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# Pointwise Gradient Estimates of Glaeser's Type

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*Dedicated to Enrico Magenes with deep admiration*

**Abstract.** – *In this paper we are concerned with gradient estimates for viscosity solutions of fully nonlinear second order elliptic equations, generalizing to the nonlinear setting the results of Yanyan Li and Louis Nirenberg about the so-called Glaeser estimate and improving the qualitative results contained in one of our preceding papers.*

## 1. – Introduction

Let $B_R$ be the $n$-dimensional ball centered at the origin with radius $R$. A few years ago Yanyan Li and Louis Nirenberg [12] proved that for *non-negative* functions $u \in C^2(B_R)$ such that $\sup_{B_R} |\Delta u| \leq M$ there exists a universal constant $C = C(n)$ such that

$$\frac{|Du(0)|}{C} \leq \begin{cases} \sqrt{2u(0)M} & \text{if } \sqrt{\dfrac{2u(0)}{M}} \leq R \\[2ex] \dfrac{u(0)}{R} + \dfrac{M}{2}R & \text{if } R \leq \sqrt{\dfrac{2u(0)}{M}} \end{cases} \tag{1.1}$$

In the same paper [12] the authors proved also the validity of an $L^p$ version of inequalities (1.1), namely that for *non-negative* functions $u \in W^{2,p}(B_R)$, such that $\|\Delta u\|_{L^p(B_R)} \leq M$ with $p > n$, there exists a positive constant $C = C(n,p)$ such that

$$\frac{|Du(0)|}{C} \leq \begin{cases} u(0)^{\frac{p-n}{2p-n}} M^{\frac{p}{2p-n}} & \text{if } \left(1 - \dfrac{n}{p}\right)^{-\frac{p}{2p-n}} \left(\dfrac{u(0)}{M}\right)^{\frac{p}{2p-n}} \leq R \\[2ex] \dfrac{u(0)}{R} + MR^{1-\frac{n}{p}} & \text{if } R \leq \left(1 - \dfrac{n}{p}\right)^{-\frac{p}{2p-n}} \left(\dfrac{u(0)}{M}\right)^{\frac{p}{2p-n}} \end{cases} \tag{1.2}$$

Their result extend the validity of the interpolation inequality

$$|u'(0)| \le \begin{cases} \sqrt{2u(0)M} & \text{if } \sqrt{\dfrac{2u(0)}{M}} \le R \\ \dfrac{u(0)}{R} + \dfrac{M}{2}R & \text{if } R \le \sqrt{\dfrac{2u(0)}{M}} \end{cases} \tag{1.3}$$

established by G. Glaeser in [8] for *non-negative* $C^2$ functions such that $|u''| \le M$ on an interval $(-R, R)$. Note that, letting $R \to +\infty$, inequalities (1.3) imply

$$|u'(0)| \le \sqrt{2u(0)\,\|u''\|_{L^\infty}}\ ,$$

a version of the classical Hadamard-Landau-Kolmogorov inequality, see [9], [10], [11] and [13], [14] for more recent results.

In a series of papers [3], [4], [5], the present authors proved several extensions of the results of [12] to *non-negative* functions $u \in C(B_R)$ satisfying, for some given elliptic function $F$, nonlinear relations on second derivatives of the form

$$|F(D^2u)| \le M \tag{1.4}$$

in the viscosity sense, see [6]. Let us recall that a mapping $F : S^n \to \mathbb{R}$, where $S^n$ is the set of symmetric $n \times n$ real matrices, is *uniformly elliptic*, see [2], if there exist positive constants $\lambda \le \Lambda$ such that

$$\lambda\|Z\| \le F(X+Z) - F(X) \le \Lambda\|Z\| \quad \forall X, Z \in S^n, \quad Z \ge 0\ . \tag{1.5}$$

Note that the special linear case $F(D^2u) = \mathrm{Tr}\,(D^2u)$ corresponds to the Li-Nirenberg setting recalled above.

The aim of this paper is to allow zero-order terms in (1.4), that is we consider functions $u$ satisfying relations such as

$$|F(D^2u) - g(u)| \le M \quad \text{in } \ B_R \tag{1.6}$$

in the viscosity sense as well as $L^p$ estimates

$$\|F(D^2u) - g(u)\|_{L^p(B_R)} \le M\,. \tag{1.7}$$

Examples of functions satisfying (1.6) and (1.7) are of course, continuous *viscosity solutions* or, respectively, $W^{2,p}$ *strong solutions* of the second order partial differential equation

$$F(D^2u) - g(u) = f(x)$$

where the right-hand side $f \in L^\infty$ or $f \in L^p$.

Concerning the zero-order nonlinear term $g$ we shall assume that

$$g : \mathbb{R}_+ \to \mathbb{R} \quad \text{is continuous and} \quad |g(s)| \le Gs \quad \text{for all } s > 0 \tag{1.8}$$

for some positive constant $G$.

Our most general result is the following:

THEOREM 1.1. – *Assume that $F$ and $g$ satisfy* (1.5), (1.8) *and let $u \in C(B_R)$ be a non-negative viscosity solution of*

$$F(D^2 u) - g(u) = f(x), \quad x \in B_R$$

*with $f \in C(B_R)$ such that*

$$\|f\|_{L^\infty(B_R)} \le M\,. \tag{1.9}$$

*Then there exist positive constants $\chi$ and $\sigma$ depending only on $n$, $\lambda$ and $\Lambda$ such that*

$$\chi \sup_{B_{R_0/2}} |Du| \le \begin{cases} \sqrt{u(0)M} & \text{if } R_* \le \min(R_G, R) \\ \dfrac{u(0)}{R} + MR & \text{if } R \le \min(R_*, R_G) \\ u(0)\sqrt{G} + \dfrac{M}{\sqrt{G}} & \text{if } R_G \le \min(R, R_*) \end{cases} \tag{1.10}$$

*where $R_* = \sqrt{u(0)/M}$, $R_G = \sqrt{\sigma/G}$ and $R_0 = \min(R_*, R_G, R)$.*

In Section 2 we perform a direct, elementary approach to the one-dimensional case; in Section 3 we consider non-negative solutions of

$$\Delta u = g(u) + f(x)$$

both in the cases $u \in C^2(B_R)$, with direct calculations for the Poisson equation, and $u \in W^{2,p}(B_R)$, using in this case the classical $L^p$ estimates for strong solutions in Sobolev spaces.
Finally, in Section 4 we use in an essential way the regularity theory of Caffarelli [1] for the proof of Theorem 1.1.

## 2. – The one-dimensional case

Throughout this section we will assume that $u : (-r, r) \to \mathbb{R}$ is a non-negative $C^2$ function such that

$$u'' - g(u) \le M \quad \text{in} \quad (-r, r) \tag{2.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$g^+(s) \le Gs, \quad s > 0\,. \tag{2.2}$$

The following mean value inequality can be seen as a weak form of the Harnack inequality:

LEMMA 2.1. – *If* (2.2) *holds with* $G < \frac{2}{r^2}$, *then the following inequality holds*

$$\fint_{-r}^{r} u \le \frac{1}{1 - \frac{1}{2} G r^2} \left( u(0) + \frac{M}{6} r^2 \right) \tag{2.3}$$

PROOF. – Taylor's formula with integral remainder gives

$$u(x) = u(0) + u'(0)\, x + \int_0^x (x - s)\, u''(s)\, ds, \quad |x| < r, \tag{2.4}$$

whence by inequality (2.1),

$$u(x) \le u(0) + u'(0)\, x + \frac{M}{2} x^2 + \int_0^x (x - s)\, g^+(u(s))\, ds\,. \tag{2.5}$$

Integrate both sides between on $(-r, r)$:

$$\begin{aligned} \int_{-r}^{r} u(t)\, dt &\le 2u(0) r + \frac{M}{3} r^3 + \int_{-r}^{r} g^+(u(s))\, \frac{(r - |s|)^2}{2}\, ds \\ &\le 2u(0) r + \frac{M}{3} r^3 + \frac{r^2}{2} \int_{-r}^{r} g^+(u(s))\, ds \end{aligned}$$

Divide now by $2r$ and use assumption (2.2) to get

$$\frac{1}{2r} \int_{-r}^{r} u(t)\, dt \le u(0) + \frac{M}{6} r^2 + \frac{G}{2} \left( \frac{1}{2r} \int_{-r}^{r} u(s)\, ds \right) r^2,$$

from which the assertion readily follows. $\square$

As a consequence, we have:

COROLLARY 2.2. – *Let G be as in* Lemma 2.1, *then*

$$|u'(0)| \le \frac{u(0)}{r} + \frac{M}{2} r + \frac{2 G r^2}{1 - \frac{G r^2}{2}} \left( \frac{u(0)}{r} + \frac{M}{6} r \right) \tag{2.6}$$

PROOF. – Assume first that $u'(0) < 0$. Using the positivity of $u$ together with assumption (1.8), inequality (2.5) with $x = r$ yields

$$
\begin{aligned}
|u'(0)| &\le \frac{u(0)}{r} + \frac{M}{2}\,r + \frac{1}{r}\int_0^r (r-s)\,g^+(u(s))\,ds \\
&\le \frac{u(0)}{r} + \frac{M}{2}\,r + \frac{G}{r}\int_0^r (r-s)\,u(s)\,ds \\
&\le \frac{u(0)}{r} + \frac{M}{2}\,r + 2Gr \fint_{-r}^{r} u
\end{aligned}
$$

The case $u'(0) > 0$ can be treated in an analogous way. At this point the statement follows from Lemma 2.1. □

REMARK 2.3. – If $M = 0$ and $r = R$, set $R_G = \sqrt{2/G}$. The estimate in Lemma 2.2 becomes in this case

$$
|u'(0)| \le \frac{1+3\sigma}{1-\sigma}\,\frac{u(0)}{\rho}
$$

where

$$
\rho = \begin{cases} R & \text{if } R \le R_G\sqrt{\sigma} \\ R_G\sqrt{\sigma} & \text{if } R > R_G\sqrt{\sigma} \end{cases}
$$

with $\sigma = \frac{2}{3}\sqrt{3} - 1$. Indeed, setting $r = tR_G$ in (2.6) with $M = 0$ one has

$$
\begin{aligned}
|u'(0)| &\le \frac{u(0)}{R_G} \inf_{\substack{0<t<R/R_G \\ 0<t<1}} \left(\frac{1}{t} + \frac{4t}{1-t^2}\right) \\
&\le \frac{u(0)}{R_G} \inf_{\substack{0<t<R/R_G \\ 0<t\le t_0}} \left(\frac{1}{t} + \frac{4t}{1-t^2}\right)
\end{aligned}
$$

where $t_0$ minimizes the function $t \to \dfrac{1}{t} + \dfrac{4t}{1-t^2}$ in the interval $(0,1)$.

Since $R_G = \sqrt{2/G}$, the assertion follows by choosing $\sigma = t_0^2$. The above formula is true also when $u(0) = 0$, since in this case $u'(0) = 0$ (recall that we are dealing with $u \ge 0$).

We are now ready to state and prove the Glaeser estimate in the 1-dimensional case. There are three possible cases for the interplay of the parameters $u(0), M, G, R$ as indicated in (2.7) below. Observe that the estimates depend continuously on $G$ as $G \to 0^+$, so that for $G = 0$ only the first two cases survive and the next statement coincides with (1.3).

Proposition 2.4. – *Let $u : (-R, R) \to \mathbb{R}$ be a non-negative $C^2$ function such that* (2.1) *holds for every $r \le R$ and* (2.2) *for some $G > 0$. Then*

$$
(2.7)\quad |u'(0)| \le \begin{cases} (1 + \frac{16}{3}\gamma^2)\sqrt{2u(0)M} & \text{if } \sqrt{\frac{2u(0)}{M}} \le \min\left(R, \frac{1}{\sqrt{G}}\right) \\ (1+4GR^2)\frac{u(0)}{R} + \left(1+\frac{4}{3}GR^2\right)\frac{M}{2}R & \text{if } R \le \min\left(\frac{1}{\sqrt{G}}, \sqrt{\frac{2u(0)}{M}}\right) \\ \frac{5}{2}\left(u(0)\sqrt{G} + \frac{2}{3}\frac{M}{\sqrt{G}}\right) & \text{if } \frac{1}{\sqrt{G}} \le \min\left(\sqrt{\frac{2u(0)}{M}}, R\right) \end{cases}
$$

*where* $\gamma = \sqrt{\frac{Gu(0)}{M}}$.

Proof. – As in Remark 2.3 we put $R_G = \sqrt{\frac{2}{G}}$. Also, we set $R_* = \sqrt{\frac{2u(0)}{M}}$ and $\gamma = \sqrt{\frac{Gu(0)}{M}} = \frac{R_*}{R_G}$. Using Corollary 2.2 with $r = tR_G = t\sqrt{\frac{2}{G}}$, from (2.6) we get

$$
(2.8)\qquad |u'(0)| \le \frac{2u(0)}{R_*} \inf_{\substack{0<t<R/R_G \\ 0<t<1}} \left[\frac{1}{2}\left(\frac{\gamma}{t} + \frac{t}{\gamma}\right) + \frac{2\,t^2}{1-t^2}\left(\frac{\gamma}{t} + \frac{t}{3\gamma}\right)\right].
$$

Hence

$$
\begin{aligned}
|u'(0)| &\le \frac{2u(0)}{R_*} \inf_{\substack{0<t<R/R_G \\ 0<t<1/\sqrt{2}}} \left[\frac{1}{2}\left(\frac{\gamma}{t} + \frac{t}{\gamma}\right) + \frac{2\,t^2}{1-t^2}\left(\frac{\gamma}{t} + \frac{t}{3\gamma}\right)\right] \\
&\le \frac{2u(0)}{R_*} \inf_{\substack{0<t<R/R_G \\ 0<t<1/\sqrt{2}}} \left[\frac{1}{2}\left(\frac{\gamma}{t} + \frac{t}{\gamma}\right) + 4t^2\left(\frac{\gamma}{t} + \frac{t}{3\gamma}\right)\right] \\
&\le \sqrt{2u(0)M} \inf_{\substack{0<t<R/R_G \\ 0<t<1/\sqrt{2}}} \left[\left(\frac{1}{2} + 4t^2\right)\frac{\gamma}{t} + \left(\frac{1}{2} + \frac{4}{3}\,t^2\right)\frac{t}{\gamma}\right],
\end{aligned}
$$

and the expression in the square bracket is minimized by $t = t_\gamma \in [\gamma, \gamma + \gamma^2]$. Choosing $t = \min\left(\gamma, \frac{R}{R_G}, \frac{\sqrt{2}}{2}\right)$ in the right-hand side, the conclusion (2.7) easily follows. □

It is worth to observe that the results of this Section also provide the Harnack inequality. Indeed, by translational invariance, inequality (2.6) implies

$$
(2.9)\qquad |u'(y)| \le 10\left(\frac{u(y)}{r} + \frac{M}{2}\,r\right)
$$

for every $y \in (-r/2, r/2)$, provided $r < \min(R, 1/\sqrt{G})$.

Integrating between 0 and $x \in (-r/2, r/2)$ and using the mean value inequality of Lemma 2.1, we get the Harnack inequality

$$u(x) \le 21\left(u(0) + \frac{M}{2}\, r^2\right). \tag{2.10}$$

Combining (2.10) and (2.9) with $y = x$, and arguing as in the proof of Proposition 2.4, finally we obtain

$$|u'(x)| \le 210\left(\frac{u(0)}{r} + \frac{M}{2}\, r\right) \tag{2.11}$$

for $|x| < r/2$. Recalling that $r < \min(R, 1/\sqrt{G})$ and arguing as in the proof of Proposition 2.4, we obtain

COROLLARY 2.5. – *Let $u : (-R, R) \to \mathbb{R}$ be a non-negative $C^2$ function such that* (2.1) *holds for every $r \le R$ and* (2.2) *for some $G > 0$. Then*

$$\frac{1}{210}\sup_{B_{r/2}} |u'| \le \begin{cases} \sqrt{2u(0)M} & \text{if } r \le \sqrt{\dfrac{2u(0)}{M}} \le \min\left(R, \dfrac{1}{\sqrt{G}}\right) \\[2ex] \dfrac{u(0)}{R} + \dfrac{M}{2} R & \text{if } r \le R \le \min\left(\dfrac{1}{\sqrt{G}}, \sqrt{\dfrac{2u(0)}{M}}\right) \\[2ex] u(0)\sqrt{G} + \dfrac{M}{2\sqrt{G}} & \text{if } r \le \dfrac{1}{\sqrt{G}} \le \min\left(\sqrt{\dfrac{2u(0)}{M}}, R\right) \end{cases} \tag{2.12}$$

## 3. – Extension to semilinear Poisson equations

We consider here a multidimensional extension of the result of the previous section.
Let $B_r$ be the $n$-dimensional ball $\{x \in \mathbb{R}^n : |x| < r\}$ and consider non-negative functions $u \in C^2(B_r)$ satisfying, for some positive constant $M$, the bound

$$|\Delta u - g(u)| \le M \quad \text{in} \quad B_r \tag{3.1}$$

The main difference with respect to the one-dimensional case is that here we need the two-sided condition (3.1) instead of the unilateral inequality (2.1).

As already claimed by Li and Nirenberg [12] for the validity of the Glaeser's inequalities for $n$-variables functions is not enough, even when $g = 0$, to assume that $\Delta u \le M$. This would lead indeed to a contradiction to the fact that there exist non-constant superharmonic functions in $\mathbb{R}^n$ for $n \ge 3$, as outlined in [5], Example 4.4.

The next Lemma is the $n$-dimensional counterpart of Lemma 2.1.

LEMMA 3.1. – *Let $u$ be a non-negative function in $C^2(B_r)$ satisfying $\Delta u - g(u) \le M$ in $B_r$. Assume also that $g$ satisfies condition* (2.2) *with $G < \dfrac{2(n+2)}{r^2}$. Then,*

$$\fint_{B_r} u \le \max_{\rho\in(0,r)} \fint_{B_\rho} u \le \frac{1}{1-\dfrac{G}{2}\dfrac{r^2}{n+2}}\left(u(0)+\frac{M}{2}\frac{r^2}{n+2}\right) \tag{3.2}$$

PROOF. – The divergence theorem and the co-area formula together with our assumptions yield for $\rho \in (0, r)$ the inequality

$$\begin{aligned}\rho^{n-1}\frac{d}{d\rho}\left(\rho^{1-n}\int_{\partial B_\rho} u\,dS\right) &= \int_{\partial B_\rho}\frac{\partial u}{\partial \nu}\,dS = \int_{B_\rho}\Delta u\,dx\\ &\le \int_{B_\rho}(M+Gu)\,dx \le M\omega_n\rho^n + G\int_{B_\rho} u\,dx\end{aligned}$$

where $\omega_n$ is the volume of the $n$-dimensional ball. Therefore,

$$\frac{d}{d\rho}\left(\rho^{1-n}\int_{\partial B_\rho} u\,dS\right) \le M\omega_n\rho + G\rho\fint_{B_\rho} u$$

Taking into account that

$$\lim_{\rho\to 0}\fint_{\partial B_\rho} u = n\omega_n u(0) \quad \text{and} \quad \lim_{\rho\to 0}\rho\fint_{B_\rho} u = 0\,,$$

and integrating with respect to $\rho$ in $(0,\sigma)$ with $\sigma < r$, we have

$$\sigma^{1-n}\int_{\partial B_\sigma} u\,dS \le n\omega_n u(0) + \frac{M}{2}\,\omega_n\sigma^2 + G\int_0^\sigma \rho\fint_{B_\rho} u\,d\rho,$$

Multiplying now by $\sigma^{n-1}$ and integrating with respect to $\sigma$ between 0 and $\tau < r$, we get

$$\int_0^\tau\left(\int_{\partial B_\sigma} u\,dS\right)d\sigma \le \omega_n u(0)\tau^n + \frac{\omega_n M\tau^{n+2}}{2(n+2)} + G\int_0^\tau\left(\sigma^{n-1}\int_0^\sigma \rho\fint_{B_\rho} u\,d\rho\right)d\sigma$$

Since the volume of $B_\tau$ is $\omega_n\tau^n$, it follows that

$$\max_{\rho\in(0,r)}\fint_{B_\rho} u \le u(0) + \frac{1}{n+2}\left[\frac{M}{2}\,r^2 + \frac{1}{2}\,Gr^2\max_{\rho\in(0,r)}\fint_{B_\rho} u\right]$$

whence the assertion for $G$ as in the statement of the Lemma. $\square$

The next technical Lemma exploits a useful consequence of the second inequality in (3.1).

LEMMA 3.2. – *Let $u$ be a non-negative function in $C^2(B_r)$ satisfying $\Delta u - g(u) \geq -M$ in $B_r$. Assume also that $g$ satisfies condition* (1.8) *with $G < \dfrac{(n+2)}{r^2}$. Then*

$$u(0) \leq \frac{1 - \dfrac{1}{2}\dfrac{Gr^2}{n+2}}{1 - \dfrac{Gr^2}{n+2}} \fint_{B_r} u + \frac{1}{1 - \dfrac{Gr^2}{n+2}} \frac{M}{2} \frac{r^2}{n+2} \tag{3.3}$$

PROOF. – Similarly to the proof of Lemma 3.1, we get

$$\fint_{B_r} u\,dx \geq u(0) - \frac{1}{n+2}\left[\frac{M}{2}r^2 + \frac{1}{2}Gr^2 \max_{\rho\in(0,r)} \fint_{B_\rho} u\right]$$

From this using (3.2), we obtain (3.3). □

LEMMA 3.3. – *Let $u \in C^2(B_R)$ be a non-negative function satisfying inequality* (3.1) *in $B_R$ with a positive constant $M$, where $g$ is a continuous function satisfying* (1.8). *If $r < \min\left(\sqrt{\dfrac{2(n+2)}{G}}, R\right)$ then*

$$\sup_{B_{r/2}} u \leq \frac{3 \cdot 2^{n-1}}{1 - \dfrac{1}{2}\dfrac{Gr^2}{n+2}} u(0) + \left(\frac{3 \cdot 2^n}{1 - \dfrac{1}{2}\dfrac{Gr^2}{n+2}} + 1\right) \frac{M}{4}\frac{r^2}{n+2}. \tag{3.4}$$

PROOF. – Let $u(x_1) = \sup_{B_{r/2}} u$. From (3.3) and (3.2) we deduce that

$$\begin{aligned}
u(x_1) &\leq \frac{1 - \dfrac{1}{8}\dfrac{Gr^2}{n+2}}{1 - \dfrac{1}{4}\dfrac{Gr^2}{n+2}} \fint_{B_{r/2}(x_1)} u + \frac{1}{1 - \dfrac{1}{4}\dfrac{Gr^2}{n+2}} \frac{M}{8}\frac{r^2}{n+2} \\
&\leq 3 \cdot 2^{n-1} \fint_{B_r} u + \frac{1}{1 - \dfrac{1}{4}\dfrac{Gr^2}{n+2}} \frac{M}{8}\frac{r^2}{n+2} \\
&\leq \frac{3 \cdot 2^{n-1}}{1 - \dfrac{1}{2}\dfrac{Gr^2}{n+2}}\left(u(0) + \frac{M}{2}\frac{r^2}{n+2}\right) + \frac{1}{1 - \dfrac{1}{4}\dfrac{Gr^2}{n+2}} \frac{M}{8}\frac{r^2}{n+2}
\end{aligned}$$

from which the assertion follows. □

PROPOSITION 3.4. – *Suppose that the assumptions of* Lemma 3.3 *are satisfied in $B_R$. Then there exists a positive constant $C = C(n)$ such that*

$$\frac{|Du(0)|}{C} \le \begin{cases} \sqrt{2u(0)M} & \text{if } R_* \le \min(R_G, R) \\ \dfrac{u(0)}{R} + MR & \text{if } R \le \min(R_*, R_G) \\ u(0)\sqrt{G} + \dfrac{M}{\sqrt{G}} & \text{if } R_G \le \min(R, R_*) \end{cases} \tag{3.5}$$

*where* $R_* = \sqrt{\dfrac{2u(0)}{M}}$ *e* $R_G = \sqrt{\dfrac{2(n+2)}{G}}$.

PROOF. – From the classical gradient estimates [7], see also [3] for the version below,

$$|Du(0)| \le \frac{1}{\sqrt{2}}\left[\frac{1}{r}\left(4n + \frac{G}{4}\, r^2\right) \sup_{B_{r/2}} u + \frac{M}{4}\, r\right]. \tag{3.6}$$

Combining (3.6) and (3.4), and taking into account that $Gr^2 < 2(n+2)$, we have

$$|Du(0)| \le \frac{1}{\sqrt{2}}\left[\left(\frac{9}{2}\, n+1\right)\left(\frac{3\cdot 2^{n-1}}{1-\dfrac{G}{2}\,\dfrac{r^2}{n+2}}\,\frac{u(0)}{r} + \left(\frac{3\cdot 2^{n}}{1-\dfrac{G}{2}\,\dfrac{r^2}{n+2}}+1\right)\frac{M}{4}\,\frac{r}{n+2}\right) + \frac{M}{4}\, r\right]$$

so that for $r < \min(R_G, R)$

$$|Du(0)| \le C\left(\frac{u(0)}{r} + Mr\right). \tag{3.7}$$

Passing to the variable $t = r/R_G$ and setting $\gamma = \dfrac{R_*}{R_G}$, we get

$$|Du(0)| \le C\sqrt{2u(0)M} \inf_{\substack{0<t\le 1 \\ 0<t\le R/R_G}} \frac{1}{2}\left(\frac{\gamma}{t} + \frac{t}{\gamma}\right)$$

with a (possibly different) constant $C$ depending only of $n$.

Finally, comparing $R_*$, $R_G$ and $R$ as in Corollary 2.5, from the above inequality we deduce (3.5). □

## 4. – $L^p$ bounds

In this Section we shall consider Sobolev functions $u \in W^{2,p}(B_R)$ such that

$$\|\Delta u - g(u)\|_{L^p(B_R)} \le M \tag{4.1}$$

for some $p > 1$.

PROPOSITION 4.1. – *Let $u \in W^{2,p}(B_R)$, $p > n$, be a non-negative function satisfying* (4.1) *with $g$ satisfying* (1.8). *Then there exist positive constants $C = C(n,p)$ and $\sigma = \sigma(n,p)$ such that*

$$\frac{|Du(0)|}{C} \leq \begin{cases} u(0)^{\frac{p-n}{2p-n}} M^{\frac{p}{2p-n}} & \text{if } R_* \leq \min(R_G, R) \\ \dfrac{u(0)}{R} + MR^{1-\frac{n}{p}} & \text{if } R \leq \min(R_*, R_G) \\ u(0)G^{\frac{1}{2}} + MG^{-\frac{p-n}{2p}} & \text{if } R_G \leq \min(R, R_*) \end{cases} \tag{4.2}$$

*where $R_* = (1 - n/p)^{\frac{p}{n-2p}} [u(0)/M]^{\frac{p}{2p-n}}$ and $R_G = \sqrt{\sigma/G}$.*

PROOF. – By assumption $u$ satisfies

$$\Delta u = f(x) + g(u)$$

where $\|f\|_{L^p(\Omega)} \leq M$ and $|g(u)| \leq Gu$.

Consider first the case $R = 1$. By Sobolev embeddings it turns out that $u \in C^{1,\alpha}(B_1)$ with $\alpha \leq 1 - \dfrac{n}{p}$ and by elliptic estimates, see [7], we also infer that

$$|Du(0)| \leq C_1 \left[ (1+G) \sup_{B_{1/2}} u + M \right] \tag{4.3}$$

where $C_1$ depends only of $n$ and $p$. Thanks to the Harnack inequality, see [7],

$$\sup_{B_{1/2}} u \leq C \left( u(0) + M + G \sup_{B_{1/2}} u \right)$$

with $C$ depending only of $n$ and $p$, so that for $G \leq \dfrac{1}{2C} \equiv \sigma$ one has

$$\sup_{B_{1/2}} u \leq C_2 \, (u(0) + M) \tag{4.4}$$

with $C_2 = C_2(n,p)$. Combining (4.3) and (4.4) for $G \leq \sigma$ we get

$$|Du(0)| \leq C \, (u(0) + M) \tag{4.5}$$

where $C = C(n,p)$ is a positive constant.

For an arbitrary $R$, let $r \in (0,R)$. The non-negative function $v(y) = u(ry)$ is a solution of the equation

$$\Delta v = r^2 f(ry) + r^2 g(v)$$

in $B_1$, where

$$\|r^2 f(ry)\|_{L^p(B_1)} \leq M r^{2-\frac{n}{p}}, \quad r^2 |g(v)| \leq G r^2 v.$$

Using the result of the case $R = 1$, if $Gr^2 < \sigma$ one has

$$|Du(0)| \le C\left(\frac{u(0)}{r} + Mr^{1-\frac{n}{p}}\right) \tag{4.6}$$

with $C = C(n, p)$. It follows that

$$|Du(0)| \le C \inf_{\substack{0<r\le R_G\\ 0<r\le R}} \left(\frac{u(0)}{r} + Mr^{1-\frac{n}{p}}\right)$$

and comparing $R_*$, $R_G$ and $R$ as before in Sections 2 and 3 the statement is proved. □

## 5. – Viscosity solutions

Let $F : \mathcal{S}^n \to \mathbb{R}$ be an uniformly elliptic operator with ellipticity constants $\Lambda \ge \lambda > 0$. In this Section we consider viscosity solutions $u \in C(B_R)$ of the fully nonlinear equation

$$F(D^2u) = f(x) + g(u) \tag{5.1}$$

in $B_R$, where $f$ is a continuous function such that

$$\|f\|_{L^\infty(B_R)} \le M \tag{5.2}$$

and $g$ is a continuous function on $\mathbb{R}_+$ satisfying (1.8).

PROPOSITION 5.1. – *Let $u \in C(B_R)$ be a non-negative function satisfying* (5.1) *where $f$ and $g$ satisfy, respectively,* (5.2) *and* (1.8). *Then there exist positive constants $C$ and $\sigma$ depending only on $n$, $\lambda$ and $\Lambda$ such that*

$$\frac{|Du(0)|}{C} \le \begin{cases} \sqrt{u(0)M} & \text{if } R_* \le \min(R_G, R) \\ \dfrac{u(0)}{R} + MR & \text{if } R \le \min(R_*, R_G) \\ u(0)\sqrt{G} + \dfrac{M}{\sqrt{G}} & \text{if } R_G \le \min(R, R_*) \end{cases} \tag{5.3}$$

*where $R_* = \sqrt{u(0)/M}$ and $R_G = \sqrt{\sigma/G}$ .*

PROOF. – The proof follows the lines of that one of Proposition 4.1; of course we have to use here results from the viscosity theory of elliptic equations instead of the corresponding ones from the $L^p$ theory. We start again with the case $R = 1$.

By assumption $|g(u)| \leq Gu$, thus for $r < \dfrac{1}{4}$

$$\int_{B_r} (f(x) + g(u))\, dx \leq \left(M + G \sup_{B_{1/4}} u\right) \omega_n r^n.$$

By the regularity theorem of Caffarelli [1] it turns out that $u \in C^{1,\alpha}(B_1)$ for every $\alpha \in (0,1)$ and, consequently,

$$|Du(0)| \leq C_1[(1+G) \sup_{B_{1/4}} u + M] \,, \tag{5.4}$$

see [5], where $C_1$ depends only of $n$, $\lambda$ and $\Lambda$.

By the Harnack inequality, see [2], we deduce, arguing as for (4.4), that there exists $\sigma = \sigma(n, \lambda, \Lambda)$ such that, if $G \leq \sigma$, one has

$$\sup_{B_{3/4}} u \leq C_2\,(u(0) + M) \tag{5.5}$$

with $C_2 = C_2(n, \lambda, \Lambda)$.

Combining the above with (5.5) we get the inequality

$$|Du(0)| \leq C\,(u(0) + M) \tag{5.6}$$

with $C$ depending only on $n$, $\lambda$ and $\Lambda$.

Arguing as for (4.6), from the case $R = 1$ we deduce, for $r < \min(\sqrt{\sigma/G}, R)$ that

$$|Du(0)| \leq C \left(\frac{u(0)}{r} + Mr\right) \tag{5.7}$$

with $C = C(n, \lambda, \Lambda)$, and we can conclude as in the proof of Proposition 4.1 comparing $R_*$, $R_G$ and $R$. $\square$

Proof of Theorem 1.1. – Looking at the proof of Proposition 1.1, we start with $R = 1$ and let $|x_0| \leq \dfrac{1}{2}$. Applying (5.4) in $B_r(x_0)$ and (5.5), provided $G \leq \sigma$ we get

$$|Du(x_0)| \leq C_1[(1+G) \sup_{B_{1/4}(x_0)} u + M] \leq C_1[(1+G) \sup_{B_{3/4}} u + M] \leq C\,(u(0) + M) \tag{5.8}$$

with a positive constant $C = C(n, \lambda, \Lambda)$.

Rescaling, for $|x_0| \leq r/2$ and $r < \min(\sqrt{\sigma/G}, R)$ we deduce that

$$|Du(x_0)| \leq C \left(\frac{u(0)}{r} + Mr\right) \tag{5.9}$$

Finally, setting $R_0 = \min(R_*, R_G, R)$, we conclude that

$$\sup_{B_{R_0/2}} |Du| \leq C \left(\frac{u(0)}{r} + Mr\right) \tag{5.10}$$

from which, resoning as before, by comparison of $R_*$, $R_G$ and $R$, we finally obtain (1.10). $\square$

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