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On a Homogenization Effect for a Two-Dimensional Diffusion Plus Transport Equation

LUC TARTAR

In memory of Enrico Magenes

Because I consider that inducing students and researchers in error is the worst possible sin for a teacher, I wondered for many years why it is so often done in the academic world, and since I felt isolated on this question, I attributed my difference in approach to my religious training: although I first rejected what I had been told in my youth, I later decided to keep the good principles of all religions, and it is my interpretation of the parable of talents that part of my duty is to transmit what I have understood about the defects of models in continuum mechanics and physics, and to develop new mathematical tools for going further on the path of understanding how the universe functions, but since I met with discrimination, slander, and threat, one may say that I was trying to swim against the current: it suggests that I was right, of course.

It is natural that mathematicians influenced by their own religion follow a different path than mine, more adapted to their own personality, and if on the occasion of his seventieth birthday, I had written in [Ta93] that in the warm atmosphere around Enrico MAGENES in Pavia, I felt as if there was an Italian branch on my family tree, I could have added that since I had been rejected by my mentors in France, and it was already clear that I had not found the friendly and peaceful environment which I had hoped for when I crossed the Atlantic, it looked as he was my only uncle left. I regret not to have told him that he was a model for me, although I was too limited by my lack of administrative abilities to try to follow his own path of action.

During my sabbatical semester at Politecnico di Milano, in the fall of 2007, I tried to advance as much as I could on the writing of my fourth book in the Springer lecture notes series of Unione Matematica Italiana, on homogenization [Ta09], and I hoped to see Enrico MAGENES each time I went to a meeting (adunanza) of Istituto Lombardo Accademia di Scienze e Lettere, but as when I saw him for the last time in Pavia, my pleasure to see him again was limited by the sadness of finding him a little weaker each time.

When I shall soon retire, I plan to often use the train connection from Neuchâtel to Milano for resuming going to the adunanze dell'Istituto Lombardo Accademia di Scienze e Lettere, and besides remembering that I certainly owe to

Enrico MAGENES having been elected Membro Straniero, I shall regret not having discussed with him about my future plans, like trying to help develop the study of science (which includes mathematics) in parts of the world where it was neglected in the past, or where it never really existed, and to teach how to avoid some of the indoctrination which developed in many places. Part of my general plan is to review the historical evolution of mechanics and physics for discussing which errors were successively made, and check if one already knows a way to correct each of them, or if one needs a theory *beyond partial differential equations* which I have tried to develop.

Here, I shall quickly describe some general ideas before discussing a homogenization question which seems promising, although its precise relation to turbulence is uncertain at the moment.

1. – Learning mechanics and physics

Problems in mechanics/physics played an important role in the past for inducing mathematicians to create new mathematical tools, but learning mechanics/physics is difficult for mathematicians, and they are easily lured in wrong directions: lacking basic knowledge in mechanics or physics is often the reason why many mathematicians fall prey to fashions, whose leaders play with terms from mechanics, physics, or engineering, but show a deluded level of understanding of these fields.

Since I was interested in doing something useful with my mathematical ability, I preferred in 1965 to go study at École Polytechnique (then in Paris on the Montagne Sainte Geneviève) instead of at École Normale Supérieure, and besides giving me the opportunity to learn analysis from Laurent SCHWARTZ (1915-2002) the first year, and "numerical analysis" from Jacques-Louis LIONS (1928-2001) the second year, I also learned some *thermodynamics* in the first year from Paul Mary Ferdinand Maurice ROY (1899-1985) and *classical mechanics* in the first year and *continuum mechanics* in the second year from Jean MANDEL (1907-1982), and I learned the various aspects of *physics* (classical, relativistic, quantic, and statistical) from many different teachers over the two years of study, and some *chemistry*.

In the fall of 1966, having already understood that I could not pursue a career of engineer because of my lack of administrative ability, I decided to do research in mathematics, choosing J.-L. LIONS as advisor since he was supposed to be more applied than his own advisor. Although L. SCHWARTZ had developed the theory of distributions which is the language for linear partial differential equations in continuum mechanics and physics, and I learned from J.-L. LIONS the mathematical tools known in the late 1960s for attacking the linear and nonlinear partial differential equations in continuum mechanics and physics, which include technical results concerning the spaces introduced by Sergei L'vovich SOBOLEV (1908-1989), I understood after a few years that my mentors were not really interested in developing a mathematical understanding of continuum mechanics or physics in the style of what I was looking for. Nevertheless, I could not have found a better environment for studying mathematics and doing research, since no mathematician from the preceding generation had developed the interests and knowledge which I acquired myself later.

Although I was never exposed to a description of how some ideas in mathematics, mechanics, and physics evolved along the centuries, I find it important to explain how the points of view in mechanics differed in the 18th, 19th and 20th centuries.

2. – Historical developments

Velocity was an easy concept, but acceleration was not, even to Galileo GALILEI (1564-1642); however, his experiments on falling bodies (from the top of the leaning tower in Pisa or along an inclined plane) or on what influences the period of a pendulum may have helped Isaac NEWTON (1643-1727) understand that force is mass times acceleration; for defining velocity and acceleration in a mathematical way, he had to invent *differential calculus*, which Gottfried Wilhelm VON LEIBNIZ (1646-1716) made more efficient.

The mathematical development of differential calculus opened the way for the study of *differential equations*, the language of *classical mechanics*, which is the *18th century point of view* in mechanics, dealing with rigid bodies (a natural simplifying assumption in *celestial mechanics*, for example), and Giuseppe Luigi LAGRANGIA (1736-1813) stands as one of its pioneers. ([1])

Differential calculus also opened the way for the study of *partial differential equations*, the language of *continuum mechanics*, which is the *19th century point of view* in mechanics, although Leonhard EULER (1707-1783) pioneered it in the 18th century, introducing both the physical *Eulerian point of view* and the mathematical *"Lagrangian" point of view* for fluids; ([2]) putting him aside, it was the work of Augustin Louis CAUCHY (1789-1857), Gabriel LAMÉ (1795-1870), Gabrio PIOLA (1794-1850), and Adhémar Jean Claude BARRÉ DE SAINT-VENANT (1797-1886), who set up the basic notions for elastic solids, while Claude Louis Marie Henri NAVIER (1785-1836), SAINT-VENANT, and George Gabriel STOKES

([1]) Since one always uses the French form of his name, Joseph Louis LAGRANGE, one usually forgets to mention that he was Italian.

([2]) If LAGRANGIA rediscovered the "Lagrangian" point of view after EULER, he discovered "Hamiltonian" systems much before William Rowan HAMILTON (1805–1865).

(1819-1903) set up the basic notions for viscous fluids. This description of 19th century subjects would not be complete without mentioning the work of James Clerk Maxwell (1831-1879) for unifying electricity and magnetism, but one usually forgets to say that it was Oliver Heaviside (1850-1925) who wrote the system of partial differential equations which one now calls the Maxwell equation, so that I call it the *Maxwell-Heaviside equation*.

The point of view of classical mechanics was somewhat wrong, since rigid bodies do not exist and real materials are slightly elastic (under a threshold): one may consider a rigid body as a limit when some elasticity coefficients tend to infinity, but this has the effect of *replacing the speed of sound by* $+\infty$.

Actually, there were various hints that the point of view of continuum mechanics is somewhat wrong too, but there seems to have been an unscientific habit of *hiding the discrepancies of the models which one uses*. Mathematicians were probably too busy trying to prove the results postulated in mechanics/physics during the preceding century for taking the time to point out at least the errors in reasoning which are usually made by practitioners: since an engineering approach is used most of the time, of inventing games which give results similar to what is observed, mathematicians should explain that A implies B is certainly not the same as B implies A, (3) so that it is silly to pretend that nature plays those invented games, and practitioners should be warned that they just put in their hypotheses what they want to find in their conclusion, which is certainly not a scientific attitude!

(3) The referee thought that I should make this point more clear to the reader. One may imagine that various games $A_1, \ldots, A_m$ each have in their conclusions something resembling a particular pattern B which nature produces in some situations, and one should analyse if some of these games are consistent with nature's laws (which physicists are supposed to find), and this should be done when finding a first game A_1 which seems to imply B (instead of adopting the naive belief that it "must" be the game played by nature). In the first paragraph of a review (in The Times Higher Education Supplement, March 23, 1984) of a book by John C. Polkinghorne *The Quantum World*, Roger Penrose wrote "Quantum theory, it may be said, has two things in its favour and only one against. First, it agrees with all the experiments. Second, it is a theory of astonishing and profound mathematical beauty. The only thing to be said against the theory is that it makes absolutely no sense." It means that the quantum mechanics game (which I find silly) is such a game A_1 which one uses by default, because one has not yet found better games $A_2, \ldots, A_m$ for shedding more light on how nature functions. When (physical) quasi-crystals were discovered, and physicists started playing a game of (mathematical) quasi-crystals based on Penrose tilings, R. Penrose should have told those theoretical physicists that they lacked physical intuition, for there are around one million atoms in the small direction of a ribbon which is about a tenth of a millimeter thick, and it is silly to assume that the way they rearrange for evacuating heat and releasing stress during the cooling process is a two-dimensional problem.

The speed of light c was first deduced from delays appearing in the measurements of the position of Io (a moon of Jupiter), by Ole Christensen RØMER (1644-1710), who completed his measurements as an assistant to the first director of the (newly founded) observatory in Paris, Giovanni Domenico CASSINI (1625-1712), who had previously proposed the finite speed of light for explaining the discrepancies, before looking for another explanation.

The reason why MAXWELL used a complicated mechanistic framework for unifying electricity and magnetism was that he believed in a "solid" aether. (4) He also noticed that perturbations in his model travel at the speed of light c, but he could not understand why, since the idea that visible light is but an electromagnetic wave (of wavelength between 0.4 and 0.8 microns) was difficult to imagine at that time.

The experiments which Albert Abraham MICHELSON (1852-1931) and Edward Williams MORLEY (1838-1923) performed around 1887 made obsolete the ideas about aether: they measured the speed of light in various directions, and, perhaps unexpectedly, the measured speed of light had the same value c in every direction. The proposition in 1889 of a contraction of physical objects in the direction of their movement by George Francis FITZ GERALD (1851-1901), made more precise in 1892 by Hendrik Antoon LORENTZ (1853-1928), who also added a change of time, forced upon Jules Henri POINCARÉ (1854-1912) to consider the transforms which send any solution of the wave equation to another solution of the wave equation, (5) and a consequence of the change of time is that *there cannot exist any instantaneous force acting at a distance* (like gravitation!), since the notion of instantaneity makes no sense, because *there is no universal time*, which is what *Poincaré's principle of relativity* is about.

A consequence of Poincaré's principle of relativity is that "particles" cannot know where the other "particles" which interact with them are "at the same time", and they just feel various fields which propagate with the speed of light c, and with which all "particles" interact: in mathematical terms, one should be dealing with systems of partial differential equations of *hyperbolic* type, probably semi-linear and only having the speed of light c as characteristic speed. However, the experiments concerning light had been interpreted as laws concerning matter, although most physicists were (and still are) stuck with 18th century ideas of "particles", i.e. they had not understood the 19th century point of view that continuum mechanics deals with waves and hyper-

(4) For reasons which I do not understand, electricity and magnetism were thought to produce transverse waves, which supposedly could not be propagated through a liquid.

(5) This is how the Lorentz group appears. Since real light is polarized and described by the Maxwell-Heaviside equation, and not by a scalar wave equation, one should study the transforms which send any solution of the Maxwell-Heaviside equation to another solution of the Maxwell-Heaviside equation, and I believe that POINCARÉ did that.

bolic systems, and that it is only by letting some of the characteristic velocities (like the speed of light c) tend to infinity that one obtains some simpler parabolic systems, which one should then not believe to be the exact laws followed by nature. ([6])

Albert EINSTEIN (1879-1955) had probably read POINCARÉ's earlier article on (special) relativity, and there were aspects which he obviously did not understand for having proposed his ideas about gravitation, which I find silly. EINSTEIN had probably seen the 1900 work of Louis Jean-Baptiste Alphonse BACHELIER (1870-1946) for modeling the buying and selling at 'La Bourse', the Paris stock exchange market, ([7]) before he presented a similar game of random jumps (in position) as a model for the real Brownian motion, which I find silly since what Robert BROWN (1773-1858) had observed under his microscope could only have been *jumps in velocity* due to interactions with "particles" too small to be seen, and it should have made real physicists question EINSTEIN's physical intuition, and wonder if he understood the basic law of *conservation of linear momentum*. EINSTEIN had also probably seen the formula $e = m\,c^2$ which POINCARÉ wrote in 1900, or which Olinto DE PRETTO (1857-1921) wrote in 1903. Thanks to his friend Marcel GROSSMANN (1878-1936) who introduced him to the work of Gregorio RICCI-CURBASTRO (1853-1925) and of Tullio LEVI-CIVITA (1873-1941), EINSTEIN impressed his fellow physicists by using mathematics which they did not know, like tensor calculus and differential geometry, and such a behaviour is probably related to the classification of sciences of Isidore Auguste Marie Francois Xavier COMTE (1798-1857), who had ranked mathematics above astronomy and then physics.

Still today, most people do not dare mentioning EINSTEIN's curious lack of physical intuition: that rays of light may be bent by changes in the index of refraction had been "understood" since the 17th century, so that imagining that a ray of light could feel something which is not on its path (like the total mass of the sun) does not make any sense, and Hannes Olof Gösta ALFVÉN (1908-1995) pointed out in the 1970s (after having received the Nobel Prize in Physics) that a few observations in the cosmos have an electromagnetic origin and not a gravitational origin. In those days, and may be so in the beginning of the 20th century too, every physics student learned that electromagnetic forces are a few order of magnitude greater than gravitational forces, and that light is but electromagnetic waves of particular wavelengths, so that a

([6]) In a model with "$c = +\infty$", it is not surprising that perturbations may travel faster than the real speed of light c.

([7]) Economic agents buying or selling stocks is not related to any known conservation laws like those governing the physical world, so that there was nothing obviously wrong in what BACHELIER had done, and he was not claiming anything about physics, as EINSTEIN did with his "fake Brownian" motion.

grazing ray could only feel the electromagnetic conditions at the boundary of the sun. ([8])

The basic refraction effect which occurs at the interface between two (transparent) materials with different indices of refraction is actually about solutions of the wave equation (or the Maxwell-Heaviside equation) with discontinuous coefficients, a question which was only understood mathematically in the mid 20th century with the theory of distributions of L. SCHWARTZ.

I believe that it was only in the late 1980s with my introduction of *H-measures* in [Ta90] that there appeared a mathematical explanation of what light rays are for the wave equation, or for the Maxwell-Heaviside equation, with smoothly varying (anisotropic) coefficients, and how light rays transport energy and momentum, because one must emphasize that the classical (formal) theory of geometrical optics is about distorted plane waves, and not about light rays! Actually, there is a mid 20th century improvement by Joseph Bishop KELLER, called GTD (*geometrical theory of diffraction*), which formally explains questions of grazing rays in computations for distorted plane waves (away from caustics), but it has not been explained mathematically yet.

3. – Homogenization

Maybe what bothered physicists was that they were used to a *scalar index of refraction*, and they did not see how a *symmetric positive definite tensor* could appear instead, apart from playing with differential geometry in the way EINSTEIN had done. Indeed, mathematicians could not explain this effect until some work in *homogenization* started, in Pisa in the late 1960s with the work of Sergio SPAGNOLO on *G-convergence* [Sp1]-[Sp2], and with his collaborators Ennio DE GIORGI (1928-1996) and Antonio MARINO afterward, and in Paris in the early 1970s with the work of Francois MURAT and myself on *H-convergence* [Ta74], [Mu], motivated by an academic problem of optimal design posed by J.-L. LIONS,

([8]) The referee found that readers may find my comments questionable, and that physicists would maintain that rays of light feel a field, but since one attributes to EINSTEIN the idea that it is the gravitational field created by the sun which bends light at its surface, it seems to me that he considered the wrong field, and that his followers forgot to learn about electromagnetism, as ALFVÉN had already suggested. Of course, EINSTEIN could hardly know what kind of plasma one finds at the surface of the sun, and it was after his death that physicists claimed that the sun functions by thermonuclear fusion, and although they started boasting in the mid 1950s that they were going to control fusion, it is still to be seen when they will succeed: the main reason why one does experiments with powerful lasers seems to be for discovering the physical laws governing plasmas at millions of degrees, since the postulated laws used by physicists seem inappropriate, although they do not hesitate to use them in their big-bang game at billions of degrees!

and thanks to the work of Enrique Évariste SANCHEZ-PALENCIA (for periodically modulated materials), it became clear (at least to me) that most 20th century mechanics and physics is about homogenization, often in difficult settings which mathematicians have not put under control yet, but which force to go *beyond partial differential equations*.

There are no restrictions of periodicity, or use of probabilities in my definition of what homogenization is, of course, since no such restrictions are present in the general mathematical approach which came out of the pioneering work of the Italian school in Pisa (S. SPAGNOLO, E. DE GIORGI, A. MARINO) or the French school in Paris (F. MURAT, L. TARTAR), as I presented it in the beginning of 1977 at Collège de France in six lectures associated to the name of Claude Antoine PECCOT (1856-1876), where I considered the subject intimately related to the theory of *compensated compactness* which I had started to develop with F. MURAT, since what homogenization is from a mathematical point of view is developing a *nonlinear microlocal theory* for dealing with partial differential equations with small scale effects (in space or/and time): the basic fields of applications in continuum mechanics are turbulence for fluids, plasticity for solids, correcting the defects of equations of state and more generally of (the so badly named) thermodynamics, so that one could develop better mathematical models from phase transitions and materials science to meteorology and climatology, and understand better the connection with smaller scales from physics, where the basic fields of applications are atomic physics, spectroscopy and scattering, and the waves named after Felix BLOCH (1905-1983), for example.

Spectroscopy is about sending a monochromatic electromagnetic wave in a gas whose properties vary on a length scale comparable to the wavelength of the wave, but also on other length scales which certainly interact with it. Of course, one expects resonance effects, but since what one knows about the microscopic scales involved is questionable, at least for serious mathematicians who can hardly believe in the silly games which physicists invented and which nature most certainly does not play, it is definitely a problem of homogenization in an hyperbolic setting for which the actual mathematical knowledge is not sufficient. ([9])

([9]) As Graeme Walter MILTON pointed out to me, 3-point correlations about oscillations are necessary for quantifying scattering phenomena, and he said that my H-measures describe the part of 2-point correlations which is scale invariant. When I introduced parametrized measures in partial differential equations of continuum mechanics in the late 1970s, it was for pointing out that my theory of *compensated compactness* (partly developed with F. MURAT) goes further, and at that time I did not know that what one now calls *Young measures* were introduced by Laurence Chisholm YOUNG (1905–2000) [Yo]; fortunately, my name is not linked to the wrong and quite silly idea that Young measures "characterize" microstructures, since the deluded people writing such nonsense also seem eager to attribute all my ideas to their friends.

Although Leonardo DA VINCI (1452-1519) had some interest in turbulent flows, nothing quantitative about instabilities was imagined before 1851, when STOKES introduced what one now calls a "Reynolds" number, about which Osborne REYNOLDS (1842-1912) only wrote in 1883, and by then it was clear that *turbulence* is created by spatial (or temporal) fluctuations of the velocity field.

Turbulence is then a question of homogenization, for a first order operator $\sum_j v_j \frac{\partial}{\partial x_j}$, [10] so that when I first considered this question in the late 1970s, I wondered what kind of effective equation would appear in the limit if one uses a sequence of velocity fields v^n which only converges weakly. Since a first order scalar equation can be considered as a degenerate elliptic equation, does one obtain an effective equation which is a partial differential equation of some degenerate elliptic type? However, a first order scalar equation is hyperbolic, and since the spectroscopy experiment is also about homogenization in an hyperbolic setting and physicists explained what they observed by rules of spontaneous absorption or emission, I guessed that it was their way of saying that an effective equation contains a memory effect, [11] so that I expected to find effective equations containing integral terms. In very special situations (chosen for applying convolution operators) it is indeed the case that some nonlocal terms appear in the effective equation, and a few examples include convolution in time, in space, or both, [12] but there is not yet a theoretical understanding in more general situations for these effective equations which go *beyond partial differential equations*.

Classical weak (or weak $\star$) convergence is only adapted to quantities which mathematically are coefficients of differential forms and physically are related to extensive quantities: the density of mass ρ, the density of linear momentum $\rho\, u$,

[10] It is natural to assume that $div(v) = 0$, so that $\sum_j v_j \frac{\partial \psi}{\partial x_j} = \sum_j \frac{\partial (v_j \psi)}{\partial x_j}$ (in the sense of distributions) for smooth functions ψ, and then use the form $\sum_j \frac{\partial (v_j \psi)}{\partial x_j}$, which has a meaning without much smoothness assumptions, i.e. in cases where some term $v_j \frac{\partial \psi}{\partial x_j}$ may not necessarily make sense.

[11] Of course, there is no need to invent a probabilistic game for explaining the form of an effective equation. Since the introduction of probabilities in physical problems has usually been a mistake in the past, because it destroyed physical reality for replacing it by postulated games, and not by games which had been shown to result from a logical reasoning about what was already known, it seems better to warn against introducing probabilities when they are not needed, which is most of the time.

[12] They were studied with a method which I developed, based on representation of functions now associated to the name of Georg PICK (1859-1942), or sometimes Rolf Herman NEVANLINNA (1895-1980) or Thomas Jan STIELTJES (1856–1894), mostly by my students, Maria Luísa MARTINS MACEDO FARIA MASCARENHAS, Kamel HAMDACHE, Nenad ANTONIĆ, and their collaborators.

the density of total energy $\rho\,\dfrac{|u|^2}{2}+\rho\,e$ are extensive quantities, but not u (unless ρ is constant), and other quantities may require different topologies of weak type. Usually, weak topologies restricted to bounded sets are metrizable, and identifying the topology of weak type adapted to a particular quantity is then just a way to explain what it means for two instances of that quantity to be near, usually for one having small scale variations and the other having none.

4. – *G*-convergence and *H*-convergence in an elliptic framework

The basic idea for S. SPAGNOLO's G-convergence is to consider a sequence of (elliptic) partial differential equations

$$
\begin{aligned}
&-div\big(a_n\,grad(u_n)\big)=f \text{ in } \Omega\subset\mathbb{R}^N,\\
&u_n\in H^1_0(\Omega),\ \text{with } f\in H^{-1}(\Omega) \text{ given},
\end{aligned}
\tag{4.1}
$$

where $a_n\in L^\infty(\Omega)$ satisfies

$$
0<\alpha\le a_n(x)\le\beta<\infty \text{ a.e. } x\in\Omega;
\tag{4.2}
$$

then, there exists a subsequence a_m independent of f, and $a^{\mathit{eff}}_{ij}\in L^\infty(\Omega)$ for $i,j=1,\ldots,N$, satisfying

$$
\begin{aligned}
&\alpha\,|\xi|^2\le\sum_{i,j=1}^N a^{\mathit{eff}}_{ij}(x)\xi_i\xi_j\le\beta\,|\xi|^2 \text{ a.e. } x\in\Omega,\ \text{for all } \xi\in\mathbb{R}^N,\\
&a^{\mathit{eff}}_{ji}(x)=a^{\mathit{eff}}_{ij}(x) \text{ a.e. } x\in\Omega,\ \text{for } i,j=1,\ldots,N,
\end{aligned}
\tag{4.3}
$$

such that

$$
\begin{aligned}
&u_m\rightharpoonup u_\infty \text{ in } H^1_0(\Omega) \text{ weak, solution of}\\
&-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}\Big(a^{\mathit{eff}}_{ij}\frac{\partial u_\infty}{\partial x_j}\Big)=f \text{ in } \Omega.
\end{aligned}
\tag{4.4}
$$

S. SPAGNOLO proved more, that if a sequence of measurable *symmetric* matrices $A^n=(A^n_{ij})_{i,j=1,\ldots,N}$ satisfies the analog of (4.3), then there exists a subsequence A^m and a measurable *symmetric* matrix A^{eff}, also satisfying the analog of (4.3), and such that for every $f\in H^{-1}(\Omega)$ the sequence $u_m\in H^1_0(\Omega)$ of solutions of $-div\big(A^m grad(u_m)\big)=f$ in Ω converges weakly in $H^1_0(\Omega)$ to $u_\infty(\Omega)$ solution of (4.4). The convergence of A^m to A^{eff} is equivalent to the convergence of Green kernels, hence its name of G-convergence, and an important property is that it is *local*.

In dimension 1, one has $\dfrac{1}{a_m}\rightharpoonup\dfrac{1}{a^{\mathit{eff}}}$ in L^∞ weak $\star$, and there is an explicit formula in dimension $N\ge 2$ in the case of layered material (i.e. A^n only depending upon (x,e) for a direction $e \in S^{N-1}$), which uses the weak $\star$ limits of precise quantities depending upon e, since the Young measure of the sequence A^m alone cannot characterize A^{eff} in dimension $N \geq 2$, and *some geometric information on the microstructures used is necessary*: it then is silly to imagine that a finite number of parameters can describe all possible micro-geometries, except in one dimension, where this is no geometry.

In the non-symmetric case, studied with F. MURAT (and he chose the name H-convergence in relation with the term homogenization), the operator $-div\big(A\, grad\cdot\big)$ does not characterize A, and (4.3) is replaced by $A^n \in \mathcal{M}(\alpha,\beta;\Omega)$ defined as

$$\begin{aligned} &\alpha\,|\xi|^2 \leq (A^n(x)\xi,\xi) \text{ a.e. } x\in\Omega,\ \text{ for all } \xi\in\mathbb{R}^N,\\ &\frac{1}{\beta}\,|A^n(x)\xi|^2 \leq (A^n(x)\xi,\xi) \text{ a.e. } x\in\Omega,\ \text{ for all } \xi\in\mathbb{R}^N; \end{aligned} \tag{4.5}$$

then, there exists a subsequence A^m, and an *effective* limit $A^{eff} \in \mathcal{M}(\alpha,\beta;\Omega)$, i.e. satisfying (4.5), such that

$$\begin{aligned} &u_m \rightharpoonup u_\infty \text{ in } H^1_{loc} \text{ weak, and}\\ &-div\big(A^m grad(u_m)\big) = f_m \to f_\infty \text{ in } H^{-1}_{loc}(\Omega) \text{ strong} \end{aligned} \tag{4.6}$$

imply

$$A^m grad(u_m) \rightharpoonup A^{eff} grad(u_\infty) \text{ in } L^2(\Omega;\mathbb{R}^N) \text{ weak,} \tag{4.7}$$

so that u_∞ satisfies an equation like (4.4) with data f_∞, independently of which boundary conditions are used (which one then must check), [13] but also

$$\big(A^m grad(u_m), grad(u_m)\big) \rightharpoonup \big(A^{eff} grad(u_\infty), grad(u_\infty)\big), \tag{4.8}$$

weakly $\star$ in $L^1_{loc}(\Omega)$ (i.e. with test functions continuous with compact support in Ω), an example of the *div-curl lemma* (first instance of the more general *compensated compactness* theory developed with F. MURAT). [14] The existence of *correctors*

(13) The independence with respect to boundary conditions had already been proved by S. SPAGNOLO in the case of G-convergence, so that F. MURAT and I were only observing how to extend his results to our generalized framework of H-convergence.

(14) In using the div-curl lemma, the approach which I followed with F. MURAT differed from that of S. SPAGNOLO, whose proof relied on a regularity theorem of Norman George MEYERS; this same regularity result permits to use test functions in L^∞ in the div-curl lemma when the fields E^m and D^m come from solving an elliptic equation like in (4.8), but in the general case the div-curl lemma is not true for piecewise continuous test functions with a discontinuity along a smooth hyper-surface; using fine results from harmonic analysis (namely a commutation theorem of Ronald Raphaël COIFMAN, Richard Howard ROCHBERG, and Guido Leopold WEISS) one may use test functions in $L^\infty \cap VMO$.

$P^m \in L^2\big(\Omega; L(\mathbb{R}^N; \mathbb{R}^N)\big)$ (depending only upon the sequence A^m) satisfying

$$P^m \rightharpoonup I \text{ in } L^2_{loc}\big(\Omega; L(\mathbb{R}^N, \mathbb{R}^N)\big) \text{ weak,}$$

$$A^m P^m \rightharpoonup A^{eff} \text{ in } L^1_{loc}\big(\Omega; L(\mathbb{R}^N, \mathbb{R}^N)\big) \text{ weak,} \tag{4.9}$$

$$grad(u_m) - P^m grad(u_\infty) \to 0 \text{ in } L^1_{loc}\big(\Omega; L(\mathbb{R}^N, \mathbb{R}^N)\big) \text{ strong,}$$

permits to describe the effect of adding lower order terms to the equation, ([15]) and to identify the Young measure of $grad(u_m)$ in a local way, using $grad(u_\infty)$ and the Young measure of P^m.

For $E^m = grad(u_m)$, satisfying $curl(E^m) = 0$ and for $D^m = A^m grad(u_m)$, satisfying $div(D^m) = f_m$, the weak convergence is natural; for A^m, the natural convergence is H-convergence.

5. – Optimal bounds

The obtention of "optimal bounds", i.e. the identification of what A^{eff} can be in terms of the Young measure of A^n, corresponds in the symmetric case (which up to now was the only physically relevant one) to characterizing the effective properties which one can obtain for mixtures by playing in the best possible way upon the proportions of the various materials used and the choices for micro-geometries, i.e. the geometrical arrangements of the various pieces at a fine scale: it is essentially a theoretical question, prior to wondering about the cost of creating efficient mixtures using intricate micro-geometries. ([16])

For finding constraints on A^{eff}, I developed in the late 1970s a general method based on compensated compactness [Ta77] (which I later made more precise by the use of my H-measures [Ta90]), which extends the initial method based on the div-curl lemma which I used with F. MURAT in the early

([15]) I introduced correctors because of a remark of Ivo M. BABUŠKA about (linearized) elasticity (in a periodic medium), that the knowledge of an effective stress is not sufficient for telling when a material may break, and one needs a multiplying factor for deducing the local stress. I then noticed that the correctors permit to find the effective equation when lower order terms like $\big(b_n, grad(u_n)\big)$ are added, and F. MURAT completed this observation by considering also terms like $div(c_n u_n)$, where one uses the correctors associated to $(A^m)^T$.

([16]) Engineers are trained to develop an intuition for finding reasonably efficient designs much before a complete scientific understanding of a question is attained. However, despite the successes of the engineering approach, it is important to continue thinking about a problem until a scientific understanding is achieved. One of my complaints about physicists is that they consider any efficient engineering model as if it is an exact law followed by nature.

1970s [Ta74]. One should not be lured by various ad hoc minimization arguments which one finds in the literature, and which are not about the question of homogenization which I consider: mentioning questions of global "energy" stored in Ω is certainly not finding bounds on $A^{eff}(x)$ valid for almost every $x \in \Omega$.

The other side of the picture is to construct particular mixtures and hope that the external constraints and the internal constructions match.

Fortunately, one often does not need to identify the exact set of all possible A^{eff}, but only a projection of it: for example, for a vector E one wants to identify the exact set of all possible $A^{eff}E$, i.e. instead of characterizing a symmetric matrix, one wants to characterize what one of its column may be: if $A^n = a_n I$ with (4.2) and

$$a_n \rightharpoonup a_\infty, \frac{1}{a_n} \rightharpoonup \frac{1}{b_\infty} \text{ in } L^\infty(\Omega) \text{ weak } \star, \tag{5.1}$$

then if a subsequence A^m H-converges to A^{eff}, one has

$$D = A^{eff}E \text{ satisfies } (D - b_\infty E, D - a_\infty E) \leq 0 \text{ a.e. in } \Omega, \tag{5.2}$$

for all $E \in \mathbb{R}^N$, i.e. D belongs to the solid ball of diameter $[b_\infty E, a_\infty E]$, and this characterization is optimal. More generally, if

$$0 < \alpha \leq b_n(x) \leq a_n(x) \leq \beta < \infty \text{ a.e. } x \in \Omega; \tag{5.3}$$

$$a_n \rightharpoonup a_\infty, \frac{1}{b_n} \rightharpoonup \frac{1}{b_\infty} \text{ in } L^\infty(\Omega) \text{ weak } \star, \tag{5.4}$$

$$\begin{aligned} &E^n \rightharpoonup E^\infty, D^n \rightharpoonup D^\infty \text{ in } L^2(\Omega; \mathbb{R}^N) \text{ weak},\\ &(E^n, D^n) \rightharpoonup (E^\infty, D^\infty) \text{ in } L^1(\Omega) \text{ weak } \star,\\ &(D^n - b_n E^n, D^n - a_n E^n) \leq 0 \text{ a.e. in } \Omega, \end{aligned} \tag{5.5}$$

then

$$(D^\infty - b_\infty E^\infty, D^\infty - a_\infty E^\infty) \leq 0 \text{ a.e. in } \Omega. \tag{5.6}$$

If $D^n = A^n E^n$, with A^n *symmetric*, then the third line of (5.5) follows from the hypothesis that the eigenvalues of A^n are in the interval $[a_n, b_n]$. Although the proof uses the method which F. MURAT and I devised in the early 1970s, where the div-curl lemma served for ensuring the second line of (5.5), I only noticed such a geometrical result in the early 1990s [Ta95].

However, I was unable to imagine a natural type of question of this kind in the non-symmetric case.

6. – A new question for the non-symmetric case

In the Fall of 2010, I heard my young colleague Gautam IYER describe some joint work [INRZ], related to “diffusion and mixing in fluid flows”: ([17]) in previous work [CKRZ], it had been observed that if v is a (smooth enough) *divergence free* velocity field in a smooth bounded open set $\Omega \subset \mathbb{R}^N$, then

$$(6.1)\qquad \begin{gathered} -\varDelta u + \sum_j v_j \frac{\partial u}{\partial x_j} = f \text{ in } \Omega, u \in H^1_0(\Omega) \text{ satisfies} \\ \|u\|_{L^\infty(\Omega)} \leq C(\Omega, p)\, \|f\|_{L^p(\Omega)} \text{ for } p > \frac{N}{2}, \end{gathered}$$

with $C(\Omega, p)$ independent of v, ([18]) and looking for a possibly optimal v gave numerical simulations suggesting the appearance of homogenization effects, as in my work with F. MURAT, summarized in [Ta98] for the occasion of a CIME (/CIM) course.

For $N = 2$, assuming that Ω is simply connected and introducing a stream function ψ such that $v = (\psi_y, -\psi_x)$, I observed that

$$(6.2)\qquad -\varDelta u + \psi_y u_x - \psi_x u_y = -div\big(A\, grad(u)\big) \text{ with } A = \begin{pmatrix} 1 & \psi \\ -\psi & 1 \end{pmatrix},$$

so that it explains the similarity with the problem which F. MURAT and I studied: assuming that $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ (and normalizing ψ by adding a constant so that $\inf_{x \in \Omega} \psi(x) = -\sup_{x \in \Omega} \psi(x)$ for example), if one considers a sequence ψ_n which stays bounded in $L^\infty(\Omega)$ but not in $H^1(\Omega)$, then A^n may H-converge to an effective matrix A^{eff} having a different form, or having the same form but with a non-smooth ψ, in which case the left side of (6.2) may not make sense, while the right side of (6.2) still keeps a meaning.

([17]) Since one too often relies on the “fake Brownian” motion ideas when talking about diffusion, either of heat according to the initial ideas of Jean-Baptiste Joseph FOURIER (1768–1830), or of matter according to the initial ideas of Adolph Eugen FICK (1829–1901), I prefer to be cautious about the physical relevance of such ideas.

([18]) It is important to notice that such estimates rely on the maximum principle, which plays no role in the general theory of homogenization. This type of result may have been known to those who extended the work of Guido STAMPACCHIA (1922-1978) on the L^q regularity of solutions of scalar second order equations with discontinuous coefficients; since I have promised to write down my own method (devised in the mid 1980s) which permits some improvements using spaces named after George Gunther LORENTZ (1910-2006), I shall soon do it and show a more general result, and I shall check at that time who was the first to notice such a bound independent of the size of v, which needs the hypothesis that it is a divergence free field, and is then not covered by some results of Giorgio TALENTI and his collaborators, who have compared the radial non-decreasing rearrangements of solutions in an open set Ω with the radially symmetric solutions of adapted problems in the ball having the same volume than Ω.

One should be *cautious about the physical relevance of models*: for fluids, (6.1) corresponds to the transport of a scalar quantity u by an incompressible (and stationary, i.e. time independent) flow v, so that if one considers a situation where there is no bound for the H^1 norm of the stream function ψ it means that there is no bound for the L^2 norm of v, proportional to kinetic energy, and since $\Delta\psi = curl(v)$ which is the vorticity of the flow v, the use of discontinuous ψ corresponds to a limiting situation where the vorticity may be quite singular. Of course, one difficulty for understanding the 3-dimensional uniqueness or regularity of weak solutions of Navier-Stokes equation, a problem initialized by the work of Jean LERAY (1906-1998) in 1930, is that one does not know how to bound the vorticity, but kinetic energy should certainly stay bounded. Also, considering a sequence of stream functions ψ_n converging only in L^∞ weak $\star$ authorizes quite unrealistic choices, since the velocity field v^n should also satisfy an equation like Navier-Stokes equation.

Nevertheless, I shall say that this model provides a second "physical" example of H-convergence for non-symmetric matrices, the other one being the treatment by G. MILTON of a classical effect named after Edwin Herbert HALL (1855-1938), also a 2-dimensional situation, (19) but here one may also consider the 3-dimensional version, with $u = curl(\psi)$, where ψ is a vector field defined up to addition of a gradient, and since

$$(6.3)\qquad \begin{aligned}(curl(\psi), grad(u)) &= \sum_j \Big(\sum_{i,k} \varepsilon_{j,i,k} \frac{\partial \psi_k}{\partial x_i}\Big) \frac{\partial u}{\partial x_j}\\ &= \sum_{i,j} \frac{\partial}{\partial x_i} \Big(B_{i,j} \frac{\partial u}{\partial x_j}\Big) \text{ with } B_{i,j} = \sum_k \varepsilon_{j,i,k}\psi_k,\end{aligned}$$

the following matrix A appears in 3-dimension

$$(6.4)\qquad A = \begin{pmatrix} 1 & -\psi_3 & \psi_2 \\ \psi_3 & 1 & -\psi_1 \\ -\psi_2 & \psi_1 & 1 \end{pmatrix}, \text{ i.e. } A\,\xi = \xi + \psi \times \xi \text{ for } \xi \in \mathbb{R}^3.$$

For H-convergence of non-symmetric matrices, F. MURAT and I introduced the notation $\mathcal{M}(\alpha,\beta;\Omega)$ for matrices satisfying (4.5), (20) which is compact for the topology of H-convergence; (21) since the matrices A of (6.2) or (6.4) satisfy

(19) The (classical) Hall effect is observed in a metallic ribbon (considered a 2-dimensional domain) in which an electric current goes through: if one applies a perpendicular magnetic field, it induces a difference of potential between the two edges of the ribbon.

(20) Notice that $\mathcal{M}(\alpha,\beta;\Omega)$ is a closed convex set of matrices, and that it means $A(x) \geq \alpha\, I$ and $\big(A(x)\big)^{-1} \geq \dfrac{1}{\beta}\, I$ (for the usual pre-order on matrices) a.e. in Ω.

(21) In the non-symmetric case, if $0 < \alpha < \gamma < \infty$, the set of A satisfying $A(x) \geq \alpha\, I$ and $|A(x)\,\xi| \leq \gamma\,|\xi|$ for all $\xi \in \mathbb{R}^N$ a.e. $x \in \Omega$ is not closed for H-convergence.

$(A\,\xi,\xi) = 0$, one has

(6.5) $$A \in \mathcal{M}(1,\beta;\Omega) \text{ with } 1 + |\psi|^2 \le \beta \text{ a.e. in } \Omega.$$

For $N = 2$, the normalization $\inf_{x\in\Omega}\psi(x) = -\sup_{x\in\Omega}\psi(x)$ consists in minimizing the L^∞ norm of ψ, hence the value of $\beta > 1$. Given $\beta > 1$, then

(6.6) $$A = I - \psi\,R_{\pi/2} \in \mathcal{M}(1,\beta;\Omega) \text{ means } ||\psi||_{L^\infty(\Omega)} \le \sqrt{\beta - 1},$$

where $R_{\pi/2}$ is the rotation of $\frac{\pi}{2}$, and for a sequence ψ_n, a subsequence A^m H-converges to $A^{\mathit{eff}} \in \mathcal{M}(1,\beta;\Omega)$. For $M \in \mathcal{M}(1,\beta)$ (i.e. no x dependence) the skew-symmetric part of M is $-\psi\,R_{\pi/2}$, independent of the orthonormal basis, and the symmetric part of M can be diagonalized in an orthonormal basis in which $M = \begin{pmatrix} \lambda_1 & \psi \\ -\psi & \lambda_2 \end{pmatrix}$, and (22)

(6.7) $$M = \begin{pmatrix} \lambda_1 & \psi \\ -\psi & \lambda_2 \end{pmatrix} \in \mathcal{M}(1,\beta) \text{ means } \lambda_1,\lambda_2 \ge 1, \text{ and}$$
$$det(M) = \lambda_1\lambda_2 + \psi_{\mathit{eff}}^2 \le \beta\,\min\{\lambda_1,\lambda_2\}.$$

The "H-closure" of the matrices of the form $I - \psi\,R_{\pi/2}$ with $||\psi||_{L^\infty(\Omega)} \le \sqrt{\beta-1}$ is a subset of $\mathcal{M}(1,\beta;\Omega)$ of the form $A^{\mathit{eff}}(x) \in \mathcal{K}_\beta$ a.e. $x \in \Omega$, for a particular $\mathcal{K}_\beta \subset \mathcal{M}(1,\beta)$ [Ta07], which has not been characterized yet, but for a given vector $E \in \mathbb{R}^2$ the general theory says that the set $\{A\,E \mid A \in \mathcal{K}_\beta\}$ is a closed convex set of $\mathbb{R}^2$. (23) However, I want to characterize such a set in a more precise situation for $||\psi_n||_{L^\infty(\Omega)} \le \sqrt{\beta - 1}$:

(6.8) $$\text{if } \psi_n \rightharpoonup \psi_\infty, \psi_n^2 \rightharpoonup \ell_\infty \text{ in } L^\infty(\Omega) \text{ weak} \star,$$
$$A^n = I - \psi_n R_{\pi/2} \text{ H-converges to } A^{\mathit{eff}},$$

then, by a convexity lemma which F. Murat and I proved in 1970, the possible pairs $(\psi_\infty, \ell_\infty)$ are characterized by

(6.9) $$\psi_\infty^2 \le \ell_\infty \le \beta - 1 \text{ a.e. in } \Omega,$$

and for the question of H-convergence one has

(6.10) $$E^m = grad(u_m) \rightharpoonup E^\infty \text{ in } L^2(\Omega;\mathbb{R}^2) \text{ weak},$$
$$D^m = A^m grad(u_m) \rightharpoonup D^\infty \text{ in } L^2(\Omega;\mathbb{R}^2) \text{ weak},$$
$$(E^m, D^m) \rightharpoonup (E^\infty, D^\infty) \text{ in } L^1(\Omega) \text{ weak} \star,$$

(22) $M \ge I$ means $\lambda_1,\lambda_2 \ge 1$, and the condition $M^{-1} \ge \frac{1}{\beta} I$ means $\frac{\lambda_1}{\lambda_1\lambda_2 + \psi^2}, \frac{\lambda_2}{\lambda_1\lambda_2+\psi^2} \ge \frac{1}{\beta}$, i.e. $\min\{\lambda_1,\lambda_2\} \ge 1$ and $det(M) = \lambda_1\lambda_2 + \psi^2 \le \beta\,\min\{\lambda_1,\lambda_2\}$.

(23) It results from the formula for layered materials written in an arbitrary basis [Ta83].

the third line following from the div-curl lemma, and my first question is to characterize D^∞ in terms of E^∞, ψ_∞, and ℓ_∞.

THEOREM 1. – *With the preceding notation, one has*

$$D^\infty = E^\infty - \psi_\infty R_{\pi/2} E^\infty + C^\infty, \tag{6.11}$$

with

$$\left| C^\infty - \frac{\varepsilon_\infty}{2}\, E^\infty \right| \leq \frac{\varepsilon_\infty}{2}\, |E^\infty| \text{ a.e. in } \Omega, \text{ with } \varepsilon_\infty = \ell_\infty - \psi_\infty^2. \tag{6.12}$$

PROOF. – In the subset ω where $\ell_\infty = \psi_\infty^2$, i.e. $\varepsilon_\infty = 0$, one has strong convergence of ψ_m to ψ_∞, i.e. in L^q for every $q \in [1, \infty)$, hence

$$\begin{aligned} &\text{in } \omega = \{x \in \Omega \mid \ell_\infty = \psi_\infty^2\}, \text{ one has} \\ &D^\infty = (I - \psi_\infty R_{\pi/2}) E^\infty = E^\infty - \psi_\infty R_{\pi/2} E^\infty. \end{aligned} \tag{6.13}$$

On $\Omega \setminus \omega$, one has $\psi_\infty^2 < \ell_\infty \leq \beta - 1$, and for applying our convexity lemma, one proceeds as follows: for vectors $v, w \in \mathbb{R}^2$ one considers the weak $\star$ limit of $(D^m, E^m) + 2(v, E^m) + 2(w, D^m)$, which is $(D^\infty, E^\infty) + 2(v, E^\infty) + 2(w, D^\infty)$, and one notices that

$$\begin{aligned} &(D^m, E^m) + 2(v, E^m) + 2(w, D^m) \\ &\quad = |E^m|^2 + 2(v, E^m) + 2(w, E^m - \psi_m R_{\pi/2} E^m) \\ &\quad = |E^m|^2 + 2(E^m, v + w + \psi_m R_{\pi/2} w) \\ &\geq -|v + w + \psi_m R_{\pi/2} w|^2 \\ &= -|v + w|^2 - 2\psi_m (v, R_{\pi/2} w) - \psi_m^2 |w|^2, \text{ a.e. in } \Omega, \end{aligned} \tag{6.14}$$

and taking the limit one obtains

$$\begin{aligned} &(D^\infty, E^\infty) + 2(v, E^\infty) + 2(w, D^\infty) \\ &\geq -|v + w|^2 - 2\psi_\infty (v, R_{\pi/2} w) - \ell_\infty |w|^2 \\ &= -|v + w + \psi_\infty R_{\pi/2} w|^2 - (\ell_\infty - \psi_\infty^2)\, |w|^2, \text{ a.e. in } \Omega, \end{aligned} \tag{6.15}$$

which is true for all $v, w \in \mathbb{R}^2$; using $z = v + w + \psi_\infty R_{\pi/2} w$ instead of v, one rewrites (6.15) as

$$\begin{aligned} &(D^\infty, E^\infty) + 2(E^\infty, z - w - \psi_\infty R_{\pi/2} w) + 2(D^\infty, w) + |z|^2 \\ &\quad + (\ell_\infty - \psi_\infty^2)\, |w|^2 \geq 0, \text{ a.e. in } \Omega, \text{ for all } w, z \in \mathbb{R}^2, \end{aligned} \tag{6.16}$$

and for minimizing the left side, one takes $z = -E^\infty$, so that

$$\begin{aligned} &(D^\infty - E^\infty, E^\infty) + 2(E^\infty, -w - \psi_\infty R_{\pi/2} w) + 2(D^\infty, w) \\ &\quad + (\ell_\infty - \psi_\infty^2)\, |w|^2 \geq 0, \text{ a.e. in } \Omega, \text{ for all } w \in \mathbb{R}^2; \end{aligned} \tag{6.17}$$

one transforms (6.17) by defining the correction C^∞ by (6.11), so that (6.17) becomes

$$(C^\infty, E^\infty) + 2(w, C^\infty) + (\ell_\infty - \psi_\infty^2)\,|w|^2 \geq 0, \text{ a.e. in } \Omega, \tag{6.18}$$

for all $w \in \mathbb{R}^2$: on ω, where $\ell_\infty = \psi_\infty^2$, one has $C^\infty = 0$, and on $\Omega \setminus \omega$, where $\ell_\infty > \psi_\infty^2$, one minimizes the left side by taking $(\ell_\infty - \psi_\infty^2)\,w + C^\infty = 0$, which gives

$$(C^\infty, E^\infty) - \frac{1}{\ell_\infty - \psi_\infty^2}\,|C^\infty|^2 \geq 0, \text{ a.e. in } \Omega \setminus \omega, \tag{6.19}$$

which localizes C^∞ in the closed disc defined by (6.12), noticing that (6.12) is true on ω, where $\varepsilon_\infty = 0$.

For constructing particular cases, one first uses the formula for layers in the direction x_1, and in an arbitrary direction by rotating the result; one then uses information from the general theory of homogenization, that a set which one is looking for is convex, that the topology of H-convergence is local, and that its restriction to any set $\mathcal{M}(\alpha, \beta; \Omega)$ is metrizable.

THEOREM 2. – *Given* $\beta > 1$, $\psi_\infty, \ell_\infty \in L^\infty(\Omega)$ *satisfying* (6.9), *i.e.* $\psi_\infty^2 \leq \ell_\infty \leq \beta - 1$ *a.e. in* Ω, *and given a measurable function* θ_∞ *from* Ω *into the unit circle* $\mathbb{S}^1$, *there exists a sequence* ψ_n *such that* $\|\psi_n\|_{L^\infty(\Omega)} \leq \sqrt{\beta - 1}$ *and* (6.8) *holds, i.e.* $\psi_n \rightharpoonup \psi_\infty$ *in* $L^\infty(\Omega)$ *weak* $\star$, $\psi_n^2 \rightharpoonup \ell_\infty$ *in* $L^\infty(\Omega)$ *weak* $\star$, *and* $A^n = I - \psi_n R_{\pi/2}$ *H-converges to* A^{eff}, *with the particular value*

$$A^{\mathit{eff}} = I - \psi_\infty R_{\pi/2} + \varepsilon_\infty \begin{pmatrix} \sin^2\theta_\infty & \sin\theta_\infty \cos\theta_\infty \\ \sin\theta_\infty \cos\theta_\infty & \cos^2\theta_\infty \end{pmatrix} \text{ a.e. in } \Omega, \tag{6.20}$$

where $\varepsilon_\infty = \ell_\infty - \psi_\infty^2$.

PROOF. – One starts by approaching $(\psi_\infty, \ell_\infty, \theta_\infty)$ by a sequence $(\psi^\delta, \ell^\delta, \theta^\delta)$ converging "strongly" (i.e. in $L^\infty(\Omega)$ weak $\star$ and $L^1(\Omega)$ strong) by cutting $\mathbb{R}^2$ in small disjoint open rectangles of sides $\leq \delta$, plus their boundaries of zero measure, as introduced by Henri Léon LEBESGUE (1875-1941) and, for each such rectangle ω which intersects Ω in a set of positive Lebesgue measure, replacing each function ψ_∞, ℓ_∞, and θ_∞ by its average on $\omega \cap \Omega$, observing that $(\psi_\infty, \ell_\infty)$ takes its values in a (closed) convex set, and paying attention to choose the representation of θ_∞ taking values in $[0, \pi)$ for also having a convex set (since one only needs θ_∞ modulo π). Since ℓ^δ is an average of ℓ_∞, which is $\geq (\psi_\infty)^2$, and the average of $(\psi_\infty)^2$ is $\geq (\psi^\delta)^2$, one deduces that $\ell^\delta = (\psi^\delta)^2 + \varepsilon^\delta$ with $\varepsilon^\delta \geq 0$.

One first proves the theorem for $(\psi^\delta, \ell^\delta, \theta^\delta)$, showing the existence of a sequence ψ_n^δ having the desired properties: for a given δ, one works separately on each $\omega \cap \Omega$ having positive measure, taking advantage of the local character of

H-convergence, ([24]) and then one uses the formula for the effective matrix in the case of layered materials: for example, if $\theta^\delta = 0$, one uses a sequence of smooth functions $\psi_n(x_1)$ which converges to ψ^δ in L^∞ weak $\star$, with ψ_n^2 converging to $\ell^\delta = (\psi^\delta)^2 + \varepsilon^\delta$ in L^∞ weak $\star$, and (6.9) is the characterization of which constraints result from $\|\psi_n\|_{L^\infty} \leq \sqrt{\beta - 1}$; the formula (valid for functions of x_1 alone) tells that

$$
\begin{aligned}
&\frac{1}{A^{\mathit{eff}}_{11}} \text{ is the } L^\infty \text{ weak } \star \text{ limit of } \frac{1}{A^n_{11}}, \text{ here } A^{\mathit{eff}}_{11} = 1,\\
&\frac{A^{\mathit{eff}}_{12}}{A^{\mathit{eff}}_{11}} \text{ is the } L^\infty \text{ weak } \star \text{ limit of } \frac{A^n_{12}}{A^n_{11}} \text{ here } A^{\mathit{eff}}_{12} = \psi^\delta,\\
&\frac{A^{\mathit{eff}}_{21}}{A^{\mathit{eff}}_{11}} \text{ is the } L^\infty \text{ weak } \star \text{ limit of } \frac{A^n_{21}}{A^n_{11}} \text{ here } A^{\mathit{eff}}_{21} = -\psi^\delta,\\
&A^{\mathit{eff}}_{22} - \frac{A^{\mathit{eff}}_{21}A^{\mathit{eff}}_{12}}{A^{\mathit{eff}}_{11}} \text{ is the } L^\infty \text{ weak } \star \text{ limit of } A^n_{22} - \frac{A^n_{21}A^n_{12}}{A^n_{11}},\\
&\qquad\qquad\text{here } A^{\mathit{eff}}_{22} = 1 + \ell^\delta - (\psi^\delta)^2 = 1 + \varepsilon^\delta,
\end{aligned}
\tag{6.21}
$$

i.e.

$$
\text{using } \psi_n(x_1) \text{ gives } A^{\mathit{eff}} = \begin{pmatrix} 1 & \psi^\delta \\ -\psi^\delta & 1 + \varepsilon^\delta \end{pmatrix},
\tag{6.22}
$$

so that using $\psi_n(\cos\theta^\delta x_1 + \sin\theta^\delta x_2)$ gives

$$
A^{\mathit{eff}} = I - \psi^\delta R_{\pi/2} + \varepsilon^\delta \begin{pmatrix} \sin^2\theta^\delta & \sin\theta^\delta\cos\theta^\delta \\ \sin\theta^\delta\cos\theta^\delta & \cos^2\theta^\delta \end{pmatrix}.
\tag{6.23}
$$

One then deduces the theorem by the argument of Georg Ferdinand Ludwig Philipp CANTOR (1845-1918) for choosing a diagonal subsequence, which is valid

([24]) I choose the rectangles ω to be open since F. MURAT and I first used open sets for proving the local character of H-convergence (by applying our div-curl lemma), but the result is true for ω any measurable set, by following the original proof of S. SPAGNOLO for the local character of G-convergence, which uses a regularity theorem of N. MEYERS, whose proof relies on more technical results: it uses the famous convolution theorem of Alberto Pedro CALDERÓN (1920-1998) and Antoni Szczepan ZYGMUND (1900-1992) for proving regularity in the constant coefficient case (with data in $W^{-1,p}$ with $p \neq 2$), a perturbation argument valid in the spaces now named after Stefan BANACH (1892-1945) although their use had been advocated earlier by Frigyes RIESZ (1880-956), followed by an application of the (real method) of interpolation in those spaces, developed in particular by J.-L. LIONS and Jaak PEETRE, following ideas of Emilio GAGLIARDO (1930-2008) and extending an original method of Marcel RIESZ (1886-1969).

here because the topology of H-convergence is metrizable (since one restricts attention to $\mathcal{M}(1,\beta;\Omega)$), and the strong convergence of θ^δ and ψ^δ (toward θ^∞ and ψ^∞) implies the H-convergence of A^{eff}.

One can construct a larger set of matrices A^{eff} than those given by (6.20), by layering materials obtained in this way in other directions, and then one will have to address the question of whether the set obtained is optimal or not; I have not studied this question, and my interest here is to show that Theorem 1 is optimal, in the ways stated in Theorem 3 and Theorem 4.

THEOREM 3. – *Given* $E \in \mathbb{R}^2$ *and* $D \in L^\infty(\Omega;\mathbb{R}^2)$ *satisfying*

$$D = E - \psi_\infty R_{\pi/2} E + C \text{ with } \left|C - \frac{\varepsilon_\infty}{2}\,E\right| \le \frac{\varepsilon_\infty}{2}\,|E| \text{ a.e. in } \Omega, \tag{6.24}$$

with $\psi_\infty, \varepsilon_\infty \in L^\infty(\Omega)$ *satisfying* $\psi_\infty^2 \le \ell_\infty = \psi_\infty^2 + \varepsilon_\infty \le \beta - 1$ *a.e. in* Ω *(and* $\beta > 1$*), there exists a sequence* ψ_n *such that* $\|\psi_n\|_{L^\infty(\Omega)} \le \sqrt{\beta - 1}$, $\psi_n \rightharpoonup \psi_\infty$ *in* $L^\infty(\Omega)$ *weak* $\star$, $\psi_n^2 \rightharpoonup \ell_\infty$ *in* $L^\infty(\Omega)$ *weak* $\star$, *and* $A^n = I - \psi_n R_{\pi/2}$ *H-converges to* A^{eff}, *with*

$$A^{eff} E = D \text{ a.e. in } \Omega. \tag{6.25}$$

PROOF. – One approaches ψ_∞ by a sequence ψ^δ which is constant in each small rectangle ω intersecting Ω, by taking averages, but it is now ε_∞ which one approaches by a sequence ε^δ by averaging, and one then defines ℓ^δ by $(\psi^\delta)^2 + \varepsilon^\delta$, noticing that $\varepsilon^\delta \ge 0$ and $(\psi^\delta)^2 + \varepsilon^\delta \le \beta - 1$ by convexity. One also defines C^δ and D^δ by averaging C and D on each $\omega \cap \Omega$, so that these sequences converge in $L^\infty(\Omega)$ weak $\star$ and $L^1(\Omega)$ strong, and one has

$$D^\delta = E - \psi^\delta R_{\pi/2} E + C^\delta \text{ in every rectangle } \omega \text{ intersecting } \Omega, \tag{6.26}$$

because E is constant, and then one notices that one also has

$$\left|C^\delta - \frac{\varepsilon^\delta}{2}\,E\right| \le \frac{\varepsilon^\delta}{2}\,|E| \text{ in every rectangle } \omega \text{ intersecting } \Omega, \tag{6.27}$$

because the set of $(c,\varepsilon) \in \mathbb{R}^2 \times \mathbb{R}$ satisfying $\left|c - \frac{\varepsilon}{2}\,E\right| \le \frac{\varepsilon}{2}\,|E|$ is convex, since the function $(c,\varepsilon) \mapsto \left|c - \frac{\varepsilon}{2}\,E\right| - \frac{\varepsilon}{2}\,|E|$ is convex in (c,ε).

For a given ω intersecting Ω one can then apply Theorem 2 and create a sequence $\psi_{n,\delta}$ converging to ψ^δ with $\psi_{n,\delta}^2$ converging to $(\psi^\delta)^2 + \varepsilon^\delta$ and $A^{n,\delta} = I - \psi_n^\delta R_{\pi/2}$ H-converging to various $A^{eff,\delta}$ given by the analog of (6.20) using ψ^δ and ε^δ, but one has the choice of which value θ_∞ to use. One actually chooses two different values θ_∞^1 and θ_∞^2, which give A_1^{eff} and A_2^{eff}; then one chooses a unit vector $e \in \mathbb{R}^2$ and one uses an iterated homogenization result consisting in using in $\omega \cap \Omega$ the two new materials A_1^{eff} and A_2^{eff} in layers perpendicular to e and in proportions η_1 and η_2 (non-negative with sum equal to 1), and this uses the local character of H-convergence and the metrizability on the set $\mathcal{M}(1,\beta;\omega\cap\Omega)$ for using a Cantor diagonal subsequence. The effective result is given by a formula, which says the same thing than the formula for layering already used, but which I expressed in 1983 in arbitrary directions for the purpose of reiterating the process, and the formula (first written in [Ta83]) is

$$(6.28)\quad A^{eff} = \eta_1 A_1^{eff} + \eta_2 A_2^{eff} - \eta_1\eta_2 (A_2^{eff} - A_1^{eff})\,\frac{e\otimes e}{\eta_2(A_1^{eff}e,e)+\eta_1(A_2^{eff}e,e)}\,(A_2^{eff}-A_1^{eff}),$$

so that

$$(6.29)\qquad \text{if } \big((A_2^{eff}-A_1^{eff})E,e\big)=0, \text{ then one has } A^{eff}E=\eta_1A_1^{eff}E+\eta_2A_2^{eff}E.$$

Since (6.20) gives

$$(6.30)\qquad A^{eff}E = E - \psi^\delta R_{\pi/2}E + c(\theta_\infty)$$
$$\text{with } c(\theta_\infty) \text{ belonging to the circle } \left|c-\frac{\varepsilon^\delta}{2}\,E\right| = \frac{\varepsilon^\delta}{2}|E|,$$

and one obtains all the circle when θ changes from 0 to π, one can then choose θ_∞^1 and θ_∞^2 and the proportions η_1 and η_2 for obtaining

$$(6.31)\qquad \eta_1 c(\theta_\infty^1)+\eta_2 c(\theta_\infty^2) = C^\delta,$$

since C^δ is inside the circle, because of (6.27). This proves the particular case of the result for the piece-wise constant functions indexed by δ, and letting δ tend to 0 and extracting an adapted Cantor diagonal subsequence proves Theorem 3.

One can be a little more precise, and generalize Theorem 3 in the following way.

THEOREM 4. – *Given $E\in L^2(\Omega;\mathbb{R}^2)$ and $D\in L^2(\Omega;\mathbb{R}^2)$ satisfying*

$$(6.32)\qquad D = E-\psi_\infty R_{\pi/2}E + C \text{ with } \left|C-\frac{\varepsilon_\infty}{2}\,E\right| \le \frac{\varepsilon_\infty}{2}\,|E| \text{ a.e. in } \Omega,$$

with $\psi_\infty,\varepsilon_\infty\in L^\infty(\Omega)$ satisfying $\psi_\infty^2\le\ell_\infty=\psi_\infty^2+\varepsilon_\infty\le\beta-1$ a.e. in Ω (and $\beta>1$), there exists a sequence ψ_n such that $\|\psi_n\|_{L^\infty(\Omega)}\le\sqrt{\beta-1}$, $\psi_n\rightharpoonup\psi_\infty$ in $L^\infty(\Omega)$ weak $\star$, $\psi_n^2\rightharpoonup\ell_\infty$ in $L^\infty(\Omega)$ weak $\star$, and $A^n=I-\psi_nR_{\pi/2}$ H-converges to A^{eff}, with

$$(6.33)\qquad A^{eff}E = D \text{ a.e. in } \Omega.$$

PROOF. – One approaches E by E^δ by averaging on $\omega\cap\Omega$ for each rectangle ω intersecting Ω on a set of positive Lebesgue measure, but since one wants to keep ε_∞ for applying Theorem 3, one defines D^δ by

$$(6.34)\qquad D^\delta = E^\delta - \psi_\infty R_{\pi/2}E^\delta + C^\delta \text{ a.e. in } \Omega,$$

where one has to choose C^δ in a compatible way, i.e. satisfying

$$\left|C^\delta - \frac{\varepsilon_\infty}{2}\, E^\delta\right| \le \frac{\varepsilon_\infty}{2}\, |E^\delta| \text{ a.e. in } \Omega: \tag{6.35}$$

because $|C| \le \varepsilon_\infty |E|$ a.e. in Ω, C vanishes (a.e.) where $\varepsilon_\infty = 0$ and one may consider $\gamma = \dfrac{C}{\varepsilon_\infty}$ by defining it to be 0 where $\varepsilon_\infty = 0$, and on $\omega \cap \Omega$ one defines $C^\delta = \varepsilon_\infty \gamma^\delta$, where γ^δ is the average of γ, so that the desired inequality is a consequence of convexity as before. C^δ may not be constant on each rectangle, but it satisfies (6.35), and Theorem 3 applies, hence there exists a sequence $\psi_{n,\delta}$ such that $\|\psi_{n,\delta}\|_{L^\infty(\Omega)} \le \sqrt{\beta - 1}$, $\psi_{n,\delta} \rightharpoonup \psi_\infty$ in $L^\infty(\Omega)$ weak $\star$, $\psi^2_{n,\delta} \rightharpoonup \ell_\infty$ in $L^\infty(\Omega)$ weak $\star$, and $A^{n,\delta} = I - \psi_{n,\delta} R_{\pi/2}$ H-converges to $A^{\mathit{eff},\delta}$, with

$$A^{\mathit{eff},\delta} E^\delta = D^\delta \text{ a.e. in } \Omega. \tag{6.36}$$

One will then let δ tend to 0 and extract a Cantor diagonal subsequence, and the proof will be complete if one observes that C^δ converges strongly to C in $L^2(\Omega; \mathbb{R}^2)$, so that D^δ converges strongly to D in $L^2(\Omega; \mathbb{R}^2)$ by (6.34): indeed, one has $\gamma \in L^2(\Omega; \mathbb{R}^2)$ since $|\gamma| \le |E|$ a.e. in Ω, so that γ^δ converges strongly to γ, and since $\varepsilon_\infty \in L^\infty(\Omega)$, $C^\delta = \varepsilon_\infty \gamma^\delta$ converges strongly to $\varepsilon_\infty \gamma = C$.

We have now a way to answer a natural question, but I have not studied the question whether the inequalities obtained in Theorem 5 are the best possible or not.

THEOREM 5. – *If a sequence ψ_n such that $\|\psi_n\|_{L^\infty(\Omega)} \le \sqrt{\beta - 1}$ satisfies $\psi_n \rightharpoonup \psi_\infty$ in $L^\infty(\Omega)$ weak $\star$, $\psi_n^2 \rightharpoonup \ell_\infty = \psi_\infty^2 + \varepsilon_\infty$ in $L^\infty(\Omega)$ weak $\star$, and $A^n = I - \psi_n R_{\pi/2}$ H-converges to A^{eff}, with*

$$A^{\mathit{eff}} = M^{\mathit{eff}} - \psi^{\mathit{eff}} R_{\pi/2} \text{ with } M^{\mathit{eff}} \text{ symmetric, with } \lambda_1 \le \lambda_2, \tag{6.37}$$

then one has

$$1 \le \lambda_1 \le \lambda_2 \le 1 + \varepsilon_\infty \text{ a.e. in } \Omega, \tag{6.38}$$

and

$$(\psi^{\mathit{eff}} - \psi_\infty)^2 \le (1 + \varepsilon_\infty - \lambda_2)(\lambda_1 - 1) \text{ a.e. in } \Omega. \tag{6.39}$$

PROOF. – For a symmetric 2×2 matrix M with eigenvalues $\lambda_1 \le \lambda_2$, one looks at the relative position of a unit vector e and its image $M\,e$, which are known once one knows the length $|M\,e|$ and the cosine of the angle $\theta \in [0, \pi]$ between e and $M\,e$ (when $M\,e \ne 0$), for example by considering $(M\,e, e)$, which is $\cos\theta\, |M\,e|\, |e|$. If one represents this configuration in a plane with an orthonormal basis by plotting $(|M\,e| \cos\theta, |M\,e| \sin\theta)$, it is easy to see that when the unit vector e varies one finds the whole upper part of the circle centered at $\frac{\lambda_1+\lambda_2}{2}$ and with radius $\frac{\lambda_2-\lambda_1}{2}$. ([25])

When a unit vector E varies, (6.38) then implies that $A^{eff}E$ lies on a circle with center at $P = \left(\frac{\lambda_1+\lambda_2}{2}, \psi^{eff}\right)$ and radius $\rho = \frac{\lambda_2-\lambda_1}{2}$, and one must write that this circle falls inside the disc described by (6.12), with center at $P_0 = \left(1+\frac{\varepsilon_\infty}{2}, \psi_\infty\right)$ and radius $\rho_0 = \frac{\varepsilon_\infty}{2}$. This gives a first condition $1 \le \lambda_1 \le \lambda_2 \le 1+\varepsilon_\infty$, which is (6.38), and then one wants to write that $|P-P_0|+\rho \le \rho_0$, but since (6.38) implies $\rho \le \rho_0$, one writes $|P-P_0|^2 \le (\rho_0-\rho)^2$, which gives a second condition

$$(6.40) \qquad (\psi^{eff}-\psi_\infty)^2 \le \left(\frac{\varepsilon_\infty+\lambda_1-\lambda_2}{2}\right)^2 - \left(1+\frac{\varepsilon_\infty}{2}-\frac{\lambda_1+\lambda_2}{2}\right)^2,$$

which after factorizing the difference of squares is (6.39).

Another natural question is to identify the union of the discs defined by (6.24) when ψ_∞ and $\ell_\infty = \psi_\infty^2+\varepsilon_\infty$ vary, i.e. satisfying (6.9): of course, it is a subset of $\{ME \mid M \in \mathcal{M}(1,\beta)\}$.

THEOREM 6. – *The union of the discs defined by* (6.24) *when* ψ_∞ *and* $\ell_\infty = \psi_\infty^2+\varepsilon_\infty$ *vary, i.e. satisfying* (6.9), *is exactly* $\{ME \mid M \in \mathcal{M}(1,\beta)\}$.

PROOF. – Choosing E as a unit vector and completing the orthonormal basis, and denoting (ξ,η) the coordinates in this basis, the image by $M \in \mathcal{M}(1,\beta)$ of E is a vector (ξ,η) such that $\xi \ge 1$ and $\xi \ge \frac{\xi^2+\eta^2}{\beta}$, i.e. $\xi^2+\eta^2 \le \beta\,\xi$, a disc centered at $\frac{\beta}{2}$ and radius $\frac{\beta}{2}$. For $-\sqrt{\beta-1}<\psi_\infty<\sqrt{\beta-1}$ the value of ε_∞ can be any value between 0 and $\beta-1-\psi_\infty^2$, ([26]) so that the disc increases with ε_∞ and for the maximum value $\varepsilon_\infty = \beta-1-\psi_\infty^2$, the disc is tangent to both the line $\xi = 1$ and the circle $\xi^2+\eta^2 = \beta\,\xi$, and one way to see it is to compute the distance from the center of the disc $\left(1+\frac{\varepsilon_\infty}{2}, \psi_\infty\right)$ to the center $\left(\frac{\beta}{2}, 0\right)$ and check that it is $\frac{\beta}{2}-\frac{\varepsilon_\infty}{2}$; one then wants to convince oneself that by an argument of continuity, this family of bi-tangent discs sweeps the whole region defined by $\xi \ge 1$ and $\xi^2+\eta^2 \le \beta\,\xi$, one way is to show that the segment joining the two points of tangency sweeps this region, and for doing

([25]) For a symmetric $n \times n$ matrix M with eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ (and $n \ge 3$), if $A_j = (\lambda_j, 0)$ for $j = 1, \ldots, n$, the set obtained for $(|M\,e|\cos\theta, |M\,e|\sin\theta)$ is the whole upper part of the closed disc of diameter A_1A_n, in which one has removed the open discs of diameter A_1A_2, A_2A_3, up to $A_{n-1}A_n$.

([26]) For $\psi_\infty = \pm\sqrt{\beta-1}$, the disc is degenerate and reduced to a point.

this one uses $|t| < \sqrt{\beta - 1}$ and the following 3 points

$$P_0 = \Big(\frac{\beta}{2}, 0\Big), P_1 = (1, t), P_2 = \Big(\frac{\beta}{1+t^2}, \frac{\beta\, t}{1+t^2}\Big). \tag{6.41}$$

Obviously, P_1 and P_2 are aligned with the origin O, P_1 is on the line $\xi = 1$ and inside the disc $\xi^2 + \eta^2 \le \beta\,\xi$, P_2 is on the circle $\xi^2 + \eta^2 = \beta\,\xi$ and inside the half space $\xi \ge 1$. Moreover, for $\varepsilon = \beta - 1 - t^2$ (which is > 0) the point

$$\begin{aligned} P_3 &= \frac{1+t^2}{\beta}\Big(\frac{\beta}{1+t^2}, \frac{\beta\, t}{1+t^2}\Big) + \frac{\beta - 1 - t^2}{\beta}\Big(\frac{\beta}{2}, 0\Big) \\ &= \Big(1 + \frac{\varepsilon}{2}, t\Big) = \theta\, P_2 + (1-\theta)\, P_0 \text{ with } \theta = \frac{1+t^2}{\beta} \end{aligned} \tag{6.42}$$

is the center of the desired disc, at distance $\frac{\varepsilon}{2}$ of P_1 and at distance $\frac{(1-\theta)\beta}{2} = \frac{\varepsilon}{2}$ of P_2, and since the segment P_1P_2 is just the continuation of OP_1, it obviously sweeps the correct domain.

One should notice that Theorem 6 does not say that $\mathcal{K}_\beta$ coincides with $\mathcal{M}(1, \beta)$, but only that these two sets of matrices have the same projection obtained by taking their value on any given vector $E \in \mathbb{R}^2$.

7. – Conclusion

If something related to turbulent flows could be deduced from such a study, it would certainly be better not to consider an equation for a scalar unknown, but an equation for a velocity field u of an incompressible fluid, so that besides an added constraint $div(u) = 0$, the equation would also contain the gradient of a "pressure". One should certainly consider a time dependent problem, but also a 3-dimensional situation, since turbulence is known to be a specifically 3-dimensional effect.

The original approach of Andrey Nikolayevich Kolmogorov (1903-1987), who had considered *developed* and *isotropic* turbulence, had at least two defects.

Even if one could guess a good equation for developed turbulence, it would not tell much about the evolution of turbulent flows. It suffices to witness how popular the "fake mechanics/physics" of gradient flows is for wondering why so few mathematicians insist in correctly describing the physical phenomena which some pretend to be interested in, and in explaining the silly character of many studies: knowing the evolution equation governing a physical phenomenon permits to deduce its stationary solutions, as well as the stability of such stationary solutions, but knowing about a principle (often not so physical) which gives the stationary solutions does not permit to deduce either the stability of such stationary solutions, or what the evolution equation for such a phenomenon is.

The defect of considering only isotropic materials comes from a lack of mathematical understanding about what nature produces, but homogenization permits to deduce in a mathematical way what nature may produce at a larger scale if one knows which equations to use in small homogeneous (isotropic) materials, and which interface conditions one should use at an interface between two different materials, so that, knowing about a scalar index of refraction, one may infer a question about the wave equation and the existence of anisotropic materials, whose index of refraction is a symmetric positive definite matrix.

However, Christiaan HUYGENS (1629-1695) was unable to explain an effect of *double refraction* (one incident ray, but two refracted rays) through a crystal of Iceland spar (calcite), which had been discovered by Rasmus BARTHOLIN (1625-1698). Although Étienne Louis MALUS (1775-1812) published a theory of double refraction of light in crystals in 1810, one year after publishing his observations about *polarized light*, it is probably not clear to everyone that these effects are *not related to a scalar wave equation*, but to Maxwell-Heaviside equation, and that anisotropy for such an equation is different from what happens for a scalar wave equation. ([27])

It is then important to have an open mind, for the case where the traditional equations used for describing a phenomenon might not be the good ones, and it is useful that mathematicians consider more general equations than those which have appeared in applications: in inventing H-convergence, F. MURAT and I did a typical work of mathematicians, and such a generalization of the work of S. SPAGNOLO was not mandated by physical considerations, but it was important that we had thought about such a general case, so that the analysis which I have shown above had become almost natural to me.

Qualitatively, it is important that the symmetric part M^{eff} may have different eigenvalues, since the usual ideas about "turbulent viscosity" suffer from the precise defect than I mentioned above, i.e. one should avoid being stuck on the idea that viscosity is a scalar.

Although I have worked above with stream functions ψ which are much too irregular for classical physical interpretations, I find interesting that ψ^{eff} may be different from ψ_∞: in some way, one should expect that in turbulent flows the constitutive equations may have unexpected forms.

It is also important to recall that one defect of continuum mechanics is that *equations of states are only approximations*, and one should then stay open

([27]) The constitutive equations $D = \varepsilon E$ for a scalar *dielectric permittivity* ε, and $B = \mu H$ for a scalar *magnetic susceptibility* μ correspond to an isotropic material with waves propagating at speed $\frac{1}{\sqrt{\varepsilon\mu}}$, and by homogenization both ε and μ become positive definite symmetric matrices, but if they do not have a common basis of eigenvectors, the effects of propagation of waves are not like those of a scalar wave equation.

about the idea that materials may change their microstructures, and that permits some variations of physical quantities at a macroscopic level, and one should then express that a physical quantity belongs to a set, which one may want to think as smaller and smaller the more information one has about micro-geometries (or mesoscopic arrangements).

Of course, a lot more should be done than what I described here, but one should realize that very few "optimal results" are known, so that there is still a lot of clarifying work to do.

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