \ \forall q^\star \in V,$$

and $q(0) = q_0$.

REMARK 2.3. – The variational inequality (2.8) expresses the fact that h_q is an element of the subdifferential of the positive part function $q \mapsto ([q - q_s]^+, 1)$, which is clearly a lower semicontinuous convex function from V into $\mathbb{R}$. Moreover, we have

$$\partial([q - q_s]^+, 1) = H(q - q_s).$$

At first glance, (2.8) seems a complicated way to simply say that $h_q \in H(q - q_s)$, but, as it will be clear in the Paragraph 3.2 below, this formulation is much handier when passage-to-the-limit operations have to be performed.

REMARK 2.4. – *A priori*, $H(q - q_s)$ is a nonempty closed convex set in V^*, but, in fact, it is a subset of the unit ball of $L^\infty(\mathcal{M})$. Indeed, if $h_q \in H(q - q_s)$ and

$q(x, y, p) \neq q_s$, then

$$h_q(x, y, p) = \frac{q(x, y, p) - q_s}{|q(x, y, p) - q_s|} \tag{2.9}$$

while if $q(x, y, p) = q_s$, we have

$$h_q(x, y, p) \in [0, 1]. \tag{2.10}$$

3. – An approximated problem

In this section, we construct a family of problems which approximate problem (**P**) in a suitable sense. In this way, the limit of such approximated solutions will be shown to be a solution to (**P**), in the sense made precise by Definition 2.2. The proofs are based on *a priori* estimates and compactness arguments, and the variational inequality (2.8) plays an essential role.

3.1 – *Problem* (**P**$_\varepsilon$)

For $\varepsilon \in (0, 1]$, define the real functions

$$H_\varepsilon(r) = \begin{cases} 0, & r \leq 0, \\ r/\varepsilon, & r \in (0, \varepsilon], \\ 1, & r > \varepsilon, \end{cases} \qquad K_\varepsilon(r) = \begin{cases} 0, & r \leq 0, \\ r^2/2\varepsilon, & r \in (0, \varepsilon], \\ r - \varepsilon/2, & r > \varepsilon. \end{cases}$$

It is straightforward to check that $K'_\varepsilon = H_\varepsilon$,

$$|H_\varepsilon(r_1) - H_\varepsilon(r_2)| \leq \frac{1}{\varepsilon}|r_1 - r_2|, \qquad \forall r_1, r_2 \in \mathbb{R} \tag{3.1}$$

and

$$|K_\varepsilon(r_1) - K_\varepsilon(r_2)| \leq |r_1 - r_2|, \qquad \forall r_1, r_2 \in \mathbb{R}. \tag{3.2}$$

Moreover,

$$|K_\varepsilon(r) - r| \leq \frac{\varepsilon}{2}, \qquad \forall r \geq 0. \tag{3.3}$$

We then consider the following family of problems, depending on the parameter ε,

$$\begin{cases} \partial_t q + Aq + B(\boldsymbol{u}(t), q) = H_\varepsilon(q - q_s)f(t), \\ q(0) = q_0, \end{cases} \tag{$\mathbf{P}_\varepsilon$}$$

in order to establish an existence result for the limiting situation described in problem (**P**). The proof of the following theorem can be deduced by a standard

Galerkin approximation procedure, together with the subsequent Lemmas 3.2, 3.3 and 3.4.

THEOREM 3.1. – *Let* $\mathcal{T}_\star > 0$ *and* $f \in L^2(0, \mathcal{T}_\star; H)$. *For every* $\varepsilon > 0$, *there exists a unique solution*

$$q_\varepsilon \in C([0, \mathcal{T}_\star]; H) \cap L^2(0, \mathcal{T}_\star; V)$$

to ($\mathbf{P}_\varepsilon$), *such that*

$$\partial_t q_\varepsilon \in L^2(0, \mathcal{T}_\star; V^*).$$

We start by proving uniqueness and continuous dependence on the initial data.

LEMMA 3.2. – *Let* $\varepsilon > 0$ *be fixed. If* $q^1_\varepsilon(t), q^2_\varepsilon(t)$ *are two solutions to* ($\mathbf{P}_\varepsilon$) *with initial data* $q^1, q^2 \in H$, *we have*

$$|q^1_\varepsilon(t) - q^2_\varepsilon(t)|^2 \le \mathrm{e}^{C_\varepsilon \mathcal{T}_\star} |q^1 - q^2|^2, \qquad \forall t \in [0, \mathcal{T}_\star],$$

where

$$C_\varepsilon = \frac{2}{\varepsilon} \|f\|_{L^2(0,\mathcal{T}_\star;H)}.$$

PROOF. – The difference $\bar{q}_\varepsilon(t) = q^1_\varepsilon(t) - q^2_\varepsilon(t)$ solves the system

$$\begin{cases} \partial_t \bar{q}_\varepsilon + A\bar{q}_\varepsilon + B(\boldsymbol{u}(t), \bar{q}_\varepsilon) = \big[H_\varepsilon(q^1_\varepsilon - q_s) - H_\varepsilon(q^2_\varepsilon - q_s)\big] f(t), \\ \bar{q}_\varepsilon(0) = q^1 - q^2. \end{cases} \tag{3.4}$$

Multiplying the above equation by $\bar{q}_\varepsilon$ in H and using the orthogonality property (2.3), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{q}_\varepsilon|^2 + \|\bar{q}_\varepsilon\|^2 = \langle \big[H_\varepsilon(q^1_\varepsilon - q_s) - H_\varepsilon(q^2_\varepsilon - q_s)\big] f(t), \bar{q}_\varepsilon \rangle.$$

Hence, (3.1) entails

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{q}_\varepsilon|^2 \le \frac{1}{\varepsilon} |f(t)|\, |\bar{q}_\varepsilon|^2,$$

and the conclusion follows from the Gronwall lemma. □

The following two lemmas provide bounds on q_ε and on its time derivative $\partial_t q_\varepsilon$. Unlike the previous continuous dependence estimate, it is now crucial that such bounds are independent of ε.

LEMMA 3.3. – *Let* q_ε *be the solution to* ($\mathbf{P}_\varepsilon$). *Then*

$$|q_\varepsilon(t)|^2 \le |q_0|^2 \mathrm{e}^{-\lambda_1 t} + C_1, \qquad \forall t \in [0, \mathcal{T}_\star], \tag{3.5}$$

and

$$\int_0^{\mathcal{T}_\star} \|q_\varepsilon(s)\|^2 \mathrm{d}s \leq \mathcal{Q}_1(|q_0|), \tag{3.6}$$

where C_1 *is a positive constant and* $\mathcal{Q}_1(\,\cdot\,)$ *is a positive increasing function, both independent of* ε*, which can be explicitly computed.*

PROOF. – Multiplying the first equation of $(\mathbf{P}_\varepsilon)$ by q_ε in H and exploiting (2.3), we obtain the energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|q_\varepsilon|^2 + \|q_\varepsilon\|^2 = \langle H_\varepsilon(q_\varepsilon - q_s)f(t), q_\varepsilon\rangle.$$

A quick estimation of the right hand side gives

$$\langle H_\varepsilon(q_\varepsilon - q_s)f(t), q_\varepsilon\rangle \leq |f(t)|\,|q_\varepsilon| \leq \frac{1}{\sqrt{\lambda_1}}|f(t)|\,\|q_\varepsilon\| \leq \frac{1}{2\lambda_1}|f(t)|^2 + \frac{1}{2}\|q_\varepsilon\|^2,$$

and, consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}|q_\varepsilon|^2 + \|q_\varepsilon\|^2 \leq \frac{1}{\lambda_1}|f(t)|^2. \tag{3.7}$$

By the Poincaré inequality, we can write the above expression as

$$\frac{\mathrm{d}}{\mathrm{d}t}|q_\varepsilon|^2 + \lambda_1|q_\varepsilon|^2 \leq \frac{1}{\lambda_1}|f(t)|^2,$$

and the standard Gronwall lemma entails

$$|q_\varepsilon(t)|^2 \leq |q_0|^2 \mathrm{e}^{-\lambda_1 t} + C_1, \tag{3.8}$$

where

$$C_1 = \frac{1}{\lambda_1}\|f\|^2_{L^2(0,\mathcal{T}_\star;H)}.$$

To obtain the second estimate, we integrate (3.7) on $(0,\mathcal{T}_\star)$, getting

$$\int_0^{\mathcal{T}_\star} \|q_\varepsilon(s)\|^2 \mathrm{d}s \leq \mathcal{Q}_1(|q_0|).$$

where

$$\mathcal{Q}_1(|q_0|) = C_1 + |q_0|^2.$$

□

In particular, estimate (3.5) implies that $q_\varepsilon \in L^\infty(0,\mathcal{T}_\star;H)$, with

$$\|q_\varepsilon\|_{L^\infty(0,\mathcal{T}_\star;H)} \leq \mathcal{Q}_1(|q_0|). \tag{3.9}$$

LEMMA 3.4. – *Let q_ε be the solution to* $(\mathbf{P}_\varepsilon)$. *Then*

$$\int_0^{\mathcal{T}_\star} \|\partial_t q_\varepsilon(s)\|_*^2 \mathrm{d}s \le \mathcal{Q}_2(|q_0|), \tag{3.10}$$

where $\mathcal{Q}_2(\,\cdot\,)$ is a positive increasing function independent of ε, which can be explicitly computed.

PROOF. – Let $q^\star \in V$ with $\|q^\star\| \le 1$. From $(\mathbf{P}_\varepsilon)$, we have

$$\langle \partial_t q_\varepsilon, q^\star \rangle = \langle H_\varepsilon(q_\varepsilon - q_s) f(t), q^\star \rangle - ((q_\varepsilon, q^\star)) - b(\boldsymbol{u}(t), q_\varepsilon, q^\star).$$

Owing to (2.6) and using the uniform bound (3.9), we infer that

$$|\langle \partial_t q_\varepsilon, q^\star \rangle| \le |f(t)| + \|q_\varepsilon\| + C\mathcal{Q}_1(|q_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}} \|q_\varepsilon\|^{1/2},$$

which implies

$$\begin{aligned}\|\partial_t q_\varepsilon\|_*^2 &\le \big(|f(t)| + \|q_\varepsilon\| + C\mathcal{Q}_1(|q_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}\|q_\varepsilon\|^{1/2}\big)^2 \\ &\le \Big(|f(t)| + \frac{3}{2}\|q_\varepsilon\| + \frac{1}{2}C^2\mathcal{Q}_1(|q_0|)^2\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2\Big)^2 \\ &\le 3|f(t)|^2 + \frac{27}{4}\|q_\varepsilon\|^2 + \frac{3}{4}C^4\mathcal{Q}_1(|q_0|)^4\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^4.\end{aligned}$$

Using the integral control (3.6) and the fact that $\boldsymbol{u} \in L^4(0, \mathcal{T}_\star; \boldsymbol{V})$, we easily obtain

$$\int_0^{\mathcal{T}_\star} \|\partial_t q_\varepsilon(s)\|_*^2 \mathrm{d}s \le 3\|f\|^2_{L^2(0,\mathcal{T}_\star;H)} + \frac{27}{4}\mathcal{Q}_1(|q_0|) + \frac{3}{4}C^4\mathcal{Q}_1(|q_0|)^4\|\boldsymbol{u}\|^4_{L^4(0,\mathcal{T}_\star;\boldsymbol{V})} := \mathcal{Q}_2(|q_0|),$$

concluding the proof. □

3.2 – *Passage to the limit (I)*

According to the above lemmas, we see that the sequence $\{q_\varepsilon\}$ is bounded in $L^2(0, \mathcal{T}_\star; V)$ and $\{\partial_t q_\varepsilon\}$ is bounded in $L^2(0, \mathcal{T}_\star; V^*)$. Consequently, there exists a subsequence, which we do not relabel, and a function $q \in L^2(0, \mathcal{T}_\star; V)$ with $\partial_t q \in L^2(0, \mathcal{T}_\star; V^*)$ such that

- $q_\varepsilon \rightharpoonup q$ weakly in $L^2(0, \mathcal{T}_\star; V)$,
- $\partial_t q_\varepsilon \rightharpoonup \partial_t q$ weakly in $L^2(0, \mathcal{T}_\star; V^*)$,
- $q_\varepsilon \to q$ strongly in $L^2(0, \mathcal{T}_\star; H)$, as the embedding

  $$\mathcal{W} = \{q : q \in L^2(0, \mathcal{T}_\star; V), \partial_t q \in L^2(0, \mathcal{T}_\star; V^*)\} \subset L^2(0, \mathcal{T}_\star; H)$$

  is compact (see e.g. [14]).

Notice also that $q \in C([0,\mathcal{T}_\star],H)$, establishing the regularity required by Definition 2.2. If we multiply the first equation of $(\mathbf{P}_\varepsilon)$ by any $q^\star \in L^2(0,\mathcal{T}_\star;V)$ and integrate in t, we find

$$\begin{aligned}\int_0^{\mathcal{T}_\star} \langle \partial_t q_\varepsilon, q^\star\rangle \mathrm{d}t + \int_0^{\mathcal{T}_\star} ((q_\varepsilon, q^\star))\mathrm{d}t + \int_0^{\mathcal{T}_\star} b(\boldsymbol{u}(t), q_\varepsilon, q^\star)\mathrm{d}t \\ = \int_0^{\mathcal{T}_\star} \langle H_\varepsilon(q_\varepsilon - q_s) f(t), q^\star\rangle \mathrm{d}t.\end{aligned} \tag{3.11}$$

Now, since $H_\varepsilon(q_\varepsilon - q_s)$ is bounded in $L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$, we see that

- $H_\varepsilon(q_\varepsilon - q_s) \rightharpoonup h_q$ weak-$*$ in $L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$,

for some $h_q \in L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$. Since in particular $fq^\star \in L^1(\mathcal{M}\times[0,\mathcal{T}_\star])$, we can pass to the limit as $\varepsilon\to 0$ in each of the terms of equation (3.11), finding

$$\int_0^{\mathcal{T}_\star} \langle \partial_t q, q^\star\rangle \mathrm{d}t + \int_0^{\mathcal{T}_\star} ((q, q^\star))\mathrm{d}t + \int_0^{\mathcal{T}_\star} b(\boldsymbol{u}(t), q, q^\star)\mathrm{d}t = \int_0^{\mathcal{T}_\star} \langle h_q f(t), q^\star\rangle \mathrm{d}t.$$

This equality holds for all functions $q^\star \in L^2(0,\mathcal{T}_\star;V)$. Hence in particular

$$\langle \partial_t q, q^\star\rangle + ((q, q^\star)) + b(\boldsymbol{u}(t), q, q^\star) = \langle h_q f(t), q^\star\rangle,$$

for each $q^\star \in V$ and a.e. $t\in(0,\mathcal{T}_\star]$. Notice also that $q \in C([0,\mathcal{T}_\star],H)$, establishing the regularity required by Definition 2.2. It remains to show that h_q belongs to $H(q-q_s)$ in the weak sense specified by the variational inequality (2.8). To this end, notice that, for every $\varepsilon>0$, the following approximate variational inequality holds

$$\int_0^{\mathcal{T}_\star} (K_\varepsilon(q^\star - q_s), 1)\mathrm{d}t - \int_0^{\mathcal{T}_\star} (K_\varepsilon(q_\varepsilon - q_s), 1)\mathrm{d}t \geq \int_0^{\mathcal{T}_\star} \langle H_\varepsilon(q_\varepsilon - q_s), q^\star - q_\varepsilon\rangle \mathrm{d}t, \tag{3.12}$$

for each $q^\star \in L^2(0,\mathcal{T}_\star;V)$, since $H_\varepsilon(q_\varepsilon - q_s)$ is the Gâteaux derivative of the convex function

$$\int_0^{\mathcal{T}_\star} (K_\varepsilon(\,\cdot\,), 1)\mathrm{d}t : L^2(0,\mathcal{T}_\star;V)\to\mathbb{R}$$

at the point $q_\varepsilon - q_s$. From the weak-$*$ convergence $H_\varepsilon(q_\varepsilon - q_s)\to h_q$ in $L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$ and the strong convergence $q_\varepsilon\to q$ in $L^2(0,\mathcal{T}_\star;H)$ we find that

$$\int_0^{\mathcal{T}_\star} \langle H_\varepsilon(q_\varepsilon - q_s), q_\varepsilon - q^\star\rangle \mathrm{d}t \to \int_0^{\mathcal{T}_\star} \langle h_q, q - q^\star\rangle \mathrm{d}t, \qquad \forall q^\star \in L^2(0,\mathcal{T}_\star;V),$$

as $\varepsilon \to 0$. Moreover, owing to (3.2) and (3.3), we observe that

$$\left| \int_0^{\mathcal{T}_\star} (K_\varepsilon(q_\varepsilon - q_s), 1)\mathrm{d}t - \int_0^{\mathcal{T}_\star} ([q - q_s]^+, 1)\mathrm{d}t \right|$$

$$\leq \int_0^{\mathcal{T}_\star} (|K_\varepsilon(q_\varepsilon - q_s) - K_\varepsilon(q - q_s)|, 1)\mathrm{d}t + \int_0^{\mathcal{T}_\star} (|K_\varepsilon(q - q_s) - [q - q_s]^+|, 1)\mathrm{d}t$$

$$\leq |\mathcal{M}|^{1/2}\mathcal{T}_\star^{1/2}\|q_\varepsilon - q\|_{L^2(0,\mathcal{T}_\star;H)} + \frac{\varepsilon}{2}|\mathcal{M}|\mathcal{T}_\star.$$

Therefore,

$$\lim_{\varepsilon \to 0} \int_0^{\mathcal{T}_\star} (K_\varepsilon(q_\varepsilon - q_s), 1)\mathrm{d}t = \int_0^{\mathcal{T}_\star} ([q - q_s]^+, 1)\mathrm{d}t.$$

From the calculation above, it is also clear that

$$\lim_{\varepsilon \to 0} \int_0^{\mathcal{T}_\star} (K_\varepsilon(q^\star - q_s), 1)\mathrm{d}t = \int_0^{\mathcal{T}_\star} ([q^\star - q_s]^+, 1)\mathrm{d}t, \qquad q^\star \in L^2(0, \mathcal{T}_\star; V).$$

Consequently, we can pass to the limit as $\varepsilon \to 0$ in (3.12), concluding that

$$\int_0^{\mathcal{T}_\star} \langle [q^\star - q_s)]^+, 1\rangle \mathrm{d}t - \int_0^{\mathcal{T}_\star} \langle [q - q_s]^+, 1\rangle \mathrm{d}t \geq \int_0^{\mathcal{T}_\star} \langle h_q, q^\star - q\rangle \mathrm{d}t, \qquad \forall q^\star \in L^2(0, \mathcal{T}_\star; V).$$

Again, in particular this implies that

$$([q^\star - q_s]^+, 1) - ([q - q_s]^+, 1) \geq \langle h_q, q^\star - q\rangle,$$

for every $q^\star \in V$ and a.e. $t \in (0, \mathcal{T}_\star]$, as desired. We have proved

THEOREM 3.5. – *Let $\boldsymbol{u} \in L^4(0, \mathcal{T}_\star; \boldsymbol{V})$ and $f \in L^2(0, \mathcal{T}_\star; H)$ be given. For any $q_0 \in H$ and any $\mathcal{T}_\star > 0$, the problem* $(\mathbf{P})$ *admits at least one solution $q \in L^2(0, \mathcal{T}_\star; V) \cap C([0, \mathcal{T}_\star]; H)$, such that $\partial_t q \in L^2(0, \mathcal{T}_\star; V^*)$.*

3.3 – *A uniqueness result*

Without any further assumptions, uniqueness of solutions is available only for the approximated problem $(\mathbf{P}_\varepsilon)$. It is known (cf. [5]) that even in the one-dimensional case with $\boldsymbol{u} \equiv 0$ and $f = \lambda > 0$, uniqueness of solutions may fail in general. On the other hand, considering stronger notions of solution (e.g. C^1 solutions and C^1 initial data satisfying certain symmetries and monotonicity conditions) may help in this sense [4, 5, 6]. The biggest issue here is that problem (**P**) fails to be in the class of the so-called evolution problems associated to multivalued monotone operators, as the operator $H(\,\cdot\,)$ is not monotone, as its monotonicity depends on the sign of f. One way to recover uniqueness is to require the forcing term to be negative almost everywhere in $\mathcal{M} \times [0, \mathcal{T}_\star]$.

THEOREM 3.6. – *Let $f \leq 0$ a.e. in $\mathcal{M} \times [0, \mathcal{T}_\star]$. Then there exists at most one solution to* (**P**). *Moreover, if $q_1(t), q_2(t)$ are two solutions to* (**P**) *with initial data $q_1, q_2 \in H$, we have the continuous dependence estimate*

$$|q^1(t) - q^2(t)| \leq |q^1 - q^2|, \qquad \forall t \in [0, \mathcal{T}_\star].$$

PROOF. – The difference $\bar{q}(t) = q_1(t) - q_2(t)$ solves the system

$$\begin{cases} \partial_t \bar{q} + A\bar{q} + B(\boldsymbol{u}(t), \bar{q}) = [h_{q_1} - h_{q_2}] f(t), \\ \bar{q}(0) = q_1 - q_2. \end{cases} \tag{3.13}$$

Taking the scalar product in H of the above equation by $\bar{q}$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{q}|^2 + \|\bar{q}\|^2 - \langle [h_{q_1} - h_{q_2}] f(t), \bar{q}\rangle = 0.$$

From the characterization (2.9)-(2.10), we obtain that

$$[h_{q_1} - h_{q_2}][q_1 - q_2] \geq 0$$

a.e. in $\mathcal{M} \times [0, \mathcal{T}_\star]$. It follows that

$$-\langle [h_{q_1} - h_{q_2}] f(t), \bar{q}\rangle \geq 0.$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{q}|^2 + \|\bar{q}\|^2 \leq 0,$$

and the Poincaré inequality together with the Gronwall lemma entails the desired result. □

REMARK 3.7. – The above assumption on f seems to have been adopted *ad hoc* to obtain uniqueness of solutions to (**P**). Actually, this assumption reflects the physics of the problem under study, as it will be clear below.

3.4 – *A bound in $L^\infty(\mathcal{M})$*

In a system of moist air, the specific humidity is the (dimensionless) ratio of the mass of water vapor to the total mass of the system. One therefore expects that, if the initial distribution of specific humidity q_0 is such that

$$q_0 \in [0,1], \qquad \text{a.e. in } \mathcal{M},$$

then also the resulting solution q to equation (**P**) will enjoy the analogous property

$$q \in [0,1], \qquad \text{a.e. in } \mathcal{M} \times [0, \mathcal{T}_\star].$$

Exploiting the truncation method of Stampacchia, we now prove that this is in fact the case.

THEOREM 3.8. – *Let* $f \leq 0$ *a.e. in* $\mathcal{M} \times [0, \mathcal{T}_\star]$, *and suppose that* $q_0 \in L^\infty(\mathcal{M})$ *with*

$$0 \leq q_0 \leq 1, \qquad \text{a.e. in } \mathcal{M}.$$

Then $q(t) \in L^\infty(\mathcal{M})$ *for a.e.* $t \in [0, \mathcal{T}_\star]$ *and*

$$0 \leq q \leq 1, \qquad \text{a.e. in } \mathcal{M} \times [0, \mathcal{T}_\star].$$

PROOF. – The result will be proved once we show that $q^- \equiv 0$ and $[q-1]^+ \equiv 0$. We preliminarily notice that if q is negative, then $q^- h_q = 0$, since $h_q = 0$ if $q < q_s$ and $q_s \in [0,1]$. Analogously, if q is positive, then again $q^- h_q = 0$, being $q^- = 0$ itself. A multiplication of (**P**) by q^- in H leads then to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|q^-|^2 + \|q^-\|^2 - b(\boldsymbol{u}(t), q, q^-) = 0.$$

If we write q as the difference $q^+ - q^-$, we can use the orthogonality property (2.3) and the fact that $\nabla q^+ = 0$ whenever $q < 0$ to deduce that

$$b(\boldsymbol{u}(t), q, q^-) = b(\boldsymbol{u}(t), q^+, q^-) - b(\boldsymbol{u}(t), q^-, q^-) = 0.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}|q^-|^2 \leq 0,$$

which implies, together with the assumption $q_0^- = 0$, that $q^- = 0$. To prove the upper bound $q \leq 1$, we proceed in a similar way. We multiply (**P**) by $[q-1]^+$ in H and use the fact that $b(\boldsymbol{u}(t), q, [q-1]^+) = b(\boldsymbol{u}(t), q-1, [q-1]^+) = 0$ to infer the energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|[q-1]^+|^2 + \|[q-1]^+\|^2 = \langle h_q f(t), [q-1]^+ \rangle.$$

As h_q and $[q-1]^+$ are positive functions, the assumption $f(t) \leq 0$ forces the right hand side of the above equation to be negative. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}|[q-1]^+|^2 \leq 0,$$

and therefore $[q-1]^+ = 0$, concluding the proof. □

4. – The coupled system

We now deal with the full coupled system temperature-specific humidity. After introducing the physical model, we proceed with the mathematical setting for weak solutions in a similar way to the one exploited for the equation of specific humidity alone. In this section, we also collect all the assumptions on the data and on the nonlinear terms.

4.1 – *The physical model*

Let

$$\mathcal{M} = \mathcal{M}' \times (p_0, p_1)$$

be a cylindrical domain, where $\mathcal{M}' \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\mathcal{M}'$ and $0 < p_0 < p_1$ are real numbers. Given $\mathcal{T}_\star > 0$, a velocity vector field $\boldsymbol{v} : \mathcal{M} \times [0, \mathcal{T}_\star] \to \mathbb{R}^2$ and a function $\omega : \mathcal{M} \times [0, \mathcal{T}_\star] \to \mathbb{R}$, we consider the system of equations

$$\frac{\partial T}{\partial t} + L_1 T + \boldsymbol{v}(t) \cdot \nabla T + \omega(t) \frac{\partial T}{\partial p} - \frac{R}{c_p p} \omega(t)\, T \in \omega^-(t) H(q - q_s) \varphi(T), \tag{4.1}$$

$$\frac{\partial q}{\partial t} + L_2 q + \boldsymbol{v}(t) \cdot \nabla q + \omega(t) \frac{\partial q}{\partial p} \in -\omega^-(t) H(q - q_s) F(T), \tag{4.2}$$

describing the evolution of the temperature T and the specific humidity q in large scale dynamics models of the atmosphere [7, 10]. In order to keep the notation as simple as possible, we will again highlight only the time-dependencies of $\boldsymbol{v}$ and ω, and omit the others. As customary in the PEs of the atmosphere, equations (4.1)-(4.2) are written in the (x, y, p, t) coordinate system, in which the pressure p is used as the vertical coordinate. In the physical problem, $\boldsymbol{u} = (\boldsymbol{v}, \omega)$ is the velocity of the fluid, where

$$\omega = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{4.3}$$

is the corresponding vertical velocity in the (x, y, p) coordinate system, and we here assume it is a given datum. We set $\nabla = (\partial_x, \partial_y)$ and $\Delta = \partial_x^2 + \partial_y^2$ to be the horizontal gradient and Laplacian operators, respectively. In this way, the heat and vapor diffusion operators L_1 and L_2 are defined [7, 10, 15] as

$$L_1 = -\mu_1 \Delta - \nu_1 \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\overline{T}} \right)^2 \frac{\partial}{\partial p} \right], \tag{4.4}$$

$$L_2 = -\mu_2 \Delta - \nu_2 \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\overline{T}} \right)^2 \frac{\partial}{\partial p} \right], \tag{4.5}$$

where μ_i, ν_i, g, R, c_p are positive constants and $\overline{T} = \overline{T}(p)$ is the average temperature over the isobar with pressure p. Concerning the right hand side of (4.1)-(4.2), φ and F are nonlinear functions of the temperature field, ω^- refers to the negative part of ω, while the Heaviside graph $H(r)$ produces different behaviors whether we consider condensation ($q > q_s$) and upward motion ($\omega < 0$) regimes or not.

We partition the boundary of $\mathcal{M}$ as

$$\begin{aligned}
\Gamma_i &= \{(x, y, p) \in \overline{\mathcal{M}} : p = p_1\},\\
\Gamma_u &= \{(x, y, p) \in \overline{\mathcal{M}} : p = p_0\},\\
\Gamma_\ell &= \{(x, y, p) \in \overline{\mathcal{M}} : (x, y) \in \partial\mathcal{M},\ p_0 \le p \le p_1\},
\end{aligned}$$

in order to equip system (4.1)-(4.2) with the following physically reasonable boundary conditions [15]:

$$\begin{aligned}
&\text{on } \Gamma_i : \frac{\partial T}{\partial p} = \alpha(T_* - T), \quad \frac{\partial q}{\partial p} = \beta(q_* - q),\\
&\text{on } \Gamma_u : \frac{\partial T}{\partial p} = 0, \quad q = 0, \\
&\text{on } \Gamma_\ell : \frac{\partial T}{\partial n} = 0, \quad \frac{\partial q}{\partial n} = 0.
\end{aligned} \tag{4.6}$$

Here, n is the outward normal vector to Γ_ℓ, the functions $T_*(x, y)$ and $q_*(x, y)$ are typical temperature and specific humidity distributions at the bottom surface of the atmosphere, and α, β are given positive constants. Finally, we supplement our system with the initial conditions

$$T(x, y, p, 0) = T_0(x, y, p), \qquad q(x, y, p, 0) = q_0(x, y, p). \tag{4.7}$$

4.2 – *Mathematical setting*

As in the previous sections, H will denote the space $L^2(\mathcal{M})$ with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. Due to our boundary conditions, it is convenient to equip the natural space for T

$$V_T = H^1(\mathcal{M})$$

with the scalar product

$$((T, T^\star))_\Gamma = (\nabla T, \nabla T^\star) + (\partial_p T, \partial_p T^\star) + \int_{\Gamma_i} T T^\star \mathrm{d}\Gamma_i.$$

In the same fashion, we define

$$V_q = \{q \in H^1(\mathcal{M}) : \, q = 0 \text{ on } \Gamma_u\},$$

equipped with the inner product

$$((q, q^\star)) = (\nabla q, \nabla q^\star) + (\partial_p q, \partial_p q^\star).$$

It is worth noticing that the above scalar products are equivalent to the standard $H^1(\mathcal{M})$-inner products thanks to the Poincaré inequality and its generalizations, and with the norms

$$\|T\|_\Gamma^2 = ((T,T))_\Gamma, \qquad \|q\|^2 = ((q,q)),$$

the spaces V_T and V_q are closed subspaces of $H^1(\mathcal{M})$. Finally, we define the product spaces

$$\mathcal{H} = H \times H, \qquad \mathcal{V} = V_T \times V_q.$$

With some abuse of notation, we will keep the symbols $(\cdot,\cdot)$ and $|\cdot|$ (resp.$((\cdot,\cdot))$ and $\|\cdot\|$) also for the norm and the scalar product on $\mathcal{H}$ (resp. $\mathcal{V}$), as no confusion will arise throughout the article. In the same way, the symbol $\langle\cdot,\cdot\rangle$ will indicate the duality pairing between any of the spaces V_T, V_q and $\mathcal{V}$ and their duals.

4.3 – *Weak formulation*

Having in mind the set of boundary conditions (4.6), we observe the following. If $T, T^\star \in V_T$, then an integration by parts yields

$$\langle L_1 T, T^\star\rangle = \mu_1(\nabla T, \nabla T^\star) + \nu_1 \int_{\mathcal{M}} \left(\frac{gp}{R\overline{T}}\right)^2 \partial_p T \partial_p T^\star \mathrm{d}\mathcal{M}$$
$$+ \nu_1 \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{T}}\right)^2 \alpha (T - T_*) T^\star \mathrm{d}\Gamma_i.$$

In the same manner, if $q, q^\star \in V_q$, we obtain

$$\langle L_2 q, q^\star\rangle = \mu_2(\nabla q, \nabla q^\star) + \nu_2 \int_{\mathcal{M}} \left(\frac{gp}{R\overline{T}}\right)^2 \partial_p q \partial_p q^\star \mathrm{d}\mathcal{M} + \nu_2 \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{T}}\right)^2 \beta (q - q_*) q^\star \mathrm{d}\Gamma_i.$$

For $T, T^\star \in V_T$, $q, q^\star \in V_q$, $U = (T, q)$, $U^\star = (T^\star, q^\star) \in \mathcal{V}$ and $\boldsymbol{u} = (\boldsymbol{v}, \omega) \in \boldsymbol{V}$, we define the bilinear and trilinear forms

$$a_T(T,T^\star) = \mu_1(\nabla T, \nabla T^\star) + \nu_1 \int_{\mathcal{M}} \left(\frac{gp}{R\overline{T}}\right)^2 \partial_p T \partial_p T^\star \mathrm{d}\mathcal{M} + \nu_1 \alpha \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{\overline{T}}}\right)^2 T T^\star \mathrm{d}\Gamma_i,$$

$$a_q(q,q^\star) = \mu_2(\nabla q, \nabla q^\star) + \nu_2 \int_{\mathcal{M}} \left(\frac{gp}{R\overline{T}}\right)^2 \partial_p q \partial_p q^\star \mathrm{d}\mathcal{M} + \nu_2 \beta \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{\overline{T}}}\right)^2 q q^\star \mathrm{d}\Gamma_i,$$

$$a(U,U^\star) = a_T(T,T^\star) + a_q(q,q^\star),$$

$$b_T(\boldsymbol{u},T,T^\star) = \int_{\mathcal{M}} (\boldsymbol{v}\cdot\nabla T + \omega\partial_p T) T^\star \mathrm{d}\mathcal{M},$$

$$b_q(\boldsymbol{u},q,q^\star) = \int_{\mathcal{M}} (\boldsymbol{v}\cdot\nabla q + \omega\partial_p q) q^\star \mathrm{d}\mathcal{M},$$

$$b(\boldsymbol{u},U,U^\star) = b_T(\boldsymbol{u},T,T^\star) + b_q(\boldsymbol{u},q,q^\star),$$

$$d_T(\omega,T,T^\star) = \int_{\mathcal{M}} \frac{R}{c_p p} \omega T T^\star \mathrm{d}\mathcal{M}.$$

Analogously, we define the linear functionals

$$\ell_T(T^\star) = \nu_1 \alpha \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{\overline{T}}}\right)^2 T_* T^\star \mathrm{d}\Gamma_i,$$

$$\ell_q(q^\star) = \nu_2 \beta \int_{\Gamma_i} \left(\frac{gp_1}{R\overline{\overline{T}}}\right)^2 q_* q^\star \mathrm{d}\Gamma_i,$$

$$\ell(U^\star) = \ell_T(T^\star) + \ell_q(q^\star).$$

The definition of a weak solution to our problem is then similar to that of problem (**P**).

DEFINITION 4.1. – *Let* $(T_0,q_0)\in\mathcal{H}$ *and* $\mathcal{T}_\star>0$ *be given. A vector* $U=(T,q)\in L^2(0,\mathcal{T}_\star;\mathcal{V})\cap C([0,\mathcal{T}_\star];\mathcal{H})$ *is a solution to system* (4.1)-(4.2) *if* $(\partial_t T,\partial_t q)\in L^2(0,\mathcal{T}_\star;\mathcal{V}^*)$ *and, for almost every* $t\in(0,\mathcal{T}_\star]$ *and every* $U^\star\in\mathcal{V}$,

$$\begin{aligned}&\langle\partial_t T,T^\star\rangle + \langle\partial_t q,q^\star\rangle + a(U,U^\star) + b(\boldsymbol{u}(t),U,U^\star)\\ &\quad = d_T(\omega(t),T,T^\star) + \ell(U^\star) + \langle\omega^-(t)h_q\varphi(T),T^\star\rangle - \langle\omega^-(t)h_qF(T),q^\star\rangle,\end{aligned} \tag{4.8}$$

for some $h_q\in L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$ *which satisfies the variational inequality*

$$([q^\star-q_s]^+,1)-([q-q_s]^+,1)\geq\langle h_q,q^\star-q\rangle,\quad \text{a.e. } t\in(0,\mathcal{T}_\star],\quad \forall q^\star\in V_q, \tag{4.9}$$

and $U(0)=(T_0,q_0)$.

As before, (4.9) means that $h_q\in H(q-q_s)$ and thus $0\leq h_q\leq 1$ almost everywhere.

4.4 – *The nonlinearities φ and F*

According to [10], we can define the nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ by

$$F(\zeta) = \frac{q_s \zeta}{p} \left(\frac{RL(\zeta) - c_p R_v \zeta}{c_p R_v \zeta^2 + q_s L(\zeta)^2} \right),$$

and

$$\varphi(\zeta) = \frac{L(\zeta)}{c_p} F(\zeta),$$

where

$$L(\zeta) = c_1 - c_2 \zeta,$$

with c_1, c_2, R, c_p, R_v strictly positive constants (see Remark 4.2). From direct calculations and the fact that $p \in [p_0, p_1]$, we see that F is a globally Lipschitz bounded function, namely

$$|F(\zeta_1) - F(\zeta_2)| \leq C|\zeta_1 - \zeta_2|, \qquad \forall \zeta_1, \zeta_2 \in \mathbb{R}, \tag{4.10}$$

and

$$|F(\zeta)| \leq C, \qquad \forall \zeta \in \mathbb{R}. \tag{4.11}$$

Moreover, φ is globally Lipschitz as well, i.e.

$$|\varphi(\zeta_1) - \varphi(\zeta_2)| \leq C|\zeta_1 - \zeta_2|, \qquad \forall \zeta_1, \zeta_2 \in \mathbb{R}. \tag{4.12}$$

As $F(0) = 0$, we deduce that $\varphi(0) = 0$ and therefore

$$|\varphi(\zeta)| \leq C|\zeta|, \qquad \forall \zeta \in \mathbb{R}. \tag{4.13}$$

REMARK 4.2. – A concrete realization of F and φ can be found in [10]. We have

$$F(T) = \frac{q_s T}{p} \left(\frac{L(T)R - c_p R_v T}{c_p R_v T^2 + q_s L(T)^2} \right).$$

In the equation above, R is the gas constant for dry air, R_v is the gas constant for water vapor and c_p is the specific heat of dry air at constant pressure, while

$$L(T) = 2.5008 \times 10^6 - 2.3 \times 10^3 T$$

is the latent heat of vaporization. Notice that $F(T) \geq 0$ whenever $L(T)R - c_p R_v T \geq 0$, and the latter is an affine decreasing function of T. Moreover,

$$F(T_0) = 0 \quad \text{for} \quad T_0 = \frac{2.5008 \times 10^6 R}{2.3 \times 10^3 R + c_p R_v}.$$

For usual values of the above constants, namely (see [7])

$$R = 287\,JK^{-1}kg^{-1}, \quad R_v = 461.50\,JK^{-1}kg^{-1}, \quad c_p = 1004\,JK^{-1}kg^{-1},$$

it turns out that $T_0 \sim 638K$. Hence, $F(T) \geq 0$ for $T < 638K$, which is a very reasonable upper bound on the temperature in the atmosphere. This justifies the hypothesis $f \leq 0$ a.e. in Theorem 3.6. Indeed, assuming that T is known for the first equation in (2.1), the forcing term may be written in terms of F as

$$f = -F(T)\omega^{-}.$$

Therefore, if we assume T to be bounded from above by T_0, it turns out that $F(T) \geq 0$. Consequently, $f \leq 0$, and uniqueness of solution to the specific humidity problem (2.1) is then assured.

4.5 – *Assumptions on the data*

Throughout the paper, the boundary data appearing in (4.6) will be assumed to satisfy

$$T_*, q_* \in L^2(\Gamma_i).$$

Concerning the average temperature $\overline{T}(p)$ appearing in (4.4)-(4.5), we will require the existence of two positive constants $\overline{T}_*, \overline{T}^*$ such that

$$\overline{T}_* \leq \overline{T}(p) \leq \overline{T}^*. \tag{4.14}$$

As in the previous sections, the velocity vector field $\boldsymbol{u}$ will be given, time-dependent and satisfying

$$\boldsymbol{u} \in L^4(0, \mathcal{T}_\star; \boldsymbol{V}) \cap L^\infty(0, \mathcal{T}_\star, \boldsymbol{H}). \tag{4.15}$$

5. – Existence results for the coupled system

In this section, we prove the existence of solutions to problem (4.1)-(4.2). Since the idea of the proof is similar to that of Section 3, we will only highlight the main differences. The main result reads as follows.

THEOREM 5.1. – *Let* $\boldsymbol{u} \in L^4(0, \mathcal{T}_\star; \boldsymbol{V}) \cap L^\infty(0, \mathcal{T}_\star, \boldsymbol{H})$ *be given. For any* $(T_0, q_0) \in \mathcal{H}$ *and any* $\mathcal{T}_\star > 0$, *problem* (4.1)-(4.2) *admits at least one solution* $U = (T, q) \in L^2(0, \mathcal{T}_\star; \mathcal{V}) \cap C([0, \mathcal{T}_\star]; \mathcal{H})$, *such that* $\partial_t U \in L^2(0, \mathcal{T}_\star; \mathcal{V}^*)$.

5.1 – *Preliminary estimates*

We start by the usual estimates on the (tri-,bi-)linear functionals defined in the previous section. We report the following lemma without proof.

LEMMA 5.2. – *Let* $(U, U^\star) \in \mathcal{V}$ *and* $\boldsymbol{u} \in \boldsymbol{V}$. *There exist positive constants* κ_i, K_i *such that the following hold.*

$$|a_T(T, T^\star)| \leq K_1 \|T\|_\Gamma \|T^\star\|_\Gamma,$$

$$a_T(T, T) \geq \kappa_1 \|T\|_\Gamma^2,$$

$$|a_q(q, q^\star)| \leq K_2 \|q\| \|q^\star\|,$$

$$a_q(q, q) \geq \kappa_2 \|q\|^2,$$

$$b_T(\boldsymbol{u}, T, T) = 0,$$

$$b_q(\boldsymbol{u}, q, q) = 0,$$

$$|b(\boldsymbol{u}, U, U^\star)| \leq C \|\boldsymbol{u}\|_{\boldsymbol{V}} |U|^{1/2} \|U\|^{1/2} \|U^\star\|,$$

$$|d_T(\omega, T, T^\star)| \leq K_3 |\omega| |T|^{1/4} \|T\|_\Gamma^{3/4} |T^\star|^{1/4} \|T^\star\|_\Gamma^{3/4},$$

$$|\ell_T(T^\star)| \leq K_4 |T^\star|,$$

$$|\ell_q(q^\star)| \leq K_5 |q^\star|.$$

In particular, we infer the existence of positive constants K, κ such that

$$|a(U, U^\star)| \leq K \|U\| \|U^\star\| \quad \text{and} \quad a(U, U) \geq \kappa \|U\|^2, \tag{5.1}$$

for any $U, U^\star \in \mathcal{V}$.

5.2 – *The approximate solutions*

We aim to approximate the singular problem (4.8)-(4.9) in the exact same way as we did for problem (**P**). The corresponding ε-approximation then reads

$$\begin{aligned} \langle \partial_t T, T^\star \rangle + \langle \partial_t q, q^\star \rangle + a(U, U^\star) + b(\boldsymbol{u}(t), U, U^\star) = d_T(\omega(t), T, T^\star) \\ + \ell(U^\star) + \langle \omega^-(t) H_\varepsilon(q - q_s) \varphi(T), T^\star \rangle - \langle \omega^-(t) H_\varepsilon(q - q_s) F(T), q^\star \rangle, \end{aligned} \tag{5.2}$$

where we no longer need any variational inequalities, as the map $H_\varepsilon(\,\cdot\,)$ is now well-defined and continuous. We then have

THEOREM 5.3. – *Let* $(T_0, q_0) \in \mathcal{H}$ *and* $\mathcal{T}_\star > 0$ *be given and* $\boldsymbol{u}$ *satisfying* (4.15). *For every* $\varepsilon > 0$, *there exists at least one solution*

$$U_\varepsilon = (T_\varepsilon, q_\varepsilon) \in C([0,\mathcal{T}_\star];\mathcal{H}) \cap L^2(0,\mathcal{T}_\star;\mathcal{V})$$

to (5.2), *such that*

$$\partial_t U_\varepsilon = (\partial_t T_\varepsilon, \partial_t q_\varepsilon) \in L^2(0,\mathcal{T}_\star;\mathcal{V}^*).$$

Unlike before, we will only prove an *existence* result, as uniqueness seems more complicated, and, in fact, not needed at this stage. Indeed, once a solution U_ε is proved to exist and it satisfies analogous bounds as those of the solution to $(\mathbf{P}_\varepsilon)$, we will be able to implement the limiting procedure in the exact same way as we did in Section 3. One more time, we will only show how to obtain the usual energy estimates needed to set up the standard Galerkin procedure used to prove Theorem 5.3.

LEMMA 5.4. – *Let* $U_\varepsilon = (T_\varepsilon, q_\varepsilon)$ *be a solution to* (5.2). *Then*

$$|U_\varepsilon(t)|^2 \le \mathrm{e}^{C\mathcal{T}_\star}(|U_0|^2 + C\mathcal{T}_\star), \qquad \forall t \in [0,\mathcal{T}_\star], \tag{5.3}$$

and

$$\int_0^{\mathcal{T}_\star} \|U_\varepsilon(s)\|^2 \mathrm{d}s \le \mathcal{Q}(|U_0|), \tag{5.4}$$

with $\mathcal{Q}(\,\cdot\,)$ *independent of* ε.

PROOF. – Setting $U^\star = U_\varepsilon$ in (5.2), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U_\varepsilon|^2 + a(U_\varepsilon, U_\varepsilon) &= d_T(\omega(t), T_\varepsilon, T_\varepsilon) + \ell(U_\varepsilon)\\ &\quad + \langle \omega^-(t) H_\varepsilon(q_\varepsilon - q_s)\varphi(T_\varepsilon), T_\varepsilon\rangle - \langle \omega^-(t) H_\varepsilon(q_\varepsilon - q_s) F(T_\varepsilon), q_\varepsilon\rangle\end{aligned}$$

From (5.1) and using Lemma 5.2, we then deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U_\varepsilon|^2 + \kappa\|U_\varepsilon\|^2 &\le C|U_\varepsilon|^2 + C + d_T(\omega(t), T_\varepsilon, T_\varepsilon)\\ &\quad + \langle \omega^-(t) H_\varepsilon(q_\varepsilon - q_s)\varphi(T_\varepsilon), T_\varepsilon\rangle - \langle \omega^-(t) H_\varepsilon(q_\varepsilon - q_s) F(T_\varepsilon), q_\varepsilon\rangle.\end{aligned}$$

The trilinear term d_T can be estimated taking advantadge of Lemma 5.2 and Young inequality:

$$d_T(\omega(t), T_\varepsilon, T_\varepsilon) \le K_3|\omega(t)|\,|T_\varepsilon|^{1/2}\|T_\varepsilon\|_\Gamma^{3/2} \le C\|\boldsymbol{u}(t)\|_{\boldsymbol{H}}^4|T_\varepsilon|^2 + \frac{\kappa}{4}\|T_\varepsilon\|_\Gamma^2.$$

Using (4.13), it follows that

$$\begin{aligned}\langle \omega^-(t)H_\varepsilon(q_\varepsilon - q_s)\varphi(T_\varepsilon), T_\varepsilon\rangle &\leq C(|\omega(t)|, |T_\varepsilon|^2) \leq C|\omega(t)| \|T_\varepsilon\|^2_{L^4(\mathcal{M})} \\ &\leq C|\omega(t)| \, |T_\varepsilon|^{1/2} \|T_\varepsilon\|^{3/2}_\Gamma \leq C|\omega(t)|^4 |T_\varepsilon|^2 + \frac{\kappa}{8}\|T_\varepsilon\|^2_\Gamma \\ &\leq C\|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}} |U_\varepsilon|^2 + \frac{\kappa}{8}\|U_\varepsilon\|^2,\end{aligned}$$

where we took advantage of the inequality

$$\|T_\varepsilon\|_{L^4(\mathcal{M})} \leq C|T_\varepsilon|^{1/4} \|T_\varepsilon\|^{3/4}_\Gamma.$$

The second nonlinear term is easy. Indeed, from (4.12), we have

$$-\langle \omega^-(t)H_\varepsilon(q_\varepsilon - q_s)F(T_\varepsilon), q_\varepsilon\rangle \leq C|\omega(t)| \, |q_\varepsilon| \leq C\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}} + \frac{\kappa}{8}\|U_\varepsilon\|^2.$$

Therefore, we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|U_\varepsilon|^2 + \kappa\|U_\varepsilon\|^2 \leq C(1 + \|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}})|U_\varepsilon|^2 + C(1 + \|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}}). \tag{5.5}$$

On one hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}|U_\varepsilon|^2 \leq C(1 + \|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}})|U_\varepsilon|^2 + C(1 + \|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}}),$$

so that estimate (5.3) follows from the Gronwall Lemma, together with the assumption $\boldsymbol{u} \in L^\infty(0, \mathcal{T}_\star; \boldsymbol{H})$. With (5.3) at hand, it is a standard matter to deduce (5.4) from (5.5). □

Again, estimate (5.3) implies that $U_\varepsilon \in L^\infty(0, \mathcal{T}_\star; \mathcal{H})$, with

$$\|U_\varepsilon\|_{L^\infty(0,\mathcal{T}_\star;\mathcal{H})} \leq \mathcal{Q}(|U_0|), \tag{5.6}$$

with $\mathcal{Q}(\,\cdot\,)$ independent of ε.

LEMMA 5.5. – *Let U_ε be a solution to* (5.2). *Then*

$$\int_0^{\mathcal{T}_\star} \|\partial_t U_\varepsilon(s)\|^2_* \mathrm{d}s \leq \mathcal{Q}(|U_0|), \tag{5.7}$$

with $\mathcal{Q}(\,\cdot\,)$ independent of ε.

PROOF. – Let $U^\star \in \mathcal{V}$ with $\|U^\star\| \leq 1$. From (5.2), we have

$$\begin{aligned}\langle \partial_t U_\varepsilon, U^\star\rangle &= -a(U_\varepsilon, U^\star) - b(\boldsymbol{u}(t), U_\varepsilon, U^\star) + d_T(\omega(t), T_\varepsilon, T^\star) + \ell(U^\star) \\ &\quad + \langle \omega^-(t)H_\varepsilon(q_\varepsilon - q_s)\varphi(T_\varepsilon), T^\star\rangle - \langle \omega^-(t)H_\varepsilon(q_\varepsilon - q_s)F(T_\varepsilon), q^\star\rangle.\end{aligned}$$

Now, Lemma 5.2 and (5.6) ensure that

$$\begin{aligned}|-a(U_\varepsilon,U^\star)-b(\boldsymbol{u}(t),U_\varepsilon,U^\star)+\ell(U^\star)| &\le C\|U_\varepsilon\|+\mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}\|U_\varepsilon\|^{1/2}+C\\ &\le C\|U_\varepsilon\|+\mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2+C\end{aligned}$$

Concerning the other terms, we have

$$|d_T(\omega(t),T_\varepsilon,T^\star)|\le C\|\omega(t)\|_{L^3(\mathcal{M})}|T_\varepsilon|\|T^\star\|_{L^6(\mathcal{M})}\le \mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}},$$

Also, in view of (4.12), (4.13) and (5.6), we infer the bounds

$$|\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)\varphi(T_\varepsilon),T^\star\rangle|\le C\|\omega(t)\|_{L^3(\mathcal{M})}|T_\varepsilon|\|T^\star\|_{L^6(\mathcal{M})}\le \mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}},$$

and

$$|\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)F(T_\varepsilon),q^\star\rangle|\le C\|\boldsymbol{u}(t)\|_{\boldsymbol{H}}.$$

Collecting the above inequalities, we end up with

$$|\langle\partial_t U_\varepsilon,U^\star\rangle|\le C\|U_\varepsilon\|+\mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2+C\|\boldsymbol{u}(t)\|_{\boldsymbol{H}}+C,$$

which implies that

$$\|\partial_t U_\varepsilon\|_*\le C\|U_\varepsilon\|+\mathcal{Q}(|U_0|)\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2+C\|\boldsymbol{u}(t)\|_{\boldsymbol{H}}+C.$$

Thus, (5.7) is then a consequence of the assumption $\boldsymbol{u}\in L^\infty(0,\mathcal{T}_\star;\boldsymbol{H})\cap L^4(0,\mathcal{T}_\star;\boldsymbol{V})$ and the previous Lemma 5.4. □

5.3 – *Passage to the limit (II)*

The procedure to pass to the limit as $\varepsilon\to 0$ in the approximate equation (5.2) is now very similar to that of Section 3.2. The main difference consists in handling the nonlinear terms, as the convergence of all the linear terms and the verification of the variational inequality (4.9) for the $L^\infty(\mathcal{M})$ weak-$*$ limit of $H_\varepsilon(q_\varepsilon-q_s)$ goes through in the exact same way as for problem $(\mathbf{P}_\varepsilon)$.

From the previous paragraph, the sequence $\{U_\varepsilon\}$ is bounded in $L^2(0,\mathcal{T}_\star;\mathcal{V})$, $\{\partial_t U_\varepsilon\}$ is bounded in $L^2(0,\mathcal{T}_\star;\mathcal{V}^*)$, and $\{H_\varepsilon(q_\varepsilon-q_s)\}$ is bounded in $L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$. Therefore, up to subsequences, we obtain the following convergences:

- $U_\varepsilon\rightharpoonup U$ weakly in $L^2(0,\mathcal{T}_\star;\mathcal{V})$,
- $\partial_t U_\varepsilon\rightharpoonup \partial_t U$ weakly in $L^2(0,\mathcal{T}_\star;\mathcal{V}^*)$,
- $U_\varepsilon\to U$ strongly in $L^2(0,\mathcal{T}_\star;\mathcal{H})$, and therefore (up to a subsequence) almost everywhere on $[0,\mathcal{T}_\star]$. Since $U_\varepsilon\in C([0,\mathcal{T}_\star];H)$ for every ε, by the dominated convergence theorem we deduce strong convergence in $L^p(0,\mathcal{T}_\star;H)$ for

every $p \in [1,\infty)$, namely

$$\lim_{\varepsilon\to 0}\|T_\varepsilon - T\|_{L^p(0,\mathcal{T}_\star;H)} = 0, \qquad \forall p \in [1,\infty). \tag{5.8}$$

- $H_\varepsilon(q_\varepsilon - q_s) \rightharpoonup h_q$ weak-$*$ in $L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$,

for some function $U=(T,q)\in L^2(0,\mathcal{T}_\star;\mathcal{V})$ satisfying $\partial_t U \in L^2(0,\mathcal{T}_\star;\mathcal{V}^*)$ and some $h_q \in L^\infty(\mathcal{M}\times[0,\mathcal{T}_\star])$. Notice that, as before, $U \in C([0,\mathcal{T}_\star];\mathcal{H})$. We want to show that

$$\int_0^{\mathcal{T}_\star}\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)\varphi(T_\varepsilon),T^\star\rangle\mathrm{d}t \to \int_0^{\mathcal{T}_\star}\langle\omega^-(t)h_q\varphi(T),T^\star\rangle\mathrm{d}t,$$

for all $T^\star \in L^2(0,\mathcal{T}_\star;V_T)$. Let us first write

$$\begin{aligned}
&\int_0^{\mathcal{T}_\star}\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)\varphi(T_\varepsilon),T^\star\rangle - \langle\omega^-(t)h_q\varphi(T),T^\star\rangle\mathrm{d}t\\
&\quad= \int_0^{\mathcal{T}_\star}\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)\big[\varphi(T_\varepsilon)-\varphi(T)\big],T^\star\rangle\mathrm{d}t\\
&\qquad+\int_0^{\mathcal{T}_\star}\langle\omega^-(t)\big[H_\varepsilon(q_\varepsilon-q_s)-h_q\big]\varphi(T),T^\star\rangle\mathrm{d}t.
\end{aligned}$$

On one hand, using (4.13) and (5.8) we have

$$\begin{aligned}
&\left|\int_0^{\mathcal{T}_\star}\langle\omega^-(t)H_\varepsilon(q_\varepsilon-q_s)\big[\varphi(T_\varepsilon)-\varphi(T)\big],T^\star\rangle\mathrm{d}t\right|\\
&\quad\le\int_0^{\mathcal{T}_\star}\int_{\mathcal{M}}|\omega(t)||\varphi(T_\varepsilon)-\varphi(T)||T^\star|\mathrm{d}\mathcal{M}\mathrm{d}t\\
&\quad\le C\int_0^{\mathcal{T}_\star}\int_{\mathcal{M}}|\omega(t)||T_\varepsilon-T||T^\star|\mathrm{d}\mathcal{M}\mathrm{d}t\\
&\quad\le C\int_0^{\mathcal{T}_\star}\|\omega(t)\|_{L^3(\mathcal{M})}|T_\varepsilon-T|\|T^\star\|_{L^6(\mathcal{M})}\mathrm{d}t\\
&\quad\le C\int_0^{\mathcal{T}_\star}|\omega(t)|^{1/2}\|\omega(t)\|^{1/2}|T_\varepsilon-T|\|T^\star\|\mathrm{d}t\\
&\quad\le C\|\boldsymbol{u}\|^{1/2}_{L^\infty(0,\mathcal{T}_\star;H)}\|\|\boldsymbol{u}\|^{1/2}_{L^2(0,\mathcal{T}_\star;V)}\|T_\varepsilon-T\|_{L^4(0,\mathcal{T}_\star;H)}\|T^\star\|_{L^2(0,\mathcal{T}_\star;V_T)}.
\end{aligned}$$

On the other hand, since $T \in C([0,\mathcal{T}_\star];H)$,

$$\int_0^{\mathcal{T}_\star}\int_{\mathcal{M}} |\omega^-(t)||\varphi(T)||T^\star| \mathrm{d}\mathcal{M}\mathrm{d}t \le C \int_0^{\mathcal{T}_\star}\int_{\mathcal{M}} |\omega^-(t)||T||T^\star| \mathrm{d}\mathcal{M}\mathrm{d}t$$

$$\le C \int_0^{\mathcal{T}_\star} \|\omega(t)\|_{L^3(\mathcal{M})}|T|\|T^\star\|_{L^6(\mathcal{M})}\mathrm{d}t$$

$$\le C \int_0^{\mathcal{T}_\star} \|\omega(t)\|_{\boldsymbol{V}}|T|\|T^\star\|_{\Gamma}\mathrm{d}t$$

$$\le C\|T\|_{C([0,\mathcal{T}_\star];H)}\|\boldsymbol{u}\|_{L^4(0,\mathcal{T}_\star;\boldsymbol{V})}\|T^\star\|_{L^2(0,\mathcal{T}_\star;V_T)},$$

we see that $\omega^-\varphi(T)T^\star \in L^1(\mathcal{M}\times[0,\mathcal{T}_\star])$. Therefore, the weak-$*$ convergence of $H_\varepsilon(q_\varepsilon - q_s)$ is enough to pass to the limit. Hence, the required convergence is proven.

The trilinear form d_T can be handled similarly, while the second nonlinear term is even easier, as the nonlinearity F is a uniformly bounded and globally Lipschitz function. Thus, the proof of Theorem 5.1 is achieved.

5.4 – *Positivity of solutions*

As in Paragraph 3.4, we establish the positivity of T and q, provided the initial and boundary data are assumed to be positive.

PROPOSITION 5.6. – *Suppose T_0, q_0, T_*, q_* are positive functions. Then $U(t) = (T(t), q(t)) \ge 0$ for all $t \in [0,\mathcal{T}_\star]$.*

PROOF. – Setting $U^\star = -U^- = (-T^-, -q^-)$ in (4.8), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U^-|^2 + a(U^-,U^-)$$
$$= d_T(\omega(t),T^-,T^-) - \ell(U^-) - \langle\omega^-(t)h_q\varphi(T),T^-\rangle + \langle\omega^-(t)h_qF(T),q^-\rangle.$$

From the nonnegativity of T_* and q_* we readily see that $\ell(U^-)\ge 0$, while Lemma 5.2 ensures that

$$d_T(\omega(t),T^-,T^-) \le C|\omega(t)|^4|T^-|^2 + \frac{\kappa}{4}\|T^-\|_\Gamma^2.$$

The first nonlinear term can be estimated as before as

$$\begin{aligned}|\langle\omega^-(t)h_q\varphi(T),T^-\rangle| &\leq C\int_{\mathcal{M}}|\omega(t)||T||T^-|\mathrm{d}\mathcal{M}=C\int_{\mathcal{M}}|\omega(t)||T^-|^2\mathrm{d}\mathcal{M}\\ &\leq C|\omega(t)|\|T^-\|^2_{L^4(\mathcal{M})}\leq C|\omega(t)||T^-|^{1/2}\|T^-\|^{3/2}_{\Gamma}\\ &\leq C|\omega(t)|^4|T^-|^2+\frac{\kappa}{4}\|T^-\|^2_{\Gamma}\leq C\|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}}|U^-|^2+\frac{\kappa}{4}\|U^-\|^2.\end{aligned}$$

As in the proof of Theorem 3.8, the second nonlinear term is identically zero, in view of the fact that $q^-h_q=0$. Therefore, a further application of Lemma 5.2 provides us with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|U^-|^2+\kappa\|U^-\|^2\leq C(1+\|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}})|U^-|^2.$$

Since $\boldsymbol{u}\in L^\infty(0,\mathcal{T}_\star;\boldsymbol{H})$ and $U^-(0)=0$, the Gronwall inequality allows us to conclude the proof. □

5.5 – *Concluding remarks*

REMARK 5.7. – We were unable to use the maximum principle to derive an upper bound for the temperature $T\leq M$. Although we might be perhaps able to overcome this technical difficulty, it appears, after further investigation, that the model that we consider has a fundamental limitation. Indeed, although this model plays an important role in studies on humidity [10, 11], it is a simplified model, the first one (simplest one) accounting for saturation. The water (rain) which is formed "quits" the system and condensation is not accounted for. In particular, the system seems to keep adding energy, because the loss of energy corresponding to the transformation of vapor into water is not taken into account. A more satisfactory model will, at least, involve three components, air, vapor and liquid water or a suitable mechanism of loss of energy. Such systems will be studied in a subsequent work.

REMARK 5.8 (On the coupling with the fluid equations). – In a future work we intend to couple the equations above with the fluid equations for $\boldsymbol{u}$ given by either the Navier-Stokes equations or the PEs. The main question is then whether the assumption made on $\boldsymbol{u}$ in Theorem 5.1, namely $\boldsymbol{u}\in L^4(0,\mathcal{T}_\star;\boldsymbol{V})\cap L^\infty(0,\mathcal{T}_\star,\boldsymbol{H})$, is consistent with the estimates actually available. As said, two cases are of interest.

1) Coupling with the Navier-Stokes equations. Since we are in space dimension three, we can only expect estimates and existence of $\boldsymbol{u}\in L^4(0,\mathcal{T}_\star;\boldsymbol{V})\cap L^\infty(0,\mathcal{T}_\star,\boldsymbol{H})$ on a small time interval $[0,\mathcal{T}_\star]$. On that interval though $\boldsymbol{u}\in L^\infty(0,\mathcal{T}_\star,\boldsymbol{V})$ (strong solutions) and our hypothesis is satisfied.

2) More interesting for us is the case where we couple these equations with the three dimensional PEs [15, 18]. From the results of Cao and Titi [1] and Kobelkov [12], we infer that the first two components of the velocity vector field $\boldsymbol{u} = (\boldsymbol{v}, \omega)$ satisfy $\boldsymbol{v} \in L^2(0, \mathcal{T}_\star; H^2) \cap L^\infty(0, \mathcal{T}_\star, H^1)$, for general domains. A delicate calculation reproducing that in Lemma 3.1 of [18] then shows that $\omega \in L^4(0, \mathcal{T}_\star; H^1)$. Again, $\boldsymbol{u}$ is seen to fulfill our assumptions. Note also that, in the space periodic case, a solution $\boldsymbol{v}$ to the primitive equations can be found in any space $L^2(0, \mathcal{T}_\star; H^{m+1}) \cap L^\infty(0, \mathcal{T}_\star, H^m)$ for every $m \geq 0$, provided natural assumptions are made on the initial data. Hence, the whole vector $\boldsymbol{u}$ belongs to such spaces as well (see [17, 18]).

The remarks above are only meant to show that the hypothesis made on $\boldsymbol{u}$ are realistic. It does not mean that it would be an easy exercise to couple the equations for T and q with those for $\boldsymbol{u}$, as several challenging additional terms actually appear in the coupling. We intend to address this question in a future work [2].

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