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Viscosity Solutions of Two-Phase Free Boundary Problems for Elliptic and Parabolic Operators

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To Enrico Magenes, a great master

1. – Introduction

The obstacle problems, the Stefan problem, variational inequalities and, in general, free boundary problems were among the main research topics of the group of mathematicians in Pavia whose leading figure was Enrico Magenes. In this brief survey we present some recent results concerning free boundary problems, for both elliptic and parabolic operators, strongly connected with some of the results achieved by Magenes and his school.

These kind of problems arise in several context and range from constraint energy minimization to phase transitions, from Finance to flow in porous media. We will be mainly concerned with problems in which the state variable can assume two phases and the condition across the free boundary is expressed by an energy balance involving the fluxes from both sides. Among the several concepts of solutions we use the notion of viscosity solution introduced by L. Caffarelli in the seminal papers [9], [10], which seems to be the most appropriate to study optimal regularity of both solutions and free boundaries. Our aim is to describe some of the main results, obtained in the last two decades.

At the same time, we shall single out some questions that are still open, also adding some clues about the typical difficulties one has to face trying to get an answer.

2. – Elliptic Free Boundary Problems

Starting from the elliptic case, we are interested in the following free boundary problem (*f.b.p.* in the sequel) and in its weak solutions in the sense of viscosity.

In the unit ball $B_1 = B_1(0) \subset \mathbb{R}^n$ we are given a continuous function u satisfying

i)

(1) $$\mathcal{L}^1 u = 0 \qquad \text{in } \Omega^+(u) = \{x \in B_1 : u(x) > 0\}$$

(2) $$\mathcal{L}^2 u = 0 \qquad \text{in } \Omega^-(u) = \{x \in B_1 : u(x) \leq 0\}^0.$$

Here, $\mathcal{L}^1$ and $\mathcal{L}^2$ are uniformly elliptic operators with ellipticity constant $\Lambda > 0$, i.e.

$$\Lambda I \le A^j(x) \le \Lambda^{-1} I \qquad \forall x \in B_1,$$

of one of the following type ($j = 1, 2$):

$$\mathcal{L}^j u = \mathrm{Tr}\big(A^j(x) D^2 u\big) + b^j(x) \cdot \nabla u$$

or

(3) $$\mathcal{L}^j u = \Phi^j\big(x, \nabla u, D^2 u\big)$$

or

$$\mathcal{L}^j u = \mathrm{div}\big(A^j(x) \nabla u\big) + b^j(x) \cdot \nabla u,$$

where

$$A^j(x) = \Big(a^j_{ik}(x)\Big), b^j(x) = \Big(b^j_1(x), \dots, b^j_n(x)\Big)$$

and $D^2 u$ is the Hessian matrix of u.

ii) Along

$$F(u) \equiv \partial \Omega^+(u) \cap B_1$$

(the *free boundary*), the following condition holds:

a) if at $x_0 \in F(u)$ there is a ball B such that

$$B \subset \Omega^+(u),\ \overline{B} \cap \Omega^+(u) = \{x_0\}$$

then, near x_0,

(4) $$u^+(x) \ge \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \qquad \text{in } B, \qquad (\alpha > 0),$$

(5) $$u^-(x) \le \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \qquad \text{in } \mathcal{C}B, \qquad (\beta \ge 0),$$

with equality along every nontangential domain in both cases, and

(6) $$\alpha \le G(\beta, \nu, x);$$

b) if at $x_0 \in F(u)$ there is a ball B such that

$$B \subset \Omega^-(u),\ \overline{B} \cap \Omega^-(u) = \{x_0\}$$

then, near x_0

(7) $$u^-(x) \ge \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \qquad \text{in } B \qquad (\beta > 0)$$

(8) $$u^+(x) \le \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \qquad \text{in } \mathcal{C}B \qquad (\alpha \ge 0)$$

with equality along every nontangential domain in both cases and

$$\alpha \geq G(\beta, \nu, x) \tag{9}$$

where ν is the interior unit normal to ∂B at x_0

The conditions (4)-(9), express the free boundary relation $u_\nu^+ = G(|u_\nu^-|, \nu, x)$ in a viscosity sense; in case a) (resp. b)) we say that $x_0 \in F(u)$ is a *right (resp. left) regular point.* In particular, conditions (6) and (9) correspond to a supersolution and subsolution condition, respectively. Accordingly, we call u a *viscosity solution of our f.b.p.* (see [13], Ch. 4).

The main hypotheses on G are:

i) The function

$$z \longmapsto G = G(z, x, \nu)$$

is continuous, strictly increasing on $[0, +\infty)$;

ii) for some $N > 0$, the function

$$z \longmapsto z^{-N} G(z, x, \nu)$$

is decreasing in $(0, +\infty)$.

Typical examples come from constraint minimization problems as

$$\min \int_{B_1} \left\{ a_{ij}(x) u_{x_i} u_{x_j} + q(x) \chi_{\{u>0\}} \right\}$$

over $u \in g + H_0^1(B_1)$, with $q \geq c > 0$ a.e. Here the free boundary condition takes the form

$$\left(u_{\nu^*}^+\right)^2 - \left(u_{\nu^*}^-\right)^2 = q(x)$$

where ν^* denotes the conormal derivative in the direction of the positive phase.

Other examples arise from singular perturbation theory, e.g. as a limiting problem when $\varepsilon \to 0$ for

$$\Delta u_\varepsilon + b(x) \cdot \nabla u_\varepsilon = \beta_\varepsilon(u)$$

where

$$\operatorname{supp}(\beta) \subset [0, 1], \int_{\mathbb{R}} \beta = 1, \beta_\varepsilon(s) = \varepsilon^{-1} \beta(s/\varepsilon).$$

Here the free boundary condition of the limiting problem takes the form

$$\left(u_\nu^+\right)^2 = 2.$$

The main issues arising in a f.b.p. are existence and optimal regularity. Due to the highly nonlinear nature of the problems, uniqueness has hardly to be expected (see [1]).

Clearly, given the jump of the gradients across the free boundary, the optimal regularity of the solution is Lipschitz continuity.

The real challenge, not only from a purely mathematical point of view, is the analysis of the free boundary, also in order to establish how classical the viscosity solutions are. By a classical solution u we mean a C^1-function up to the free boundary $F(u)$ from both sides, satisfying the free boundary condition in a classical pointwise sense. We may split the analysis of $F(u)$ according the following scheme.

a) *Existence/uniqueness, comparison*

$$\text{b) } \textit{Regularity} \begin{cases} \text{of the solution} \\ \text{of the free bdry} \quad \begin{cases} \text{weak} \\ \text{strong} \quad \begin{cases} \textit{Lip} \Rightarrow C^{1,\alpha} \\ \textit{“flatness”} \Rightarrow \textit{more reg.} \end{cases} \end{cases} \end{cases}$$

By *weak* results we mean the basic geometric measure properties, such as $F(u)$ to be a set of finite perimeter or some density properties of points on $F(u)$.

Strong results refers to improvements of the regularity starting from certain assumptions, in the spirit of minimal surface theory. Typically in stationary problems one proves that if we assume that $F(u)$ is locally a uniformly Lipschitz graph then actually, $F(u)$ is locally a $C^{1,\alpha}$ graph. We will refer to this kind of result as *Lipschitz implies* $C^{1,\alpha}$.

Another reasonable and sometimes necessary (e.g. in obstacle problems) starting assumption is the *flatness of* $F(u)$. This kind of condition may be given in several ways. For instance, a surface is *ε-flat* in a neighborhood of one of its point if after a blow-up can be trapped within two hyperplanes at distance ε. This happens for example near a differentiability point. Another way to state a flatness condition appears in Theorem 3 below.

Improvement of flatness of the free boundary leads usually to its Lipschitz continuity or directly to its $C^{1,\alpha}$ regularity. One can then apply the classical result of Kinderlehrer and Nirenberg to get C^∞ or even analytic regularity under the same hypotheses on the data.

3. – Existence and weak regularity of $F(u)$

Let us consider now the existence theory and weak regularity. In [11] Caffarelli considers the case

$$\mathcal{L}^1 = \mathcal{L}^2 = \operatorname{div}(A(x)\nabla)$$

with A Hölder continuous and the non-degeneracy condition $G(0, \nu, x) \geq c > 0$. His main result is the following

THEOREM 1. – *Given a Lipschitz domain Ω and $g \in C(\partial\Omega)$, there exists a viscosity solution $u \in C(\overline{\Omega})$ with $u = g$ on $\partial\Omega$. Moreover, u is Lipschitz in Ω, $\Omega^+(u)$ is a set of finite perimeter and*

$$0 < \alpha_1 \leq \frac{u^+(x)}{dist(x, F(u))} \leq \alpha_2. \tag{10}$$

Note the linear growth of the positive part of the solution, expressed by the left inequality in (10). Caffarelli uses a Perron method constructing a *minimal solution* given by

$$u(x) = \inf_{v \in S} v(x)$$

where S is a class of continuous supersolutions v in $\overline{\Omega}$ such that $v \geq g$ on $\partial\Omega$ and that $F(v)$ is regular from the left. Similarly one can construct a maximal solution with analogous properties. From a philosophical point of view, this result implies the existence of universal regularity and nondegeneracy properties for these kind of problems. This cannot be expected for instance in parabolic two phase problems as we will see in the sequel.

This existence result has been extended by P.Y. Wang in [31] to the case $\mathcal{L}^1 = \mathcal{L}^2 = F(D^2u)$ with F concave.

Here is one open question.

OPEN PROBLEM 1. – *Existence and weak regularity for $\mathcal{L}^1 = \mathcal{L}^2 = F(D^2u, Du)$, with F non concave in the Hessian matrix and even for the linear case $\mathcal{L}^1 = \mathcal{L}^2 = Tr(A(x)D^2u) + b(x) \cdot \nabla u$.*

Let us see what constitutes the main obstruction in extending Caffarelli's method. The key tool is the following *monotonicity formula* of Alt, Caffarelli and Friedman (see [2]).

THEOREM 2. – *Let $u = u^+ - u^-$ be such that div $(A(x)\nabla u^\pm) \geq 0$ in B_1, with $u^+(0) = u^-(0) = 0$. Assume that A is Hölder continuous with exponent α. Then for, say, $0 < r \leq 1/2$, and some constant $c(n) > 0$, the function*

$$\Phi(r) \equiv r^{-4} e^{-cr^\alpha} \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}}\, dx \int_{B_r} \frac{|\nabla u^-|^2}{|x|^{n-2}}\, dx$$

is increasing and

$$\Phi(r) \leq c(n) \|u\|^4_{L^\infty(B_1)}.$$

Observe that if the supports of u^+ and u^- where separated by a smooth surface with normal ν at $x = 0$ then, by taking the limit as $r \to 0$, we could deduce that

$$\left(u_\nu^+(0)\right)^2 \left(u_\nu^-(0)\right)^2 \leq \Phi(1/2)$$

so that, "morally" $\Phi(r)$ gives a control in average of the product of the normal derivatives of u at the origin.

It seems hard to extend the above theorem to non-divergence form operators. It can be proved in some special cases under additional hypotheses; for instance, one can prove the following result ([22]):

THEOREM 3. – *For* $1/2 \leq \beta \leq 1$ *and* $k = 1, 2$ *let* $u_k \in C^{0,\beta}(B_1(0)) \cap C^2(B_1(0) \cap \Omega^+(u_k))$ *be non negative functions satisfying the following conditions:* $u_1 u_2 = 0$, $u_1(0) = u_2(0) = 0$,

$$Lu_k \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u_k(x) \geq 0$$

in $\Omega^+(u_k) = \{x \in B_1(0) : u_k > 0\}$.

Suppose $a_{ij} \in C^{0,\alpha}(B_1(0))$, $0 < \alpha \leq 1$ *and moreover that* u_k *enjoy the estimates:*

$$c_1 \frac{u_k(y)}{d_y} \leq |\nabla u_k(y)| \leq c_2 \frac{u_k(y)}{d_y}, ::::| D^2 u_k(y) | \leq c_3 \frac{u_k(y)}{d_y^2} \tag{11}$$

in $\Omega^+(u_k)$, $k = 1, 2$, *where* d_y *is the distance from the zero set. Then for* $0 < r < \dfrac{1}{2}$

$$\Phi(r) \equiv \frac{e^{Cr^\alpha}}{r^4} \int_{B_r} \frac{| \nabla u_1(y) |^2}{| y |^{n-2}} dy \int_{B_r} \frac{| \nabla u_2(y) |^2}{| y |^{n-2}} dy$$

is bounded. In particular $\Phi(r) \leq C \|u_1\|^2_{C^{0,\bar\beta}(1/2)} \|u_2\|^2_{C^{0,\bar\beta}(1/2)}$ *and* $\Phi(r)$ *is monotone increasing.*

The main assumption in the above theorem is really the first inequality in (11). For instance, if u_1, u_2 were solutions (instead of just subsolutions) and their supports were separated by a Lipschitz surface, then (11) is true. If no information on $F(u)$ is available, In view of the applications to free boundary problems, one should be able to use the free boundary condition itself, but so far no proof is available.

Thus:

OPEN PROBLEM 2. – *To prove a monotonicity formula for nonnegative subsolutions to nondivergence form operators with Hölder coefficients.*

Some related results concerning questions of uniqueness in two-phase problems are proven by Guozhen Lu and P.Y. Wang in [24]. In particular, via a very interesting comparison theorem, uniqueness of a viscosity solution is established in singular perturbation problems for a class of fully non linear operator of the form $F(D^2u, Du)$, including for instance the p-Laplace operator.

4. – Regularity: strong results

The regularity theory for the Laplace operator has been developed by L. Caffarelli in the two seminal papers [9], [10].

In particular the "*Lipschitz implies* $C^{1,\alpha}$" part is contained in [9] while the *flat implies Lipschitz* part is shown in [10]. In these papers Caffarelli sets up a general strategy to attack the regularity of the free boundary through an iterative procedure, based on interior and boundary Harnack inequalities.

Briefly the strategy of the proof in [8] consists of the following main steps. Starting from a Lipschitz graph, one shows that in a neighborhood of $F(u)$ the level sets of u are still Lipschitz graph, locally in the same direction. Then one improves the Lipschitz constant of the level sets of u away from the free boundary. Here Harnack inequality applied to directional derivatives of u plays a major role. Then the task is to carry this interior gain up to the free boundary. To this aim, Caffarelli introduces a powerful method of continuity based on the construction of a family of continuous subsolutions, on which we will came back later on. Finally, by rescaling and iterating the last two steps, one obtains a geometric decay of the Lipschitz constant, which amounts to the $C^{1,\alpha}$ regularity of $F(u)$.

After 10 years M. Feldman (see [17]) considers anisotropic operators with constant coefficients and extends to this case the results in [8].

P. Y. Wang manages to extend the results both in [8] and [9] to a class of concave fully non linear operators of the type $F(D^2u)$ (see [W2]). One year later Feldman (see [18]) considers a class of non concave fully non linear operators of the type $F(D^2u, Du)$. He shows that Lipschitz free boundaries are $C^{1,\alpha}$ thus extending to this case the results in [8].

The first papers dealing with variable coefficient operators are by Cerutti, Ferrari, Salsa (see [12]) and by Ferrari ([16]). They consider respectively, linear elliptic operators in non-divergence form and a rather general class of fully nonlinear operators $F(D^2u, Du, x)$, with Holder continuity in x, including Bellman's operators. One of the main difficulty in extending the theory to variable coefficients operator is the fact that directional derivatives do not satisfy any reasonable elliptic equation.

A refinement of the techniques in [12] leads to the following results (see [19]), where the drift coefficient is merely bounded measurable.

THEOREM 4. – *Let u be a weak solution of our free boundary problem in B_1, where $\mathcal{L}^j u = Tr(A^j(x)D^2u) + b^j(x)\cdot\nabla u$, $j=1,2$. Suppose $0\in F(u)$ and that*

i) $A^j \in C^{0,a}(C_1)$, $0<a\le 1$, $b^j\in L^\infty(B_1)$.

ii) $0<\alpha_1 \le \dfrac{u^+(x)}{dist(x,F(u))} \le \alpha_2$.

iii) $G(0,\nu,x)\ge c>0$.

There exist $0<\overline{\theta}<\pi/2$ and $\overline{\varepsilon}>0$ such that, if for $0<\varepsilon<\overline{\varepsilon}$, $F(u)$ is contained in an ε-neighborhood of a graph of a Lipschitz function $x_n=g(x')$ with

$$Lip(g)\le \tan\left(\frac{\pi}{2}-\overline{\theta}\right)$$

then $F(u)$ is a $C^{1,\alpha}$-graph in $B_{1/2}$.

Condition *ii)* expresses a linear behavior of u^+ at the free boundary while being trapped in a neighborhood of two Lipschitz graph with small Lipschitz constant is another way to express a flatness condition. Thus, *flatness* plus *linear behavior* of the positive part imply smoothness.

Under the same hypotheses on the operators, one can also prove that Lipschitz free boundary are smooth. Namely ([19]):

THEOREM 5. – *Let u be a weak solution of our free boundary problem in B_1, where $\mathcal{L}^j u = Tr(A^j(x)D^2u) + b^j(x)\cdot\nabla u$, $j=1,2$. Suppose $0\in F(u)$ and that*

i) $A^j \in C^{0,a}(C_1)$, $0<a\le 1$, $b^j\in L^\infty(B_1)$.

ii) $\Omega^+(u)=\{(x',x_n): x_n>f(x')\}$ *where f is a Lipschitz continuous function with $Lip(f)\le L$.*

iii) *$G=G(z)$ is continuous, strictly increasing and for some $N>0$, $z^{-N}G(z)$ is decreasing in $(0,+\infty)$. Then, on $B'_{1/2}\subset\mathbb{R}^{n-1}$, f is a $C^{1,\gamma}$ function with $\gamma=\gamma(n,N,L,\Lambda,a)$.*

This theorem has been recently extended to the same class of fully nonlinear operators considered in [16] by Argiolas and Ferrari (see [3]).

We draw two consequences from the above theorems.

COROLLARY 6. – *The conclusion of Theorem 5 holds if $L^1=L^2=div(A(x,u)\nabla u)$ with A Lipschitz with respect to all its arguments.*

The other application is to the minimal solution constructed in [11]:

THEOREM 7. – *Let u be the minimal viscosity solution constructed in Theorem 1. Assume $A=A(x)$ is Lipschitz. Then, if $x_0\in\partial_{red}\,\Omega^+(u)\cap B_{1/2}$, $F(u)$ is a $C^{1,\alpha}$-graph in a neighborhood of x_0.*

Thus, the theory of viscosity solution of general free boundary problems for divergence form operators can be considered quite satisfactory, at least in the case of Lipschitz coefficients.

Naturally, we pose:

OPEN PROBLEM 3. – *Regularity of the free boundary in the case of divergence form operators with Hölder coefficients.*

Let us examine what is the main difficulty in dealing with divergence form operators with Hölder coefficients.

Let us go back to the continuation method used by Caffarelli in [8] to carry up to the free boundary the interior decay on the Lipschitz constant. The key point is the construction of a family of deformations, constructed as the supremum of an harmonic function over balls of variable radius. Here is the main question: let u be a given non negative function, harmonic on its support. Let g be a smooth function, $1 \leq g \leq 2$, and define the *sup-convolution*

$$v_g(x) = \sup_{B_{g(x)}(x)} u.$$

Under which condition on g *is* v_g *subharmonic on its support?*

The answer is given by the following differential inequality:

$$g\Delta g \geq C(n)|\nabla g|^2. \tag{12}$$

The situation in the variable coefficient case is much more involved. For instance, if we have a non negative function u such that $\mathcal{L}u = \mathrm{Tr}\big(A(x)D^2u\big) + b(x) \cdot \nabla u = 0$ on its support, the condition that g has to satisfy in order to make v_g an $\mathcal{L}$-subsolution on its support takes the following form:

$$\mathcal{L}g \geq C(n,\Lambda)\left\{\frac{|\nabla g|^2 + \omega^2}{g} + \|b\|_{L^\infty}\right\} \tag{13}$$

where ω is the modulus of continuity of A computed at $\max g/\Lambda$.

There are two main draw-backs in condition (13) with respect to its constant coefficient counterpart.

The first one is fact that it is not homogeneous with respect to g; this causes the need of a delicate balance between rescaling and decay of the Lipschitz constant of the level sets of u in the iteration procedure to carry the interior gain to the free boundary.

The second one is that the proof has an intrinsic non-divergence feature and so far any attempt to find a proof for divergence form operator with Hö lder continuous coefficients has failed (see [FS2] for a more divergence form oriented proof).

A recent quite interesting result of De Silva ([14]) deals with one phase problems with right hand side. The main feature of this paper is that presents a new approach in order to improve the flatness of the free boundary and achieve $C^{1,\alpha}$ regularity. The novelty with respect to Caffarelli's method is that it avoids the use of the sup-convolutions and it is based uniquely on rescaling and Harnack inequality.

There is a strong hope that using a combinations of his method with Caffarelli's technique and the perturbation analysis in [12], [19] one should succeed to prove regularity of flat and or Lipschitz free boundaries, even with non-zero right hand side, which indeed constitutes a major extension of the theory.

OPEN PROBLEM 4. – *Optimal regularity of the solution and analysis of the free boundary for two phase problems with distributed sources.*

5. – Evolutionary Free Boundary problems

We now consider evolution free boundary problems. Formally one seeks for a function in a space-time cylinder $C_R = B_R \times (-R^2, R^2)$ such that

$$L^1 u = 0 \text{ in } \Omega^+(u) \text{ and } L^2 u = 0 \text{ in } \Omega^-(u)$$

and

$$V_\nu = -\frac{u_t^+}{|u_\nu^+|} = -G(u_\nu^+, |u_\nu^-|, \nu, x, t)$$

on $F(u) = \partial\Omega^+(u) \cap C_R$, the free boundary.

Here

$$L^j = \mathcal{L}^j - D_t$$

where

$$\mathcal{L}^j u = \mathrm{Tr}\big(A^j(x,t) D^2 u\big) + b^j(x,t) \cdot \nabla u$$

or

$$\mathcal{L}^j u = \Phi^j\big(x, t, \nabla u, D^2 u\big)$$

or

$$\mathcal{L}^j u = \mathrm{div}\big(A^j(x,t)\nabla u\big) + b^j(x,t) \cdot \nabla u$$

are uniformly elliptic operators. In the free boundary condition $\nu = \nabla u^+/|\nabla u^+|$, so that V_ν represents the speed of the free boundary in the positive phase direction. Typical example come from the classical two-phase Stefan problem where $G(u_\nu^+, |u_\nu^-|, \nu, x, t) = u_\nu^+ - |u_\nu^-|$.

We require that G is Lipschitz continuous with respect to all its arguments, strictly increasing with respect to u_ν^+ and strictly decreasing with respect to u_ν^-.

By classical supersolution resp. (subsolution) of the above problem we mean a smooth function v in both $\overline{\Omega}^+(u)$ and $\overline{\Omega}^-(u)$, $\mathcal{L}^j$-supercaloric (resp. subcaloric) whose free boundary is a smooth surface, satisfying the free boundary condition $V_\nu \geq -G\big(u_\nu^+, |u_\nu^-|, \nu, x, t\big)$ (resp. $\geq$) in a pointwise sense. The inequality in the free boundary condition reflect the fact that for a supersolution (subsolution) the speed V_ν has to be smaller (greater) than the one of a solution sharing the same data on the parabolic boundary $\partial_p C_R$ of C_R.

We shall deal with viscosity solutions that we introduce below.

DEFINITION. – *A continuous function u in C_R is a viscosity subsolution (respectively supersolution) if for every subcylinder $Q \subset C_R$ and every classical supersolution (respectively subsolution) in Q, $u \leq v$ on $\partial_p C_R$ (respectively $u \geq v$) implies $u \leq v$ in Q (respectively $u \geq v$). The function u is a viscosity solution if it is both a viscosity sub and a super solution.*

For an evolution problem we can pose the same questions on existence and regularity issues, both for the solution and the free boundary. Here other important questions are related to the asymptotic behavior for $t \to +\infty$, for instance. However, the presence of time entails new, serious difficulties. The first and obvious one is the role of time, already present in the Harnack inequality, in which the past controls the future only from below. This implies that stronger hypotheses have to be made on the geometry of the free boundary or on the starting configuration and that the strong local conclusions achieved in the elliptic case cannot be obtained.

6. – Existence and uniqueness

Existence results for viscosity solutions by I. C. Kim and N. Požar can be found in [25]. Actually, in this quite nice paper, the authors give a slightly different notion of viscosity solution and also prove a comparison theorem for sub and supersolution with strictly separated boundary data. If $L = \Delta$ and $G(a, b) = a - b$, which corresponds to the classical two-phase Stefan problem, they also prove the equivalence of the notions of viscosity solutions and weak solutions in Sobolev spaces, defined via the so called enthalpy formulation. A remarkable consequence is the uniqueness of viscosity solutions with continuous boundary data.

Having a comparison theorem at hand they can use a Perron method to construct minimal and maximal solutions.

Another recent important paper is [25], where the authors prove that the classical two-phase Stefan problem admits a unique local (in time) solution that is *analytic* in space and time.

OPEN PROBLEM. – *Can one prove weak regularity results for the minimal/maximal solutions, as in Caffarelli's Theorem* 1?

This is not a trivial question, given the strong degeneracy of the free boundary condition.

7. – Regularity

We now come to the regularity questions. Also here the understanding of the problem is well developed when $L^j = \Delta - a_j D_t$. In a series of papers Athanasopoulos, Caffarelli and Salsa obtain the following results.

Optimal regularity of the solution (see [6]). If $F(u)$ is locally a Lipschitz graph both in space and time then u is Lipschitz across $F(u)$. Note that there are counterexamples showing that in general the solution in the two phase Stefan problem is not Lipschitz (see [23]). We point out that, although the heat equation scales parabolically, the free boundary condition is invariant under Hyperbolic rescaling, so that Lipschitz continuity in space and time (rather than Lipschitz in space, $1/2$ Hölder in time) appears as an appropriate hypothesis for $F(u)$. Basically it amounts to say that the speed of $F(u)$ is finite. We will come back on this crucial aspect.

Are Lipschitz free boundaries smooth? We observed above that, in general, additional hypotheses have to be assumed on the geometry of the free boundary to achieve strong regularity results. A striking evidence of this fact is that Lipschitz free boundaries could not regularize as two counterexamples show: one is a one-phase case in dimension $n = 2$, in which $u^- \equiv 0$ (see [26]), the other one is a true two-phase Stefan problem in dimension $n = 3$ (see [7]). Thus the situation is quite different with respect to the stationary case.

Let us briefly describe the one-phase counterexample. Consider the function

$$w(\rho, \theta, t) = \rho^{g(t)} \{\cos[g(t)\theta]\}^+$$

where ρ, θ are polar coordinates in the plane and g is a decreasing function greater than 2.

If R is chosen sufficiently small, depending on g, then w is a supersolution of the one-phase Stefan problem in C_R. At the origin, $F(w)$ shows a persistent corner with an angle less than $\pi/2$, since $g > 2$, and the heat flux there is zero (from both sides of $F(w)$).

Let now u be the solution of the one-phase Stefan problem in C_R with $u = w$ on $\partial_p C_R$. Then $u \leq w$ in C_R forcing $F(u)$ to have a persistent corner at the origin as long as the angle stays less than $\pi/2$. Numerical simulations by Nochetto, Schmidt and Verdi (see [27]) seems to indicate that the critical angle should actually be greater than $\pi/2$.

A closer look to the counterexamples reveals that the obstruction to instantaneous regularization comes from two facts. The first is the simultaneous vanishing of the two fluxes from both sides of the free boundary and the second is the largeness of the Lipschitz constant in space of the free boundary.

Thus, positive results can be given along two directions. First (see [7]), if $F(u)$ is locally Lipschitz and a non-degeneracy condition of the form $u_\nu^+ + u_\nu^- \geq m > 0$ holds in a suitable weak sense, then $F(u)$ is a C^1-surface, the time sections $F_\tau(u) = F(u) \cap (t = \tau)$ are Liapunov-Dini domains and the solution is locally classical. The main strategy follows the lines of the elliptic case: improvement of the Lipschitz constant of the level sets of u away from $F(u)$, propagation of this interior gain to $F(u)$, rescaling and iteration. But things are not so simple as we will see below.

Flat free boundaries are smooth. The same result can be achieved (see [8]) if we ask that the Lipschitz constant in space is small. Indeed the flatness condition carries a sort of nondegeneracy through a variant of a Hopf principle at regular points of the free boundary. As a consequence it is possible to prove an instantaneous regularization from flat initial free boundary and to show a waiting time regularization phenomenon when the solution evolves in time towards a non degenerate steady state solution.

While the counterexamples indicates how strongly degenerate the problem is, another major source of difficulties comes from the rescaling properties of the problem. In principle, there are three types of rescaling that one could use: a *parabolic* rescaling, $u_\rho(x,t) = u(\rho x, \rho^2 t)$; a *parabolic blow up*, $u_\rho(x,t) = \rho^{-1} u(\rho x, \rho^2 t)$; a *hyperbolic blow up*, $u_\rho(x,t) = \rho^{-1} u(\rho x, \rho t)$. Here we assume that $\rho \to 0$.

All these rescalings have advantages and disadvantages. The parabolic rescaling leaves both the equation and (only if G is linear) the free boundary condition unchanged; on the other hand, it progressively deteriorates the nondegeneracy conditions.

After a parabolic blow-up, the equation and the nondegeneracy condition remains unaltered, but the free boundary condition progressively degenerates, preventing from any gain in regularity.

The hyperbolic blow-up leaves nondegeneracy and the free boundary conditions unchanged. There are two drawbacks. A minor one is that the coefficient of $D_t u_\rho$ is vanishing, disconnecting more and more the various time levels. However this inconvenient is kept under control during the iterations by the space-time monotonicity properties of u. A more serious problem is that every estimate have to be done in hyperbolic geometry. This forces the use of an intermediate pseudohyperbolic blow-up, tailored to overcome the above difficulty, which leads to a weaker than $C^{1,\alpha}$ regularity of the free boundary as indicated in Theorem 8 below.

The extension of the above results to variable coefficients runs into extra complications, much more serious than in the elliptic counterpart. Since directional derivatives do not satisfy any kind of reasonable equation, one has to resort to a quite delicate perturbation arguments.

In [21] Ferrari and Salsa prove the optimal regularity (Lipschitz continuity) of the solution under non-degeneracy and flatness conditions. The key point is a control of u_t by the spatial gradient, which is already a quite delicate estimate in the case of the heat equation and that takes the following form:

$$|u_t(x,t)| \leq c d_{x,t}^{a-1} D_n u(x,t) \tag{14}$$

for some $a \in (0,1]$, where $d_{x,t}$ denotes the distance of (x,t) from $F(u)$. Note that in the case of time independent operator this estimates carries out as in [8] and no flatness nor any non-degeneracy condition is needed.

The analysis of the free boundary requires a potential analysis apparatus, developed in two very nice papers by R. Argiolas and A. Grimaldi (see [4] and [5]). Among other results, they prove that the L-caloric measure in a Lipschitz (non cylindrical) domain Ω in $\mathbb{R}^{n+1}$ is an A_∞ weight with respect to surface measure on $\partial\Omega$, a crucial fact, assuring a common interior gain in the Lipschitz constant of the level set of the solution from both side.

In a recent paper, still to appear, Ferrari and Salsa (see [23]) prove that flat free boundaries are indeed smooth, without requiring an a priori nondegeneracy condition. Precisely, the following theorem holds:

THEOREM 8. – *Let u be a viscosity solution to our f.b.p. in $C_2 = B_2' \times (-2,2)$ and set $M = \sup_{C_2} |u|$, $u^+(e_n, 3/2) = m^+ > 0$, $u^-(e_n, 3/2) = m^- \geq 0$.*

Assume that the free boundary $F(u)$ is given by the graph of a function $x_n = f(x',t)$ such that $(0,0) \in F(u)$ and for every $(x,t),(y,s) \in C_2 \cap F(u)$,

$$| f(x',t) - f(y',s) | \leq L | x' - y' | + L_0 | t - s | .$$

Then, if L is small enough, depending only on n, $\|A\|_{C^{0,\alpha}}$, $\|\mathbf{b}\|_\infty$, L_0, λ, Λ, $m^\pm/M$, the following conclusions hold in C_1 :

(i) If $G = G(a,b)$ is a linear function, then $F(u)$ is a C^1 graph in space and time. Moreover, there exist positive constants c_1, c_2 depending only on n,

$\|A\|_{C^{0,\alpha}}$, $\|\mathbf{b}\|_\infty$, L_0, λ, Λ, $m^\pm/M$, *such that, for every* $(x', x_n, t), (y', y_n, s) \in F(u)$:

$$| \nabla_{x'} f(x',t) - \nabla_{x'} f(y',s)| \leq c_1 (\log |x'-y'|)^{-4/3} \; ;$$
$$| D_t f(x',t) - D_t f(x',s)| \leq c_1 (\log |t-s'|)^{-1/3}.$$

(ii) If G is non linear, the same conclusions hold if $\left|\frac{\Lambda}{\lambda}-1\right|$ *and* $\| b \|_\infty$ *are sufficiently close to* 0, *depending on* n, α, L_0 *and* c_G.

As a consequence u is a classical solution that is $\in C^1(\overline{\Omega}^+(u)) \cap C^1(\overline{\Omega}^-(u))$ *and on* $F(u)$:

$$u_\nu^+ \geq c_2 > 0$$

The overall strategy of the proof of Theorem 8 follows the papers [7] and [9] by using a perturbation technique from the constant coefficients, based on three key facts. One is estimates (14) which allows to extend Hopf's principle at regular points of $F(u)$ thus recovering nondegeneracy.

The second one is the construction of a family of continuous deformations as in the elliptic case, but with the crucial difference that the *inf/sup-convolutions*

$$v_{g(x,t)}(x,t) = \sup_{B_{g(x,t)}(x)} u(x,t)$$

becomes $\mathcal{L}$-*super/sub-solutions* for a family of operators with close coefficients, not for a single operator. Precisely, the following theorem holds:

THEOREM 9. – *Let u be a solution of our f.b.p. in a cylinder C for the operator* $L_{B,\mathbf{b}}$. *Assume that* $L_{A,\mathbf{b}'}$ *is another operator in the same class. Let* ε_0 *be as in Lemma and* $\phi \in C^2(\overline{C}_R)$ *be a strictly positive function. Let* $\omega = \omega(\phi_{max})$. *Assume that:*

$$| A - B | \leq C_0 \left(\frac{\Lambda}{\lambda} - 1\right) \leq \omega, \left\|\mathbf{b}-\mathbf{b}'\right\|_\infty \leq \omega,$$

and that in a smaller cylinder $C' \subset C$, *with* $d(\mathcal{C}', \mathcal{C}) \geq \rho \gg \varepsilon_0$,

$$D_t \phi \geq 0,$$

and

$$\mathcal{P}^-(\phi) - c_1 \partial_t \phi \geq C \, \frac{| \nabla \phi |^2 + \omega^2}{\phi} + c_2 (\, | \, \nabla \phi \, | + \omega). \tag{15}$$

for some positive constants C_0, C, c_1, c_2 *depending only on* $n, \lambda, \Lambda, \rho$. *Then, in both* $\Omega^\pm(v_{\phi,\tau})$ $v_{\phi,\tau}$ *is a* $L_{A,\mathbf{b}'}$*-subsolution and in both* $\Omega^\pm(w_{\phi,\tau})$ $w_{\phi,\tau}$ *is a* $L_{A,\mathbf{b}'}$*-supersolution.*

A consequence (the third key fact) is a local interior stability result, which states that under small perturbations of the coefficients flat free boundaries remain close and flat. This seems to be a new results for two phase problems.

Many questions remains open. Here we list some of them.

OPEN PROBLEM 5. – *Prove that Lipschitz, nondegenerate (but non necessarily flat) free boundaries are smooth.*

This result would allow nonlinear divergence operators of the type

$$u_t - \operatorname{div}(A(x,t,u)).$$

The next question concern the notion of flatness. For instance assume that $F(u)$ is not necessarily a graph but is ε-*flat* in the sense that it is trapped between two flat Lipschitz graphs at distance ε. This situation occurs for instance under a blow-up around a differentiability point of the free boundary. We ask:

OPEN PROBLEM 6. – *Can we expect further regularity if $F(u)$ is ε-flat, with ε small enough?*

As we mentioned in the constant coefficient case, a positive answer could be useful in establishing waiting time regularization phenomena, in the style of porous medium equations.

Other questions are related to the extension of the regularity theory to fully non linear operators or to problems with distributed sources. These questions at the moment seems to be at another order of complexity and most likely they require new methods and ideas.

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