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Lipschitz Contractions, Unique Ergodicity and Asymptotics of Markov Semigroups

Francesco Altomare - Ioan Raşa

to the memory of Professor Giovanni Prodi

Abstract. – *We are mainly concerned with the asymptotic behaviour of both discrete and continuous semigroups of Markov operators acting on the space $C(X)$ of all continuous functions on a compact metric space X. We establish a simple criterion under which such semigroups admit a unique invariant probability measure μ on X that determines their limit behaviour on $C(X)$ and on $L^p(X,\mu)$.*

The criterion involves the behaviour of the semigroups on Lipschitz continuous functions and on the relevant Lipschitz seminorms.

Finally, we discuss some applications concerning the Kantorovich operators on the hypercube and the Bernstein-Durrmeyer operators with Jacobi weights on $[0,1]$. *As a consequence we determine the limit of the iterates of these operators as well as of their corresponding Markov semigroups whose generators fall in the class of Fleming-Viot differential operators arising in population genetics.*

Introduction

One of the most interesting aspects of the theory of strongly continuous semigroups of operators concerns their asymptotic behaviour that, among other things, gives useful information about the behaviour for large time of the solutions to the Cauchy problems governed by them (see, e.g., [12], [17] and the references therein).

A similar significance can be attributed to the study of the iterates of a single linear operator because of its connections with ergodic theory and, in particular, with ergodic theorems (see, e.g., [14]).

More recently, a renewed interest also arose from the study of iterates of positive linear operators in approximation theory and related fields (see, e.g., [3, Chapter 6], [5], [13], [18], [19]).

In this paper we develop some results related to these topics for both discrete and continuous semigroups of Markov operators acting on the space $C(X)$ of all continuous functions on a compact metric space X. More precisely, we establish a simple criterion under which such semigroups admit a unique invariant probability measure μ on X that determines their limit behaviour on $C(X)$ and on $L^p(X,\mu)$. The criterion is based on the behaviour of the semigroups on Lipschitz continuous functions and on the relevant Lipschitz seminorms. Furthermore, some estimates of the rate of convergence in the limit relationships are also given.

Finally, we discuss some applications concerning the Kantorovich operators on the hypercube and the Bernstein-Durrmeyer operators with Jacobi weights on $[0,1]$. As a consequence we determine the limit of the iterates of these operators as well as of their corresponding Markov semigroups whose generators fall in the class of Fleming-Viot differential operators arising in population genetics.

1. – Preliminaries and notation

Let (X,d) be a compact metric space. We shall denote by $C(X)$ the linear space of all real-valued continuous functions on X endowed with the supremum norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \quad (f \in C(X)) \tag{1.1}$$

and the pointwise ordering, with respect to which it is a Banach lattice.

Let $\mathfrak{B}_X$ be the σ-algebra of all Borel subsets of X and denote by $M_1^+(X)$ the subset of all probability Borel measures on X.

If $\mu \in M_1^+(X)$ and $p \in [1,+\infty[$, then $L^p(X,\mu)$ stands for the linear space of all (the equivalence classes of) Borel-measurable real-valued functions on X that are p-fold μ-integrable.

The space $L^p(X,\mu)$, endowed with the natural norm

$$\|f\|_p := (\int_X |f|^p d\mu)^{1/p} \quad (f \in L^p(X,\mu)) \tag{1.2}$$

and the ordering

$$f \le g \text{ if } f(x) \le g(x) \text{ for } \mu\text{-a.e. } x \in X \tag{1.3}$$

$(f,g \in L^p(X,\mu))$, is a Banach lattice and $C(X)$ is dense in it.

As usual, we shall denote by $\mathbf{1}$ the constant function with constant value 1. If $\varphi : C(X) \to \mathbb{R}$ is a positive linear functional such that $\varphi(\mathbf{1}) = 1$, then by the Riesz representation theorem (see, e.g., [7, Section 29]), there exists a unique $\mu \in M_1^+(X)$ such that

$$\varphi(f) = \int_X f d\mu \quad \text{for every} \quad f \in C(X). \tag{1.4}$$

A Markov operator on $C(X)$ is a positive linear operator $T : C(X) \to C(X)$ such that $T(\mathbf{1}) = \mathbf{1}$. Such an operator is then continuous and $\|T\| = 1$. Moreover, by the Riesz representation theorem, there exists a family $(\mu_x)_{x\in X}$ in $M_1^+(X)$ such that

$$T(f)(x) = \int_X f d\mu_x \quad (f \in C(X), x \in X). \tag{1.5}$$

Therefore, for every $p \in [1, +\infty[$, from the Hölder inequality it turns out that

$$|T(f)(x)|^p \leq \int_X |f|^p d\mu_x = T(|f|^p)(x) \tag{1.6}$$

$(f \in C(X), x \in X)$.

Every Markov operator T on $C(X)$ admits at least one invariant probability measure, i.e., a measure $\mu \in M_1^+(X)$ such that

$$\int_X T(f) d\mu = \int_X f d\mu \quad \text{for every} \quad f \in C(X) \tag{1.7}$$

(see, e.g., [14, Section 5.1, p. 178]). On account of (1.6) we get that for every $f \in C(X)$ and $p \in [1, +\infty[$,

$$\int_X |T(f)|^p d\mu \leq \int_X T(|f|^p) d\mu = \int_X |f|^p d\mu$$

and hence T extends to a unique bounded linear operator $T_p : L^p(X, \mu) \to L^p(X, \mu)$ such that $\|T_p\| \leq 1$. Furthermore, T_p is positive as $C(X)$ is a sublattice of $L^p(X, \mu)$ and, if $1 \leq p < q < +\infty$, then $T_p = T_q$ on $L^q(X, \mu)$.

From now on, for a given $p \in [1, +\infty[$, if no confusion can arise, we shall denote by $\widetilde{T}$ the operator T_p.

In the sequel, given $\mu \in M_1^+(X)$, we shall denote by $\Lambda(\mu)$ the subset of all Markov operators T on $C(X)$ for which μ is an invariant measure.

Below we list some simple properties of this subset, that can be easily verified.

PROPOSITION 1.1. – *For $\mu \in M_1^+(X)$ the following properties hold:*

(1) *If $S, T \in \Lambda(\mu)$, then $S \circ T \in \Lambda(\mu)$ and $\widetilde{S \circ T} = \widetilde{S} \circ \widetilde{T}$ on $L^p(X, \mu)$ for every $p \in [1, +\infty[$.*

(2) *If $(T_i)_{i\in I}^{\leq}$ is a net in $\Lambda(\mu)$ and if there exists $T(f) := \lim\limits_{i\in I^{\leq}} T_i(f)$ uniformly on X for every $f \in C(X)$, then $T \in \Lambda(\mu)$ and $\widetilde{T}(f) = \lim\limits_{i\in I^{\leq}} \widetilde{T}_i(f)$ in $L^p(X, \mu)$ whenever $f \in L^p(X, \mu)$ and $1 \leq p < +\infty$.*

2. – Asymptotic behaviour of semigroups of Markov operators

In order to simultaneously treat both discrete and continuous semigroups of Markov operators, we shall introduce the symbol $\mathbb{A}$ to denote either the interval $[0, +\infty[$ or the set $\mathbb{N}$ of all positive integers. The subset $\mathbb{A}$ will be endowed with the usual ordering $\le$ inherited from $\mathbb{R}$. Without no explicit mention, we shall refer to this ordering when we shall consider converging nets $(x_\tau)^{\le}_{\tau\in\mathbb{A}}$ in some metric space, whose limit will be denoted by $\lim\limits_{\tau\to\infty} x_\tau$.

Consider a compact metric space (X, d) and set

$$\delta(X) := \sup\{d(x,y) \mid x, y \in X\} \tag{2.1}$$

and

$$Lip(X) := \{f \in C(X) \ \mid \ |f|_{Lip} := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} < +\infty\}. \tag{2.2}$$

Below we state and prove the main result of the paper.

THEOREM 2.1. – *Let $(T(\tau))^{\le}_{\tau\in\mathbb{A}}$ be a net of Markov operators on $C(X)$ such that*

(i) *(semigroup property): $T(\sigma)\circ T(\tau) = T(\sigma+\tau)$ for every $\sigma, \tau \in \mathbb{A}$;*

(ii) *(Lipschitz contraction property) For every $\tau \in \mathbb{A}$, $T(\tau)(LipX) \subset Lip(X)$ and*

$$|T(\tau)f|_{Lip} \le c(\tau)|f|_{Lip} \quad (f \in Lip(X))$$

where $c : \mathbb{A} \longrightarrow]0, +\infty[$ with $\lim\limits_{\tau\to\infty} c(\tau) = 0$.

Then

(1) *For every $f \in C(X)$ the net $(T(\tau)f)^{\le}_{\tau\in\mathbb{A}}$ converges uniformly on X to a constant function.*

(2) *There exists a unique $\mu \in M_1^+(X)$ such that $T(\tau) \in \Lambda(\mu)$ for every $\tau \in \mathbb{A}$, i.e., $\int_X T(\tau)f d\mu = \int_X f d\mu$ for each $f \in C(X)$ and $\tau \in \mathbb{A}$. Moreover,*

$$\lim_{\tau\to\infty} T(\tau)f = \int_X f d\mu \quad \textit{uniformly on } X \quad (f \in C(X)) \tag{2.3}$$

as well as

$$\lim_{\tau\to\infty} \widetilde{T}(\tau)f = \int_X f d\mu \quad \text{in} \quad L^p(X,\mu) \quad (f \in L^p(X,\mu)) \tag{2.4}$$

for every $p \in [1, +\infty[$.

(3) *For every $f \in Lip(X)$ and $\tau \in \mathbb{A}$*

$$|T(\tau)f - \int_X f d\mu| \le 2c(\tau)\delta(X)|f|_{Lip}. \tag{2.5}$$

PROOF. – Because of assumption (ii), given $f \in Lip(X)$ and $\tau \in \mathbb{A}$, for every $x, y \in X$ we get

$$|T(\tau)f(x) - T(\tau)f(y)| \leq c(\tau)\delta(X)|f|_{Lip} \tag{1}$$

and hence

$$|T(\tau)f(x)\mathbf{1} - T(\tau)f| \leq c(\tau)\delta(X)|f|_{Lip}.$$

Therefore, for $\sigma \in A$, by recalling that $T(\sigma)$ is a Markov operator and by using the semigroup property (i), we obtain

$$|T(\tau)f(x) - T(\sigma+\tau)f| \leq c(\tau)\delta(X)|f|_{Lip}$$

and hence, since $x \in X$ was arbitrarily chosen,

$$|T(\tau)f - T(\sigma+\tau)f| \leq c(\tau)\delta(X)|f|_{Lip}. \tag{2}$$

The above estimate together with the assumption on the function c show that the family $(T(\tau)f)^{\leq}_{\tau\in\mathbb{A}}$ is a Cauchy net in $C(X)$ with respect to the uniform norm and hence it is convergent.

Therefore, we may consider the mapping $T : Lip(X) \to C(X)$ defined by

$$T(f) := \lim_{\tau\to\infty} T(\tau)f \quad (f \in Lip(X)),$$

that is linear, positive and $T(\mathbf{1}) = \mathbf{1}$.

From (1) it follows that, for every $f \in Lip(X)$ and $x, y \in X$, $T(f)(x) = T(f)(y)$ so that $T(f)$ is constant. In other words, there exists a positive linear functional $\psi : Lip(X) \to \mathbb{R}$ such that

$$T(f) = \psi(f)\mathbf{1} \quad \text{for every} \quad f \in Lip(X).$$

The functional ψ extends to a positive linear functional $\varphi : C(X) \to \mathbb{R}$ such that $\varphi(\mathbf{1}) = \psi(\mathbf{1}) = 1$ and hence, by the Riesz representation theorem there exists $\mu \in M_1^+(X)$ such that $\varphi(f) = \int_X f d\mu \quad (f \in C(X))$. Consequently $\lim_{\tau\to\infty} T(\tau)f = \int_X f d\mu$ uniformly on X, provided $f \in Lip(X)$. Since $Lip(X)$ is dense in $C(X)$, the same limit relationship extends to $C(X)$ as well which shows (2.3) and, hence, (2.4) (see Proposition 1.1, part (2)).

Notice that $T(\tau) \in \Lambda(\mu)$ for every $\tau \in \mathbb{A}$ because, given $f \in C(X)$,

$$\int_X T(\tau)f d\mu = \lim_{\sigma\to\infty} T(\sigma)(T(\tau)f) = \lim_{\sigma\to\infty} T(\sigma+\tau)f = \int_X f d\mu.$$

Moreover, if $\nu \in M_1^+(X)$ and if $T(\tau) \in \Lambda(\nu)$ for each $\tau \in \mathbb{A}$, then, from (2.3) we obtain

$$\int_X f d\nu = \lim_{\tau\to\infty} \int_X T(\tau) f d\nu = \int_X f d\mu$$

for every $f \in C(X)$ and hence $\nu = \mu$.

Finally (2.5) follows from (2) letting σ to tend to ∞. □

REMARK 2.2. – 1. Note that, under the assumptions of Theorem 2.1, from (2.5) it follows that for every $r > 0$

$$\lim_{\tau\to\infty} T(\tau)f = \int_X f d\mu \tag{2.6}$$

uniformly on X and uniformly with respect to $f \in Lip(X)$, $|f|_{Lip} \le r$.

2. Consider the $K-$functionals ([11, p. 171])

$$K(f,\delta) := \inf_{g\in Lip(X)} \left\{ \| f - g \|_\infty + \delta | g |_{Lip} \right\}$$

$(f \in C(X), \delta > 0)$ and

$$\tilde{K}(f,\delta) := \inf_{g\in Lip(X)} \left\{ \| f - g \|_p + \delta | g |_{Lip} \right\}$$

$(f \in L^p(X,\mu),\ 1 \le p < +\infty,\ \delta > 0)$.

Then, from (2.5) it follows that, for every $\tau \in \mathbb{A}$

$$| T(\tau)f - \int_X f d\mu | \le 2K(f, c(\tau)\delta(X)) \qquad (f \in C(X))$$

and

$$\| \tilde{T}(\tau)f - \int_X f d\mu \|_p \le 2\tilde{K}(f, c(\tau)\delta(X)) \quad (f \in L^p(X,\mu), 1 \le p < +\infty).$$

Finally, we point out that, if X is a compact interval of $\mathbb{R}$ or the unit circle $\mathbb{T}$ of $\mathbb{R}^2$, then

$$K(f,\delta) = \frac{1}{2}\bar{\omega}(f, 2\delta) \le \omega(f, 2\delta),$$

where $\omega(f,\cdot)$ denotes the usual modulus of continuity of f and $\bar{\omega}(f,\cdot)$ denotes the least concave majorant ([11, Chapter 6, Theorem 2.1 and Chapter 2, Lemma 2.1]).

Below we discuss some consequences of Theorem 2.1. Consider a Markov operator $T : C(X) \to C(X)$ such that

$$T(Lip(X)) \subset Lip(X) \tag{2.7}$$

and assume that there exists $c \in]0,1[$ such that

$$|T(f)|_{Lip} \le c|f|_{Lip} \tag{2.8}$$

for every $f \in Lip(X)$.

For every $m \in \mathbb{N}$, denote by T^m the iterate of T of order m. Clearly $T^m(Lip(X)) \subset Lip(X)$ and

$$|T^m(f)|_{Lip} \le c^m |f|_{Lip} \quad (f \in Lip(X)). \tag{2.9}$$

Therefore, Theorem 2.1 applies to $\mathbb{A} = \mathbb{N}, T(m) = T^m$ and $c(m) = c^m$ $(m \in \mathbb{N})$ and we get

COROLLARY 2.3. – *Under assumptions* (2.7) *and* (2.8), *there exists a unique* $\mu \in M_1^+(X)$ *such that* $T \in \Lambda(\mu)$, *i.e.,* $\int_X T(f)d\mu = \int_X f d\mu$ *for every* $f \in C(X)$. *Moreover,*

$$\lim_{m\to\infty} T^m(f) = \int_X f d\mu \quad \textit{uniformly on} \quad X \tag{2.10}$$

for every $f \in C(X)$, *and, if* $p \in [1, +\infty[$ *and* $f \in L^p(X, \mu)$, *then*

$$\lim_{m\to\infty} \widetilde{T}^m(f) = \int_X f d\mu \quad \text{in} \quad L^p(X, \mu). \tag{2.11}$$

Moreover, if $f \in Lip(X)$ *and* $m \geq 1$, *then*

$$|T^m(f) - \int_X f d\mu| \leq 2c^m \delta(X)|f|_{Lip} \tag{2.12}$$

so that the limit (2.10) *is uniform with respect to* $f \in Lip(X), |f|_{Lip} \leq r \quad (r > 0)$.

REMARK 2.4. – 1. Markov operators admitting only one invariant probability measure are called uniquely ergodic ([14, Section 5.1, p. 178]). Under assumptions (2.7) and (2.8), clearly from (2.11) it turns out that for every $f \in L^p(X, \mu)$ the sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1} \widetilde{T}^k(f)\right)_{n\geq 1}$ of the Cesàro means converges to $\int_X f d\mu$ in $L^p(X, \mu)$. On the other hand, by Akcoglu's ergodic theorem ([14, Theorem 2.6, p. 190]), the sequence is μ-a.e. convergent and hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \widetilde{T}^k(f) = \int_X f d\mu \quad \mu - a.e. \tag{2.13}$$

2. Estimates of the rate of convergence in (2.10) and (2.11) can be directly obtained from Remark 2.2, 2. We omit to explicitly state them for the sake of brevity.

The problem of checking condition (ii) of Theorem 2.1 for a continuous family $(T(t))_{t\geq 0}$ of Markov operators on $C(X)$ seems to be a more delicate task, especially when one does not know an explicit description of the operators $T(t)$ as it generally happens when dealing with the C_0-semigroup generated by some linear operator $A : D(A) \subset C(X) \to C(X)$.

Below we discuss a simple situation where both Theorem 2.1 and Corollary 2.3 can be successfully applied. This situation often occurs in the theory of approximation by positive linear operators (see also Section 3).

COROLLARY 2.5. – *Consider a semigroup* $(T(t))_{t\geq 0}$ *of Markov operators on* $C(X)$ *and assume that there exists a sequence* $(L_n)_{n\geq 1}$ *of Markov operators on* $C(X)$ *such that for every* $t \geq 0$ *there exists a sequence* $(k(n))_{n\geq 1}$ *of positive integers such that* $k(n)/n \to t$ *and*

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f) \quad \textit{uniformly on} \quad X \tag{2.14}$$

for every $f \in C(X)$.

Furthermore, assume that

(i) *There exists* $\omega \in \mathbb{R}, \omega < 0$, *such that for every* $n \geq 1$, $L_n(Lip(X)) \subset Lip(X)$ *and* $|L_n(f)|_{Lip} \leq \left(1 + \dfrac{\omega}{n}\right)|f|_{Lip}$ *for every* $f \in Lip(X)$.

(ii) *There exists* $\mu \in M_1^+(X)$ *such that* $L_n \in \Lambda(\mu)$ *for every* $n \geq 1$.

Then

(1) *For every* $n \geq 1$ *and* $f \in C(X)$

$$\lim_{m\to\infty} L_n^m(f) = \int_X f d\mu \quad \textit{uniformly on} \quad X. \tag{2.15}$$

(2) *For every* $t \geq 0$, $T(t)(LipX) \subset Lip(X)$ *and* $|T(t)f|_{Lip} \leq \exp(\omega t)|f|_{Lip}$ $(f \in Lip(X))$. *Moreover,* $T(t) \in \Lambda(\mu)$ *and*

$$\lim_{t\to+\infty} T(t)f = \int_X f d\mu \quad \textit{uniformly on} \quad X \tag{2.16}$$

for every $f \in C(X)$.

(3) *If* $1 \leq p < +\infty$, $n \geq 1$ *and* $f \in L^p(X, \mu)$, *then*

$$\lim_{m\to\infty} \widetilde{L}_n^m(f) = \int_X f d\mu = \lim_{t\to+\infty} \widetilde{T}(t)f \quad \text{in} \quad L^p(X,\mu). \tag{2.17}$$

PROOF. – Each operator L_n satisfies conditions (2.7) and (2.8) so that, by Corollary 2.3, there exists a unique $\nu_n \in M_1^+(X)$ such that $L_n \in \Lambda(\nu_n)$. Therefore $\nu_n = \mu$ and hence (2.15) follows from (2.10).

From (2.14) it turns out that, given $t \geq 0, T(t)(Lip(X)) \subset Lip(X)$ and

$$|T(t)f|_{Lip} \leq \exp(\omega t)|f|_{Lip}$$

because $\lim\limits_{n\to\infty}\left(1 + \dfrac{\omega}{n}\right)^{k(n)} = \exp(\omega t)$. Therefore, Theorem 2.1 applies and hence there exists a unique $\nu \in M_1^+(X)$ such that $T(t) \in \Lambda(\nu)$ for every $t \geq 0$. On the other hand $T(t) \in \Lambda(\mu)$ for every $t \geq 0$ as pointed out in Proposition 1.1, part (2), and hence $\nu = \mu$. Accordingly, (2.3) implies (2.16) as well as (2.17) follows from (2.11) and (2.4), respectively.

REMARK 2.6. – Notice that, according to (2.5) and (2.12), if $f \in Lip(X)$, then

$$|L_n^m(f) - \int_X f d\mu| \leq 2\Big(1 + \frac{\omega}{n}\Big)^m \delta(X)|f|_{Lip} \tag{2.18}$$

and

$$|T(t)f - \int_X f d\mu| \leq 2\exp(\omega t)\,\delta(X)|f|_{Lip} \tag{2.19}$$

($n \geq 1, m \geq 1, t \geq 0$). Other estimates for arbitrary functions in $C(X)$ or in $L^p(X, \mu)$ can be obtained by applying Remark 2.2, 2.

3. – Some applications

In this section we illustrate some applications that are mainly concerned with sequences of Markov operators occurring in approximation theory and with their associated Markov semigroups.

3.1 – *Kantorovich operators on the hypercube and the associated Fleming-Viot differential operator*

Given $N \geq 1$, consider the hypercube of $\mathbb{R}^N$

$$[0,1]^N := \{(x_i)_{1\leq i\leq N} \in \mathbb{R}^N \mid 0 \leq x_i \leq 1 \text{ for every } i = 1, \dots, N\}$$

endowed with the metric induced by the l_1-norm $\|\cdot\|_1$ defined by

$$\|x\|_1 := \sum_{i=1}^N |x_i| \quad (x = (x_i)_{1\leq i\leq N} \in \mathbb{R}^N).$$

We shall denote by λ_N the usual Borel-Lebesgue measure on $[0,1]^N$ and the corresponding space $L^p([0,1]^N, \lambda_N)$ will be simply denoted by $L^p([0,1]^N)$ $(1 \leq p < +\infty)$.

Given $n \geq 1$, consider the positive linear operator $C_n : L^1([0,1]^N) \to C([0,1]^N)$ defined by

$$C_n(f)(x) = \sum_{\substack{h=(h_i)_{1\leq i\leq N}\\ h_i\in\{0,\dots,n\}}} \Big[(n+1)^N \int_{\frac{h_1}{n+1}}^{\frac{h_1+1}{n+1}} dt_1 \dots \int_{\frac{h_N}{n+1}}^{\frac{h_N+1}{n+1}} f(t_1, \dots, t_N) dt_N\Big] P_{n,h}(x) \tag{3.1}$$

where

$$P_{n,h}(x) := \prod_{i=1}^N \binom{n}{h_i} x_i^{h_i}(1 - x_i)^{n-h_i} \tag{3.2}$$

($f \in L^1([0,1]^N)$, $x = (x_i)_{1\le i\le N} \in [0,1]^N$). The operators $C_n, n \ge 1$, have been first introduced in [22] and they represent a natural multidimensional extension of the classical Kantorovich operators on $L^1([0,1])$ (see, e.g., [3, pp. 333-335]). A recent generalization of them can be found in [4] and [5].

Each operator C_n is a Markov operator on $C([0,1]^N)$ and a positive contraction on $L^p([0,1]^N), 1 \le p < +\infty$ (see [4, proof of Theorem 2.5]).

Note that, given $n \ge 1, h = (h_i)_{1\le i\le N} \in \{0,\dots,n\}^N$ and $x = (x_i)_{1\le i\le n} \in [0,1]^N$, then

$$\int\limits_{[0,1]^N} P_{n,h}(x)dx = \prod_{i=1}^{N} \binom{n}{h_i} \int\limits_0^1 t^{h_i}(1-t)^{n-h_i}dt = \frac{1}{(n+1)^N}$$

so that, if $f \in L^1([0,1]^N)$,

$$\int\limits_{[0,1]^N} C_n(f)(x)dx = \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} \int\limits_{\frac{h_1}{n+1}}^{\frac{h_1+1}{n+1}} dt_1 \dots \int\limits_{\frac{h_N}{n+1}}^{\frac{h_N+1}{n+1}} f(t_1,\dots,t_N)dt_N = \int\limits_{[0,1]^N} f(x)dx.$$

Therefore,

$$C_n \in \Lambda(\lambda_N) \quad \text{for every} \quad n \ge 1. \tag{3.3}$$

and, obviously, $C_n|_{L^p([0,1]^N)}$ coincides with the extension of $C_n|_{C([0,1]^N)}$ as discussed in Section 1. As we pointed out in [5, formula (2.22)], the operators C_n are closely related to the Bernstein operators on $C([0,1]^N)$ that are defined by

$$B_n(f)(x) := \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} f\Big(\frac{h_1}{n},\dots,\frac{h_N}{n}\Big)P_{n,h}(x) \tag{3.4}$$

($f \in C([0,1]^N), x \in [0,1]^N$).

Setting, for $f \in L^1([0,1]^N)$ and $x = (x_i)_{1\le i\le N} \in [0,1]^N$,

$$\begin{aligned} F_n(f)(x) &:= (n+1)^N \int\limits_{\frac{nx_1}{n+1}}^{\frac{nx_1+1}{n+1}} dt_1 \dots \int\limits_{\frac{nx_N}{n+1}}^{\frac{nx_N+1}{n+1}} f(t_1,\dots,t_N)dt_N \\ &= \int\limits_0^1 dt_1 \dots \int\limits_0^1 f\Big(\frac{t_1+nx_1}{n+1},\dots,\frac{t_N+nx_N}{n+1}\Big)dt_N, \end{aligned} \tag{3.5}$$

then

$$C_n(f)(x) = B_n(F_n(f))(x). \tag{3.6}$$

If $f \in Lip([0,1]^N)$, then $F_n(f) \in Lip([0,1]^N)$ and $|F_n(f)|_{Lip} \le \dfrac{n}{n+1}|f|_{Lip}$.

Therefore, from (3.6) and from [3, Corollary 6.1.22 and Section 6.3.1, p. 476] it follows that $C_n(f) \in Lip([0,1]^N)$ and

$$|C_n(f)|_{Lip} \leq \frac{n}{n+1}|f|_{Lip} \leq \Big(1 - \frac{1}{2n}\Big)|f|_{Lip}. \tag{3.7}$$

Finally, we point out that in [5, Theorems 3.2 and 3.5] it was proved that there exist a Markov C_0-semigroup $(T(t))_{t\geq 0}$ on $C([0,1]^N)$ and a positive contractive C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p([0,1]^N)$, $1 \leq p < +\infty$, such that for every $f \in C([0,1]^N)$ (resp., $f \in L^p([0,1]^N)$) and $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $k(n)/n \to t$,

$$T(t)f = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \text{uniformly on } X \tag{3.8}$$

and

$$\widetilde{T}(t)f = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \text{in } L^p([0,1]^N) \tag{3.9}$$

respectively.

Moreover, the generators of these C_0-semigroups are the closures in $C([0,1]^N)$ and in $L^p([0,1]^N)$, respectively, of the elliptic second order differential operator $A : C^2([0,1]^N) \to C([0,1]^N)$ defined by

$$Au(x) := \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}(\frac{1}{2} - x_i)\frac{\partial u}{\partial x_i}(x). \tag{3.10}$$

Such a differential operator falls in a class of Fleming-Viot operators arising in population genetics (see, e.g., [10] and [16] and the references therein).

Summing up, all the assumptions of Corollary 2.5 are satisfied and hence we obtain the next result.

THEOREM 3.1. – *The following statements hold true:*

(1) *For every* $t \geq 0$, $T(t) \in \Lambda(\lambda_N)$; *moreover* $T(t)(Lip([0,1]^N) \subset Lip(([0,1]^N)$ *and*

$$|T(t)f|_{Lip} \leq \exp(-\frac{1}{2}t)|f|_{Lip}$$

for every $f \in Lip([0,1]^N)$).

(2) *If* $f \in C([0,1]^N)$ *and* $n \geq 1$, *then*

$$\lim_{m\to\infty} C_n^m(f) = \int_{[0,1]^N} f(x)dx = \lim_{t\to+\infty} T(t)(f)$$

uniformly on $[0,1]^N$.

(3) *If* $f \in L^p([0,1]^N), 1 \leq p < +\infty$, *and* $n \geq 1$, *then*

$$\lim_{m\to\infty} C_n^m(f) = \int_{[0,1]^N} f(x)dx = \lim_{t\to+\infty} \widetilde{T}(t)(f)$$

in $L^p([0,1]^N)$.

(4) *If* $f \in Lip([0,1]^N)$, *then*

$$|C_n^m(f) - \int_{[0,1]^N} f(x)dx| \leq 4(1-\frac{1}{2n})^m |f|_{Lip}$$

and

$$|T(t)f - \int_{[0,1]^N} f(x)dx| \leq 4\exp(-\frac{t}{2})|f|_{Lip}$$

$(n \geq 1, m \geq 1, t \geq 0)$.

Other estimates for arbitrary functions in $C([0,1]^N)$ or in $L^p([0,1]^N)$ can be obtained by applying Remark 2.2, 2.

3.2 – *Bernstein-Durrmeyer operators with Jacobi weights and the associated Fleming-Viot differential operator*

Consider the interval $[0,1]$ endowed with its natural metric. Let $\alpha \geq 0$ and $\beta \geq 0$ be given real numbers and let $\mu \in M_1^+([0,1])$ the absolutely continuous measure having the normalized Jacobi weight

$$w_{\alpha,\beta}(x) := x^\alpha(1-x)^\beta / \int_0^1 t^\alpha(1-t)^\beta dt, \quad (x \in [0,1]), \tag{3.11}$$

as density with respect to the Borel-Lebesgue measure on $[0,1]$.

For each $n \geq 1, k \in \{0,1,\dots,n\}$ and $f \in L^1([0,1],\mu)$ let

$$\begin{aligned} a_{n,k}(f) :&= \frac{\int_0^1 x^k(1-x)^{n-k} f(x)d\mu(x)}{\int_0^1 x^k(1-x)^{n-k} d\mu(x)} \\ &= \frac{\int_0^1 x^{k+\alpha}(1-x)^{n-k+\beta} f(x)dx}{\int_0^1 x^{k+\alpha}(1-x)^{n-k+\beta} dx} \\ &= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)} \int_0^1 x^{k+\alpha}(1-x)^{n-k+\beta} f(x)dx, \end{aligned}$$

where Γ denotes the usual Gamma function.

Given $n \geq 1$, consider the positive linear operator $M_n : L^1([0,1],\mu) \to C([0,1])$ defined by

$$M_n(f)(x) := \sum_{k=0}^{n} a_{n,k}(f)\binom{n}{k}x^k(1-x)^{n-k}$$

$(f \in L^1([0,1],\mu),\ x \in [0,1])$.

The operators $M_n, n \geq 1$ are called the Bernstein-Durrmeyer operators with Jacobi weights $w_{\alpha,\beta}$ on $[0,1]$ (see [9]). A more general definition of Bernstein-Durrmeyer operators with Jacobi weights on a simplex of $\mathbb{R}^N$ is presented in [1], [2], [8], [21] and the references therein.

Clearly each operator M_n is a Markov operator on $C([0,1])$ and

$$\int_0^1 M_n f(x)d\mu(x) = \int_0^1 f(x)d\mu(x) \quad (f \in C([0,1]))$$

i.e., $M_n \in \Lambda(\mu)$. Moreover, each M_n is a positive linear contraction on $L^p([0,1],\mu), 1 \leq p < +\infty$ ([9, page 27]) and hence the restriction of M_n to $L^p([0,1],\mu)$ coincides with the extension of M_n, restricted to $C([0,1])$, as discussed in Section 1.

The next result describes the behaviour of the operators M_n with respect to the functions lying in $Lip([0,1])$.

PROPOSITION 3.2. – *Given $n \geq 1$ and $f \in Lip([0,1])$, then $M_n(f) \in Lip([0,1])$ and*

$$|M_n(f)|_{Lip} \leq \frac{n}{n+\alpha+\beta+2}|f|_{Lip}. \tag{3.12}$$

PROOF. – By using Lagrange's mean value theorem we find

$$|M_n(f)|_{Lip} = \|(M_n(f))'\|_\infty,\ (f \in C([0,1])). \tag{3.13}$$

A straightforward calculation leads to

$$(M_n(f))'(x) = n\sum_{j=0}^{n-1}(a_{n,j+1}(f) - a_{n,j}(f))\binom{n-1}{j}x^j(1-x)^{n-1-j} \tag{3.14}$$

$(n \geq 1, f \in C([0,1]),\ x \in [0,1])$.

Moreover, if $f \in Lip([0,1])$ and $j \in \{0,\ldots,n-1\}$, introducing the function

$$F(x) := x^{j+\alpha+1}(1-x)^{n-j+\beta} \quad (0 \leq x \leq 1),$$

we get

$$
\begin{aligned}
&a_{n,j+1}(f) - a_{n,j}(f)\\
&= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+2)\Gamma(n-j+\beta+1)}\\
&\times \int_0^1 [(n-j+\beta)x^{j+\alpha+1}(1-x)^{n-j+\beta-1} - (j+\alpha+1)x^{j+\alpha}(1-x)^{n-j+\beta}]f(x)dx\\
&= -\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+2)\Gamma(n-j+\beta+1)} \int_0^1 F'(x)f(x)dx\\
&= -\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+2)\Gamma(n-j+\beta+1)} \int_0^1 f(x)dF(x)\\
&= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(j+\alpha+2)\Gamma(n-j+\beta+1)} \int_0^1 F(x)df(x),
\end{aligned}
$$

where both last integrals are Riemann-Stieltjes integrals and, in the last equality, we applied the partial integration formula.

If we consider the last integral as limit of Riemann-Stieltjes sums, we get

$$
|\int_0^1 F(x)df(x)| \le |f|_{Lip} \int_0^1 F(x)dx = |f|_{Lip} \frac{\Gamma(j+\alpha+2)\Gamma(n-j+\beta+1)}{\Gamma(n+\alpha+\beta+3)}.
$$

Thus

$$
|a_{n,j+1}(f) - a_{n,j}(f)| \le \frac{1}{n+\alpha+\beta+2}|f|_{Lip}. \tag{3.15}
$$

Now (3.12) is a consequence of (3.13), (3.14) and (3.15). □

In order to apply Corollary 2.5, we proceed to show the inequality

$$
|M_n(f)|_{Lip} \le (1+\frac{\omega}{n})|f|_{Lip}, \ (f \in Lip([0,1])), \tag{3.16}
$$

where

$$
\omega := -\frac{\alpha+\beta+2}{\alpha+\beta+3} < 0. \tag{3.17}
$$

In fact, (3.16) can be immediately derived from (3.12), by using the elementary inequality

$$
\frac{n}{n+t} \le 1 - \frac{1}{n}\frac{t}{t+1}, \ (n \ge 1, t > 0).
$$

On the other hand, consider the differential operator

$$Au(x) := x(1-x)u''(x) + (\alpha + 1 - (\alpha + \beta + 2)x)u'(x)$$

($u \in C^2([0,1]),\ x \in [0,1]$), that is a one-dimensional Fleming-Viot differential operator as well.

On account of [6, Section 2] there exists a Markov semigroup $(T(t))_{t\geq 0}$ on $C([0,1])$ generated by the closure of A; moreover, $C^2([0,1])$ is a core of this generator. In fact, the closure of A and its domain are explicitly described in [6, Theorem 2.1]. It is also known that for all $u \in C^2([0,1])$,

$$\lim_{n\to\infty} n(M_n(u) - u) = Au, \text{ uniformly on } [0,1].$$

(See [15], [20, Section 25.2]).

Now, according to Trotter's theorem (see, e.g., [3, Theorem 1.6.7]),

$$T(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \text{ uniformly on } [0,1],$$

for every $t \geq 0, f \in C([0,1])$, and for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$.

Summing up, all the assumptions of Corollary 2.5 are satisfied, with μ defined by the density (3.11) and ω given by (3.17). Taking Remark 2.6 into account, we are in a position to state our last result.

THEOREM 3.3. – *The following statements hold true:*

(1) *For every* $t \geq 0$, $T(t) \in \mathcal{A}(\mu)$*; moreover* $T(t)(Lip([0,1])) \subset Lip([0,1])$ *and*

$$|T(t)(f)|_{Lip} \leq \exp(\omega t)|f|_{Lip} \quad (f \in Lip([0,1])).$$

(2) *If* $f \in C([0,1])$ *and* $n \geq 1$, *then*

$$\lim_{m\to\infty} M_n^m(f) = \int_0^1 f(x)d\mu(x) = \lim_{t\to+\infty} T(t)(f)$$

uniformly on $[0,1]$.

(3) *If* $f \in L^p([0,1],\mu), 1 \leq p < \infty$, *and* $n \geq 1$, *then*

$$\lim_{m\to\infty} M_n^m(f) = \int_0^1 f(x)d\mu(x) = \lim_{t\to+\infty} \widetilde{T}(t)(f)$$

in $L^p([0,1],\mu)$.

(4) *If* $f \in Lip([0,1])$, *then*

$$|M_n^m(f) - \int_0^1 f(x)d\mu(x)| \leq 2(1 + \frac{\omega}{n})^m |f|_{Lip}$$

and

$$|T(t)(f) - \int_0^1 f(x)d\mu(x)| \leq 2\ \exp(\omega t)|f|_{Lip}$$

$(n \geq 1, m \geq 1, t \geq 0)$.

REMARK 3.4. – 1) Other estimates for arbitrary functions in $C([0,1])$ or in $L^p([0,1],\mu)$ can be obtained by applying Remark 2.2, 2.

2) On account of Proposition 1.1, it is easy to verify that the family $(\widetilde{T}(t))_{t\geq 0}$ associated with the Markov semigroup $(T(t))_{t\geq 0}$ considered in Theorem 3.3 is itself a C_0-semigroup of positive contractions on $L^p([0,1],\mu), 1 \leq p < +\infty$. Moreover, its generator is an extension of the generator of $(T(t))_{t\geq 0}$ and $C^2([0,1])$ is a core for it. In [16, Theorem 2.4] the domain of the generator is determined. Finally, for every $t \geq 0$ and $f \in L^p([0,1],\mu)$

$$\widetilde{T}(t)f = \lim_{n\to\infty} M_n^{k(n)}(f) \quad \text{in } L^p([0,1],\mu),$$

where $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers such that $k(n)/n \to t$.

3) The above results can be extended to Bernstein-Durrmeyer operators with Jacobi weights on a simplex of $\mathbb{R}^N$. Details will be given in a forthcoming paper.

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