Bollettino Unione Matematica Italiana

Antonio Ambrosetti

Remarks on a Bifurcation Problem in Fluid Dynamics

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **5** (2012), n.2, p. 205–209.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_2_205_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012), 205-209

Remarks on a Bifurcation Problem in Fluid Dynamics

ANTONIO AMBROSETTI

Dedicated to the memory of Enrico Magenes

Abstract. – *We sharpen some previous results of* [2, 4], *dealing with a bifurcation problem arising in fluid dynamics.*

1. – Introduction and description of the problem

The stationary motion of a viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ is described by a Navier-Stokes type system, in which the temperature is taken into account:

$$\left\{ \begin{array}{rcl} (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \Delta \boldsymbol{u} & = & -\nabla p + \boldsymbol{h}(\vartheta, x), \quad \text{in } \Omega \\ \boldsymbol{u} \cdot \nabla \vartheta - \Delta \vartheta & = & 0, \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array} \right. \tag{1}$$

where $x \in \Omega$, $\boldsymbol{u}(x) = (u_1(x), u_2(x), u_3(x))$ is the velocity field, $p = p(x)$ and $\vartheta = \vartheta(x)$ are, respectively, the pressure and the temperature of the fluid.

System (1) is complemented with the following boundary conditions

$$\boldsymbol{u}(x) = \boldsymbol{0}, \qquad \vartheta(x) = \vartheta_0(x), \qquad x \in \partial\Omega, \tag{1'}$$

where $\boldsymbol{0}$ denotes the origin in $\mathbb{R}^3$ and ϑ_0 is given. Precisely, according to [2, 4], we will take $\vartheta_0(x) = \alpha x_3 + \alpha_0$. Hence setting $\vartheta = \zeta - \vartheta_0$ one has that $\zeta_{|\partial\Omega} = 0$ and $\Delta\vartheta = \Delta\zeta$.

Finally on $\boldsymbol{h} = (h_1, h_2, h_3)$ we assume that $h_1 = h_2 = 0$, h_3 is continuous and

$$h_3(\vartheta, x) = h(\zeta) + \vartheta_0(x) = h(\zeta) + \alpha x_3 + \alpha_0.$$

With this notation (1-1′) become

$$\left\{ \begin{array}{rcl} \sum u_i \dfrac{\partial u_j}{\partial x_i} - \Delta u_j & = & p_{x_j}, \quad j = 1, 2, \\ \sum u_i \dfrac{\partial u_3}{\partial x_i} - \Delta u_3 & = & p_{x_3} + h(\zeta) + \alpha x_3 + \alpha_0, \\ \sum u_i \dfrac{\partial \zeta}{\partial x_i} - \Delta \zeta & = & \alpha u_3, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array} \right. \tag{2}$$

and

$$(2') \qquad \begin{cases} \boldsymbol{u} &= \boldsymbol{0}, \quad \text{on } \partial\Omega, \\ \zeta &= 0, \quad \text{on } \partial\Omega. \end{cases}$$

Let us consider the Hilbert space $(H_0^1(\Omega))^3$ endowed with scalar product

$$((\boldsymbol{u},\boldsymbol{v})) = \int_\Omega \sum \frac{\partial u_k}{\partial x_i}\frac{\partial v_k}{\partial x_i} dx$$

Let N, resp.N^1, denote the closure in $(L^2(\Omega))^3$, resp. $(H_0^1(\Omega))^3$, of $\boldsymbol{v} \in C_0^\infty(\Omega)$ such that $\nabla \cdot \boldsymbol{v} = 0$. It is known that $N^\perp = \{\nabla\psi : \psi \in H^1(\Omega)\}$. In particular,

$$(3) \qquad (\nabla p, \boldsymbol{v}) = 0, \quad (\vartheta_0, v_3) = 0, \quad \forall\, \boldsymbol{v} = (v_1, v_2, v_3) \in N,$$

where we use the same symbol $(.,.)$ both for the scalar product in $(L^2(\Omega))^3$ as well as in $L^2(\Omega)$.

Using (3) we infer that the weak form of (2-2′) is

$$(4) \qquad \begin{cases} ((\boldsymbol{u},\boldsymbol{v})) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) &= (h(\zeta), v_3), \quad \forall\, \boldsymbol{v} = (v_1,v_2,v_3) \in N \\ ((\zeta,\phi)) + (\boldsymbol{u}\cdot\nabla\zeta, \phi) &= \alpha(u_3,\phi), \quad \forall\, \phi \in H_0^1, \end{cases}$$

where

$$b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \int \sum_{i,k} u_i \frac{\partial u_k}{\partial x_i} v_k dx.$$

On the other hand, if $(\boldsymbol{u},\zeta) \in E$ is a solution of (4) then there exists $p(x)$ (up to a constant), such that $(\boldsymbol{u},\zeta)$ satisfy (2). In conclusion, we can look for solutions $(\boldsymbol{u},\zeta) \in E := N^1 \times H_0^1(\Omega))$ of (4).

REMARK 1. – It is easy to check that there exist linear compact $L, \mathcal{L}$ and compact operators $B : (H_0^1(\Omega))^3 \to (H_0^1(\Omega))^3, \mathcal{B} : E \to E,$ with $B(0) = B'(0) = 0$ and $\mathcal{B}(\boldsymbol{0},0) = \mathcal{B}'(\boldsymbol{0},0) = 0$, such that (4) is equivalent to

$$(P) \qquad \begin{cases} \boldsymbol{u} &= B(\boldsymbol{u}) + Lh(\zeta), \\ \zeta &= \mathcal{B}(\boldsymbol{u},\zeta) + \alpha\mathcal{L}\boldsymbol{u}. \end{cases}$$

This will allow us to use topological degree arguments. □

2. – A bifurcation result

Suppose that

$$(h1) \qquad h(0) = 0,\ h \in C^1(\mathbb{R}) \text{ and } h'(0) = \lambda$$

If $(h1)$ holds, then for all $\lambda, \alpha \in \mathbb{R}$, (P) has the trivial solution $(\boldsymbol{u}, z) = (\boldsymbol{0}, 0)$. Problem (P) linearized at $(\boldsymbol{0}, 0)$ becomes

$$(\mathrm{P}') \qquad \begin{cases} -\Delta w_i &= 0, \quad (i = 1, 2), \\ -\Delta w_3 &= \lambda\, z \\ -\Delta z &= \alpha\, w_3. \end{cases}$$

or, in weak form

$$\begin{cases} \boldsymbol{w} &= \lambda(0, 0, K[z]), \\ z &= \alpha K[w_3], \end{cases}$$

where K denotes the inverse of $-\Delta$ on $H_0^1(\Omega)$ and $\boldsymbol{w} = (w_1.w_2, w_3)$.

Problem (P′) is equivalent to

$$(5) \qquad \begin{cases} -\Delta w_i &= 0, \quad (i = 1, 2), \\ \Delta^2 w_3 &= \lambda\, \alpha\, w_3. \end{cases}$$

In (5) $w_1, w_2 \in H_0^1(\Omega)$ while w_3 satisfies the Navier boundary conditions $w_3 = \Delta w_3 = 0$ on $\partial\Omega$. Actually $\Delta(w_3)_{|\partial\Omega} = -\lambda z_{|\partial\Omega} = 0$. Let

$$\mathcal{H} = \{\phi \in H^2(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\}$$

LEMMA 2. – *The problem*

$$(6) \qquad \Delta^2 u = \mu\, u, \qquad u \in \mathcal{H}$$

has a sequence of positive eigenvalues μ_k, $k = 1, 2, \ldots$. The first eigenvalue μ_1 is simple and the associated eigenfunction does not change sign in Ω.

PROOF. – The eigenvalues of (6) are the characteristic values of the linear operator $T : \mathcal{H} \to \mathcal{H}$ defined by

$$(T(u), v)_{\mathcal{H}} = \int_\Omega \Delta u \Delta v dx, \qquad \forall\, v \in \mathcal{H}$$

Since T is positive symmetric and compact, the result follows. □

Let μ_k denote the eigenvalues (repeated according to their multiplicity) of (6) with eigenfunction e_k. The relationship between the eigenvalues of (6) and those of (P') is stated in the following lemma.

LEMMA 3. – *For any $\alpha > 0$, $\lambda_k = \mu_k/\alpha$ is an eigenvalue of* (P′) *with eigenfunction* $(0, 0, e_k, \alpha K[e_k])$. *The multiplicity of λ_k is the same of μ_k.*

According to Remark 1, the system (P) is of the type Identity - Compact. Then the Rabinowitz global bifurcation theorem [3] applies to (P) yielding the following

THEOREM 4. – *Let* ($h1$) *hold and suppose that* μ_k *has odd multiplicity. Then* $\lambda_k = \mu_k/\alpha$ *is a bifurcation point for* (P). *Precisely, from such* λ_k *emanates a global continuum* Γ_k *which is either unbounded or meets another eigenvalue* $\lambda_j \neq \lambda_k$.

Furthermore, using the bifurcation theorem from a simple eigenvalue, see e.g. [1], we deduce

THEOREM 5. – *Let* ($h1$) *hold. Then there exists a global continuum* Γ_1 *of solutions of* (P) *emanating from* $\lambda_1 = \mu_1/\alpha$, *which satisfies the preceding alternative. The bifurcation is supercritical, resp. subcritical, provided* $h \in C^3$, $h''(0) = 0$, *and* $h'''(0) < 0$, *resp.* $h'''(0) > 0$. *Moreover,* Γ_1 *is a curve and if* $h''(0) < 0$ *then* (P) *has two nontrival solutions for all* $\lambda_1 < \lambda < \lambda_1 + \varepsilon$, $\varepsilon \sim 0$.

3. – Existence in the large

We look for solutions $(\boldsymbol{u}, z) \in E$ of (P) such that $\boldsymbol{u} \neq \boldsymbol{0}$ and $z \neq 0$. These solutions will be called *nontrivial*. We will use topological degree methods.

Consider on the homotopy $G : [0,1] \times \mathcal{H} \to \mathcal{H}$

$$G(t, \boldsymbol{u}, \zeta) = G_t(\boldsymbol{u}, \zeta) = (tB(\boldsymbol{u}) + \lambda L\zeta, tB(\boldsymbol{u}) + \lambda L\zeta)$$

If $G_t(\boldsymbol{u}, \zeta) = (\boldsymbol{0}, 0) \in E$ then

$$(P_t) \qquad \begin{cases} \boldsymbol{u} &= tB(\boldsymbol{u}) + \lambda L\zeta, \\ \zeta &= t\mathcal{B}(\boldsymbol{u}, \zeta) + \alpha\mathcal{L}\boldsymbol{u}, \end{cases}$$

If $t = 0$, (P_t) is nothing but (P′), while if $t = 1$ we get (P).

Suppose that $h \in C(\mathbb{R})$ and

$$(h2) \qquad |h(\zeta)| \leq a + b|\zeta|, \text{ for some } a, b > 0\text{-}$$

In [2] it is proved that if ($h2$) holds then for all $R \gg 1$ and all $t \in [0,1]$, any solution $(\boldsymbol{u}, \zeta) \in E$ of (P_t) satisfies the a-priori bound

$$\|\boldsymbol{u}\| + \|\zeta\| \leq R.$$

Then, by the invariance property of the Leray-Schauder topological degree it follows

$$deg(G_1, D_R, (\boldsymbol{0}, 0)) = deg(G_0, D_R, (\boldsymbol{0}, 0)),$$

where $D_r = \{(\boldsymbol{u}, \zeta) \in \mathcal{H} : \|\boldsymbol{u}\| < r,\ \|\zeta\| < r\}$. Moreover,

LEMMA 6. – *If* $(h2)$ *holds then for all* $R \gg 1$ *one has that* $deg(G_0, D_R, (\mathbf{0}, 0)) = 1$.

In addition, using Lemma 2 and 3 we infer

LEMMA 7. – *If* $(h1)$ *holds then for all* $\lambda_k < \lambda < \lambda_{k+1}$ *one has that* $deg(G_0, D_\varepsilon, (\mathbf{0}, 0)) = (-1)^{m_k}$ *where* m_k *denotes the number of eigenvalues of* Δ^2 *smaller than* μ_{k+1}.

It follows

THEOREM 8. – *Let* $(h1-2)$ *hold and suppose that* m_k *is an odd integer. Then for all* $\lambda_k < \lambda < \lambda_{k+1}$*, problem* (P) *has at least a nontrivial solution* $(\boldsymbol{u}, \zeta)$.

REMARK 9. – If $h(\zeta) = \lambda\zeta$ it is proved in [2] that (P) has only the trivial solution for $\lambda \leq \lambda_1/\alpha$. □

REFERENCES

[1] A. AMBROSETTI - G. PRODI, *A Primer of Nonlinear Analysis*, Cambridge Studies in Advanced Mathematics No. 34, Cambridge Univ. Press, 1993.

[2] A. MARINO, *Su un problema di diramazione riguardante i moti convettivi in un fluido viscoso*, Rend. Sem. Mat. Univ. Padova, **38** (1967), 199-216.

[3] P. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., **7** (1971), 487-513.

[4] W. VELTE, *Stabilitätsverhalten und Verzweigung stationärer Lösungen der Navier Stokesschen Gleichungen*, Arch. Rat. Mech. Anal., **16** (1964), 97-125.

S.I.S.S.A., via Bonomea 265, 34136 Trieste
E-mail: ambr@sissa.it

Received August 31, 2011 and in revised form September 22, 2011