Bollettino Unione Matematica Italiana

Michel Artola

A Class of Weighted Spaces

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **5** (2012), n.1, p. 125–158.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2012_9_5_1_125_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2012.

Bollettino U. M. I.
(9) **V** (2012), 125-158

A Class of Weighted Spaces

Michel Artola

In honour of Professor B. Muckenhoupt

Abstract. – *Hardy's inequality is used to give trace results, approximation and intermediate derivative properties for some weighted Banach spaces of generalized Sobolev type.*

1. – Introduction

The paper concerns previously unpublished original results by the author that systematically apply Hardy's inequality in weighted spaces. The lack of emphasis on a application in the recent book [14], devoted mainly to the inequality and its history, suggests that it might be time to publish these earlier conclusions.

The paper is dedicated to Professor Muckenhoupt to acknowledge his recognition of the method I developed for treating Hardy's inequality.

The original objective, explored while studying delay partial differential equations under the direction of J. L. Lions, was to extend to weighted Banach spaces certain interpolation results known in unweighted spaces or spaces of special weight t^{α}. The extension to intermediate derivatives in weighted Hilbert spaces was achieved in ([2], [3]), but the general extension required Hardy's inequality and certain relevant implications. The investigation was completed in 1968 with the statement and proof of a necessary and sufficient condition on the weight ω of the weighted spaces $L^p_\omega(B)$ and $L^q_\omega(B)$, defined in Section 2, in order that the Hardy operator

$$\mathcal{H} \ : \ f \longrightarrow \mathcal{H}(f) = \frac{1}{t}\int_0^t f(\sigma)d\sigma$$

is continuous from

$$L^p_\omega(B) \longrightarrow L^q_\omega(B), \ \ p, q > 1,$$

where B is a Banach space.

Talenti, Muckenhoupt, and others [14] at about the same time, but independently, obtained similar results for functions with scalar values on $(a, b) \subset R$.

The reason why my own contributions were not then submitted for publication was partly because at that time my interest included the development of the inequality not in its own right, but for the purpose of obtaining approximations, here illustrated in Section 3, that led to new properties of trace spaces described in Section 4.

Another interest, again concerning the inequality, focused on interpolation properties for derivatives in weighted Banach spaces of Sobolev type, where the derivative is the complex operator $D^{i\eta}$ defined, by a convolution whose kernel is $\frac{1}{\Gamma(-i\eta)} Pf[x_+^{-(i\eta+1)}]$, $\eta \in R$. The symbol Pf, represents "the finite part" or the "pseudo function" in the sense of Laurent Schwartz, $x_+ = max(0, x)$ and Γ is the usual Gamma function ([4], [5]).

The present paper is devoted mainly to applications of Hardy's inequality in some weighted spaces that generalize those with weights t^α discussed by J. L. Lions and J. Peetre ([16], [19]). We observe that conditions imposed on α imply that the weights implicitely belong to the Hardy class. Although the weight does not need to be specified *a priori* for our generalisation, nevertheless, it suits our purpose to impose a restriction.

The paper is constructed as follows. Section 2 contains the definition of the Hardy class $\mathcal{H}(p)$ and the proof given previously by the author of the necessary and sufficient condition for a weight to be in $\mathcal{H}(p)$ and further properties. These proofs were originally developed in the context of the classic Hardy inequality, and not in the general case considered by Muckenhoupt *et al.* Section 2 further undertakes the extension to the class $\mathcal{H}_{l,k}$ required subsequently in Section 4 to establish properties of trace spaces. An application, described in Section 2.7, is employed later in Section 3.

The spaces $W^{(m)}_{c_0,c_1}$ are introduced in Section 3, existence of traces for a special case is discussed and some inclusions are proved using complex interpolation and real methods. Approximation properties are next derived, but, in order to apply the result of Section 2.7, the derivative is subject to the assumption $c_1 \in \mathcal{H}(p_1)$. This avoids appealing to properties of intermediate derivatives that still await proof. The proofs of Proposition 3.10 and Corollary 3.11 rely in part on duality.

The discussion in Section 4, devoted to trace spaces and their properties, assumes that $c_i \in \mathcal{H}(p_i)$, $i = 0, 1$, to enable application of conclusions obtained in ([10], [16]).

Intermediate derivatives in a trace space are studied in Section 5, using the Hardy-Littlewood maximal operator which necessitates the assumption

$$c_i \in \Lambda(p_i),\ i = 0, 1 \tag{1.1}$$

where $\Lambda(p)$ is the class of Muckenhoupt [21]. Assumption (1.1) seems sufficient to extend spaces of weights t^{α} to those for which the inclusion $\Lambda(p) \subset \mathcal{H}(p)$ may be proved.

Finally, it is of interest to improve results obtained in [5] using complex and real interpolation methods.

The outcome will be reported in a forthcoming paper.

2. – A class of weighted L^p spaces.

2.1 – *Notation and definitions.*

Let B be a real or complex Banach space equipped with norm $|.|_B$, and let ω a positive locally integrable function on $\Omega \subset R^N$ ($\omega \in L^1_{loc}(R^+)$) taking values in R^+. Define the measure v, to be such that $dv = \omega(x)dx$, *where* $\omega > 0$, is a density with respect to the Lebesgue measure in R^N. Such a density ω will be called a *weight.*

We denote by L^p_ω(B) $1 \leq p \leq +\infty$, the space of functions u strongly measurable with values in B satisfying :

$$\int_\Omega |u(x)|^p_B dv < +\infty,$$

with usual modification for $p = +\infty$.

The space $L^p_\omega(B)$ becomes a Banach space when equipped with the norm

$$u \longrightarrow |u|_{L^p(B)} = \left(\int_\Omega |u(x)|^p_B dv \right)^{1/p}.$$

In what follows, we shall take $N = 1$, $\Omega = R^+ =]0, +\infty[$, and $\overline{R}^+ = [0, +\infty[$. It is also of interest to set $\omega = c^p$, where $c > 0$ satisfies

$$c \in L^p_{loc}(R^+), \ \frac{1}{c} \in L^{p'}_{loc}(\overline{R}^+) \tag{2.1}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Indeed, when $\omega = c^p$, the condition $u \in L^p_\omega(B)$ is equivalent to $cu \in L^p(B)$ using Lebesgue measure. Accordingly, we still refer to c *as a weight.*

In fact, the space $L^p_c(B)$ *always denotes the space of functions* u, *such that* $cu \in L^p(B)$. The letter ω is reserved exclusively for the density $\omega = c^p$, where c satisfies (2.1).

REMARK 2.1. – Note that the dual space of $L^p_\omega(B)$ is given by $L^{p'}_{\omega'}(B')$, and $\omega' = \omega^{1-p'} = c^{-p'}$, and B' is dual to B. Conditions (2.1), when assumed reasonable for c, by Hölder's inequality leads to

$$\forall T > 0,\ L^p_\omega(0, T; B) \subset L^1(0, T; B),$$

with continuous injective mapping.

2.2 – *The Hardy class $\mathcal{H}(p)$.*

Let B be a Banach space and f be a strongly measurable function on R^+ with values in B. Consider $L^p_\omega(R^+.B) \simeq L^p_\omega(B)$ and we use here the representation $\omega = c^p$, with c satisfying (2.1). We define the Hardy operator $\mathcal{H}$ by

$$\mathcal{H}\ :\ f \longrightarrow \mathcal{H}(f) = \frac{1}{t}\int_0^t f(\sigma)d\sigma,\ t > 0. \tag{2.2}$$

We set the

DEFINITION 2.2. – *For $p > 1$, we say that ω (or c) belongs to the Hardy class and one write: ω (resp.c) $\in \mathcal{H}(p)$), if $\mathcal{H}:\ f \longrightarrow \mathcal{H}(f)$ is continuous from $L^p_\omega(B)$ to $L^p_\omega(B)$.*

Then there is a positive constant K such that

$$\|\mathcal{H}(f)\|_{L^p_\omega(B)} \leq K\|f\|_{L^p_\omega(B)} \text{ for all } f \in L^p_\omega(B) \tag{2.3}$$

REMARK 2.3. – If $c(t) = t^\alpha,\ \theta = \alpha + \dfrac{1}{p}$, we know [12] that $c \in \mathcal{H}(p)$, with $K = \dfrac{1}{1-\theta}$ to be the best constant in (2.3).

2.3 – *A (first) necessary condition for ω (resp.c) belonging to $\mathcal{H}(p)$.*

We have the

PROPOSITION 2.4. – *Let the space $L^p_c(B)$ with c satisfying* (2.1). *Then a necessary condition to have $c \in \mathcal{H}(p)$ is*

$$\forall T_0 > 0,\ \frac{c}{t} \in L^p((T_0, +\infty); R^+). \tag{2.4}$$

PROOF. – Since step functions belong to $L^p_\omega(R^+; B)$, let $b \in B$, and if χ_τ is the characteristic function for $(0, \tau)$, consider the function

$$f_0 = \tilde{\chi}_{T_0} \otimes b = [\chi_{2T_0} - \chi_{T_0}]b$$

which is in the space $L^p_\omega(B)$ by (2.1).Then

$$\mathcal{H}(f_0) = 0, \quad 0 \le t \le T_0, \ = \frac{t - T_0}{t} b, \ t \in (T_0, 2T_0), \quad = \frac{T_0 b}{t} \ \text{if} \ t \ge 2T_0.$$

Thus, like we have assumed $\mathcal{H}(f_0) \in L^p_\omega(B)$, the announced result holds

2.4 – *A charactérization for ω (resp.c) belongings to $\mathcal{H}(p)$.*

Define for $t > 0$:

$$\Phi_p : \ t \longrightarrow \Phi_p(t) = \left[\int_t^{+\infty} \left(\frac{c(\sigma)}{\sigma} \right)^p d\sigma \right]^{1/p},$$

$$\Psi_{p'}; \ t \longrightarrow \Psi_{p'}(t) = \left[\int_0^t \frac{d\sigma}{c(\sigma)^{p'}} \right]^{1/p'}.$$

Where $1/p + 1/p' = 1$, which exist by (2.1) and Proposition 2.4.

PROPOSITION 2.5. – *Assume* (2.1) *holds. Then*

i) $c \in \mathcal{H}(p)$ *if and only if*

(2.5) $$\sup_{t>0} [\Phi_p(t).\Psi_{p'}(t)] < +\infty;$$

ii) *In* (2.3) *the constant K in the right is*

(2.6) $$K = K(p, c) = \gamma_p \sup_{t>0} \ [\Phi_p(t)\Psi_{p'}(t)], \quad \gamma_p = (p)^{1/p}(p')^{1/p'}.$$

The proof given hereafter is the original mine and, as I have already said, was obtained independently of that of Talenti [26], Tomaselli [28] and Muckenhoupt [20]. We give it here for the sake of convenience for the reader.

PROOF OF THE PROPOSITION 2.5.

α) *Necessity of the condition* (2.5).

Let $b \in B$. and consider $T_0 > 0$. From (2.1) the function $f = \chi_{T_0} c^{-p'} \otimes b$ is in $L^p_\omega(B)$ and a simple computation gives:

$$\left[\int_{R^+} (c(t)|f(t)|_B)^p dt \right]^{1/p} \le \left[\Psi_{p'}(T_0) \right]^{p'/p} |b|_B.$$

On the other hand

$$\mathcal{H}(f)(t) = \frac{b}{t}\left[\psi_{p'}(t)\right]^{p'},\ t \in (0, T_0),\ = \frac{b}{t}\left[\Psi_{p'}(T_0)\right]^{p'},\ t > T_0,$$

then, if we assume that $\omega \in \mathcal{H}(p),\ 1<p<+\infty$, there is a constant K (which not depends on T_0) with

$$\left(\int_{R^+}\left[c(t)|\mathcal{H}(f)(t)|_B\right]^p dt\right)^{1/p} \leq K\left[\Psi_{p'}(T_0)\right]^{p'/p}|b|_B,$$

from (2.3). Now a fortiori

$$\forall \varepsilon > 0,\ \left(\int_{\varepsilon}^{+\infty}\left[c(t)|\mathcal{H}(f)(t)|_B\right]^p dt\right)^{1/p} \leq K\left[\Psi_{p'}(T_0)\right]^{p'/p}|b|_B;$$

Choosing $\varepsilon = T_0$, one has simply

$$\Phi_p(T_0)\left[\Psi_{p'}(T_0)\right]^{p'} \leq K\left[\psi_{p'}(T_0)\right]^{p'/p},\ \forall T_0 > 0,$$

and finally, we obtain the condition (2.5).

β) (2.5) is a sufficient condition to have ω (or c) ∈ $\mathcal{H}(p)$.

Assume that (2.5) holds with

(2.7)
$$\sup_{t>0}\ [\Phi_p(t)\Psi_{p'}(t)] = M < +\infty,$$

and we have to prove that (2.3) holds with $K = K(c, p)$ given by (2.6).

We are going to estimate

$$I = \int_{R^+}\left|\frac{c(t)}{t}\int_0^t f(\tau)d\tau\right|_B^p dt \leq \int_{R^+}\left[\frac{c(t)}{t}\int_0^t |f(\tau)|_B d\tau\right]^p dt,\ 1<p<+\infty.$$

A natural idea is to write $|f(t)|_B = c(t)|f(t)|_B(c(t))^{-1}$ into the integral on $(0, t)$ in the right side of the last formula, before applying Hölder inequality; nevertheless, that led to a difficulty, but, if we inspect the particular case studied in the proof of necessity, we can get up it, writing:

$$|f(\tau)|_B = [g(\tau)\psi_{p'}(\tau)]^{1/p}[c(\tau)[\psi_{p'}(\tau)]^{1/p}]^{-1},\ g(\tau) = [c(\tau)[f(\tau)|_B]^p.$$

Now, applying Hölder inequality to the integral on $(0, t)$, one has

$$I \leq \int_{R^+}\left(\frac{c(t)}{t}\right)^p\left[\int_0^t g(\tau)\Psi_{p'}(\tau)d\tau\right].\left[\int_0^t\left(c(\tau)[\psi_{p'}(\tau)]^{1/p}\right)^{-p'}dt\right]^{p/p'},$$

where

$$\int_0^t [\phi_2(\tau)]^{p'} d\tau = \int_0^t \frac{1}{(c(\tau))^{p'}} \left[\Psi_{p'}(\tau)\right]^{-p'/p} d\tau = \int_0^t \left[p' \frac{d}{d\tau} \Psi_{p'}(\tau)\right] d\tau = p' \Psi_{p'}(t).$$

Thus

$$I \leq (p')^{p-1} \int_{R^+} \left[\left(\frac{c(t)}{t}\right)^p \left[\int_0^t g(\tau)\Psi_{p'}(\tau) d\tau\right] \left(\Psi_{p'}(t)\right)^{p-1} \right] dt,$$

and exchanging the order of integration, one has

$$I \leq (p')^{p-1} \int_{R^+} \left[g(t)\Psi_{p'}(t) \int_t^{+\infty} \left(\frac{c(\tau)}{\tau}\right)^p \left(\Psi_{p'}(\tau)\right)^{p-1} \right] dt.$$

Now using assumption (2.5) (i.e.: (2.7))

$$\left(\Psi_{p'}(\tau)\right)^{p-1} \leq M^{p-1} (\Phi_p(t))^{-(p-1)}$$

and noticing that

$$\left(\frac{c(\tau)}{\tau}\right)^p \left(\Phi_p(\tau)\right)^{-(p-1)} = p \frac{d}{d\tau} \Phi_p(\tau)$$

finally we obtain

$$I \leq (p'M)^{p-1} p \int_{R^+} g(t)\Psi_{p'}(t)\Phi_p(t) dt \leq (p')^{p-1} p M^p \int_{R^+} g(t) dt,$$

and (2.3) is proved if $1 < p < +\infty$.

The cases $p = 1$ *or* $p = +\infty$ are obvious and can be checked by continuity.

REMARK 2.6. – With $\omega = c^p$, condition (2.5) reads

(2.8) $$\sup_{t>0} \left\{ \left[\int_t^{+\infty} \tilde{\omega}(t) dt \right]^{1/p} \left[\int_0^t \omega^{1-p'} \right]^{1/p'} \right\} < +\infty, \ \textit{with} \ \tilde{\omega}(t) = \frac{c^p(t)}{t^p}.$$

If, $\mathcal{X}$, $\mathcal{Y}$, are two normed spaces, we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of linear continuous maps from $\mathcal{X}$ into $\mathcal{Y}$ and $\mathcal{L}(\mathcal{X}, \mathcal{X}) = \mathcal{L}(\mathcal{X})$.

Consider the integral operator $I: f \longrightarrow I(f)$; $I(f)(t) = \int_0^t f(\sigma) d\sigma$; then $\mathcal{H}(f) \in \mathcal{L}(L^p_\omega(B))$ is equivalent to $I \in \mathcal{L}(L^p_\omega(B), L^p_{\tilde{\omega}}(B))$. Speaking with integral's operator, our result for Hardy's operator is a particular case of the following problem:

"Given a pair of weights $(\omega_1,\ \omega_2)$, find conditions to have $I \in \mathcal{L}(L^p_{\omega_1}, L^p_{\omega_2})$". This was the point of wiew of Muckenhout, Talenti, Tomaselli, ... see [14]. The result is

$$\sup_{t>0}\left\{\left[\int_t^{+\infty} \omega_1(t)dt\right]^{1/p}\left[\int_0^t \omega_2(t)^{1-p'}dt\right]^{1/p'}\right\} < +\infty.$$

The proof is the same (up to some details) that in Proposition 2.5.

REMARK 2.7. – Observe that if $\phi.\omega$ is a weight, where ϕ is a positive, non increasing scalar function, then one has

$$\textit{If } \omega \in \mathcal{H}(p) \textit{ then } \phi.\omega \in \mathcal{H}(p)\ ,\ 1 \le p \le +\infty \tag{2.9}$$

Obviously

$$\int_t^{+\infty} \phi(\tau).\tilde{\omega}(\tau)d\tau \le \phi(t) \int_t^{+\infty} \tilde{\omega}(\tau)d\tau,$$

and

$$\int_0^t [\phi\omega]^{1-p'}(\tau)d\tau \le \frac{1}{\phi^{p'-1}(t)} \int_0^t \omega^{1-p'}(\tau)d\tau,$$

so that

$$\left[\int_t^{+\infty} \phi(\tau)\tilde{\omega}(\tau)d\tau\right]^{1/p}\left(\int_0^t [\phi\omega](\tau)^{1-p'}\right)^{1/p'} \le \Phi_p(t)\Psi_{p'}(t) < +\infty.$$

2.5 – *The dual operator $\mathcal{H}^*$ of $\mathcal{H}$.*

We denote $\mathcal{H}^*$ the operator defined by

$$\mathcal{H}^* :\ \forall t > 0,\ \phi(t) \longrightarrow \mathcal{H}^*(\phi)(t) = \int_t^{+\infty} \frac{\phi(\tau)}{\tau} d\tau. \tag{2.10}$$

$\mathcal{H}^*$ is the dual operator of $\mathcal{H}$. We want to check the

PROPOSITION 2.8. – *Assume that $c \in \mathcal{H}(p)$, $(1 < p < +\infty)$ holds true. Then the operator $\mathcal{H}^*$ is continuous from $L^{p'}_{1/c}(B')$ to $L^{p'}_{1/c}(B')$.*

PROOF. – Let us consider

$$J = \int_{R^+} \left[\frac{1}{c(t)} \int_t^{+\infty} \frac{|\phi(\tau)|_{B'}}{\tau} d\tau \right]^{p'} dt$$

and if we write $\dfrac{|\phi(\tau)|_{B'}}{\tau} = \gamma(\tau)[\Phi_p(\tau)]^{1/p'}.\ \dfrac{c(\tau)}{\tau}[\Phi_p(\tau)]^{-p/p'}$ where $\gamma(\tau) = \dfrac{|\phi(\tau)|_{B'}}{c(\tau)}$, Hölder inequality gives

$$\int_t^{+\infty} \frac{|\phi(\tau)|_{B'}\cdot}{\tau} dt \leq \left[\int_t^{+\infty} [\phi(\tau)]^{p'} \Phi_p(t) dt \right]^{1/p'} \left[\int_t^{+\infty} \tilde{\omega}(\tau)[\Phi_p(\tau)]^{-p/p'} d\tau \right]^{1/p'}.$$

Now, observing that

$$\tilde{\omega}(\tau)[\Phi_p(\tau)]^{-p/p'} = -p \frac{d}{d\tau}[\Phi_p(\tau)],$$

we obtain

$$J \leq p^{p'-1} \int_{R^+} \frac{1}{[c(t)]^{p'}} \int_t^{+\infty} [\gamma(\tau)]^{p'} \Phi_p(\tau) d\tau [\Phi_p(t)]^{p'-1} dt,$$

Therefore, using Fubini theorem, one has

$$J \leq \int_{R^+} [\gamma(\tau)]^{p'} \Phi_p(\tau) \left[\int_0^{\tau} \frac{1}{[c(t)]^{p'}} [\Phi_p(t)]^{p'-1} dt \right] d\tau. \tag{2.11}$$

But, as $c \in \mathcal{H}(p)$, we have $[\Phi_p(t)]^{p'-1} \leq M^{p'-1} \Psi_{p'}^{1-p'}$ and like we know (from Propositon 2.5) that $\int_0^{\tau} \dfrac{1}{[c(t)]^{p'}} [\psi_{p'}(t)]^{1-p'} dt = p' \psi_{p'}(\tau)$, thus we deduce from (2.10)

$$J \leq \frac{[Mp]^{p'}}{p-1} \int_{R^+} [\gamma(\tau)]^{p'} d\tau.$$

The result is proved.

The adjoint operator of the integral operator I is now I^* defined by $I^* : \phi(t) \longrightarrow \int_t^{+\infty} \phi(\tau) d\tau$ and (see also [20]) $I^* \in \mathcal{L}(L^q_{\omega_1},\ L^q_{\omega_2})$, $1 \leq q < +\infty$, if and only if

$$\sup_{t>0} \left| \int_0^t \omega_2(\tau) d\tau \right]^{1/q} \left[\int_t^{+\infty} \omega_1(\tau)^{1-q'} d\tau \right]^{1/q'} < +\infty$$

which still gives (2.5), with $q = p', \omega_1 = \tilde{\omega}^{-1},\ \omega_2 = \omega^{-1}$.

2.6 – *A variant :* $\mathcal{H}_{k,l}(p),\ k,l>0$*, classes.*

More generally consider, with Hardy-Littlewood- Polyà ([12] p. 330), the family of operators

$$\mathcal{H}_{k,l}:\ f\longrightarrow \mathcal{H}_{k,l}(f),\ \mathcal{H}_{k,l}(f)(t)=\frac{1}{t^{k+1}}\int_0^t \tau^l f(\tau)d\tau \tag{2.12}$$

and when $k=l$, one write $\mathcal{H}_{l,l}=\mathcal{H}_l$.

Let $p\in[1,+\infty]$ and $L^p_\alpha(R^+;R)=L^p_\alpha(R)$ a weighted Banach space whose the weight is given (following our convention) by $c(t)=t^\alpha$, then :

$$\textit{if}\ \alpha+\frac{1}{p}<l+1, \mathcal{H}_l\in\mathcal{L}(L^p_\alpha(R))\ \textit{and}\ \mathcal{H}_{k,l}\in\mathcal{L}(L^p_\alpha(R),L^p_{\alpha+k-l}(R)),$$

The result can be easily extended to spaces $L^p_\alpha(B)$ where B is a Banach space and more generally to the weighted spaces $L^p_c(B)$.

Accordingly with previous notations, we set $c_l(t)=\dfrac{c(t)}{t^l}$, $\omega_l(t)=c_l(t)^p$, *if* $k=l$, and $c_{k,l}(t)=t^k c_l(t)$, $\omega_{k,l}(t)=c^p_{k,l.}(t)$.

In what follows, writing "*iff*" for "*if and only if*", we claim

PROPOSITION 2.9. – i) *case* $k=l$: $\mathcal{H}_l\in\mathcal{L}(L^p_\omega(B))$ *iff* $\omega_l\in\mathcal{H}(p)$.
ii) *case* $k\neq l$: *one has* $\mathcal{H}_{k,l}\in\mathcal{L}(L^p_\omega(B),L^p_{\omega_{k,l}}(B))$, *iff* c_l (*or* ω_l) $\in\mathcal{H}(p)$.

Note that the result holds if $\omega\in\mathcal{H}(p)$ and $l>0$ from remark 2.7.

PROOF. – i) $\mathcal{H}_l\in\mathcal{L}(L^p_\omega(B))$ means that there is a constant $K>0$, with:

$$\int_{R^+} c^p(t)\left[\frac{1}{t^{l+1}}\int_0^t \tau^l f(\tau)d\tau\right]^p dt\le K^p\int_{R^+} c^p(t)|f(t)|^p_B dt,$$

which is exactly

$$\int_{R^+} c^p_l(t)|\mathcal{H}(g)(t)|^p_B dt\le K^p\int_{R^+} c^p_l(t)|g(t)|^p_b dt,\quad g(t)=t^l f(t).$$

ii) $\mathcal{H}_{k,l}\in\mathcal{L}(L^p_\omega(B),L^p_{\omega_k}(B))$ means that

$$\int_{R^+}\left(t^{k-l}c(t)\right)^p\left[\frac{1}{t^{k+1}}\left|\int_0^t \tau^l f(\tau)d\tau\right|_B\right]^p dt=\int_{R^+} c^p_l(t)|\mathcal{H}(g)(t)|^p_B dt\le K^p\int_{R^+} c^p_l(t)|g(t)|^p_B dt.$$

The following remark concerning $\mathcal{H}_l$ will be also usefull in Section 4.

REMARK 2.10. – One notices that one can write $\mathcal{H}_l(f)(t) = \int\limits_0^1 \sigma^l f(t\sigma)d\sigma$ therefore if f is $j-time$ differentiable with values in B, we obtain:

$$(2.13)\quad D^j(\mathcal{H}_l(f)(t) = \int\limits_0^1 \sigma^{l+j} D^j f(t\sigma)d\sigma = \frac{1}{t^{l+j+1}}\int\limits_0^t \tau^{l+j} D^j f(\tau)d\tau = \mathcal{H}_{l+j}(D^j f)(t).$$

2.7 – *A first application.*

The following proposition will be usefull in Section 3.

PROPOSITION 2.11. – *Let u be a continuously $(m-1)$-time differentiable function with values in B and let the distributional derivative $D^m u$, of order $m \geq 1$, be locally integrable and satisfying*

$$c(.)D^m u \in L^p(R^+; B),\ \textit{where}\ c \in \mathcal{H}(p),$$

and

$$D^j u(0)\ = 0,\ 0 \leq j \leq m-1$$

then

$$q_j(.)D^{m-j}u \in L^p(R^+; B),\ q_j(t) = \frac{c(t)}{t^j},\ q_j \in \mathcal{H}(p),\ 0 \leq j \leq m.$$

PROOF OF PROPOSITION 2.11. – One has

$$D^{m-1}u(t) = \int\limits_0^t D^m u(\tau)d\tau,$$

and

$$\frac{D^{m-1}u(t)}{t} = \mathcal{H}(D^m u)(t).$$

Since $c \in \mathcal{H}(p)$, one obtain the result for $j = 1$. Now, we note that, from remark 2.7 $\left(\text{with } \phi(t) = \frac{1}{t^j}\right)$:

$$q_j \in \mathcal{H}(p),\ 1 \leq j \leq m,$$

and since we can write

$$D^{m-j}u(t) = \frac{1}{(j-1)!}\int\limits_0^t (t-\tau)^{j-1} D^m u(\tau)d\tau,$$

then

$$\frac{|D^{m-j}u(t)|_B}{t^j} \leq \mathcal{H}(|D^m(u)(t)|_B),$$

therefore the result for j follows, $1 \leq j \leq m$.

3. – A class of weighted Sobolev Spaces

3.1 – *Notations and definitions*

In what follows if $\mathcal{X}$, $\mathcal{Y}$, be normed spaces , $\mathcal{X} \subset \mathcal{Y}$ means always algebric inclusion with continuous injective mapping.

Let A_0 and A_1 be two Banach spaces continuously embedded into a (real or complex) topologic vector space $\mathcal{A}$ so that

$X = A_0 \cap A_1$ is equipped with the norm $|u|_X = max(|u|_{A_0}, |u|_{A_1})$,

$Y = A_0 + A_1$ is equipped with the norm : $|u|_Y = \inf_{a=a_0+a_1} (|a_0|_{A_0} + |a_1|_{A_1})$.

of course, we have

$$X \subset A_i \subset Y, \ (i = 0, 1)$$

and we assume that

$$X \text{ is dense into } A_i \ (i = 0, 1). \tag{3.1}$$

We consider $\omega_0 = c_0^{p_0}$ and $\omega_1 = c_1^{p_1}$, p_0, $p_1 \in [1, +\infty]$, be two weight functions satisfying

$$\forall \varepsilon, t > 0, \ c_i \in L^{p_i}(]\varepsilon, t[; R^+), \ c_i^{-1} \in L^{p'_i}(]\varepsilon, t[; R^+), \ i = 0, 1. \tag{3.2}$$

and we define

$$W^{(m)}[p_0, c_0, A_0; p_1, c_1, A_1] = W^{(m)}_{c_0,c_1}$$

the space of functions u locally integrable on R^+, with $u \in L^{p_0}_{c_0}(A_0)$ and such that $D^m u \in L^{p_1}_{c_1}(A_1)$.The last condition must be understood as follows: u is m-time differentiable at the sense of distributions on R^+ with values in Y and $D^m u$ is locally integrable, so that the product with c_1 makes a sense. Indeed, since $D^m u$ is locally integrable, then $D^{m-1}u$ is absolutely continuous, hence continuous, then we can consider that u is $(m-1)$-time continuously differentiable on $R^+$with values in Y and $D^i u(t)$, $0 \leq i \leq m-1$, is defined for $t \in]0, +\infty[$. Therefore if $\lim_{t \longrightarrow 0} D^i u(t) = a$ in Y exists, we shall said that $D^i u$ has a trace $D^i u(0) = a$ at $t = 0$.

Equipped with the norm

$$u \longrightarrow \|u\|_{W^m} = max(|u|_{L^{p_0}_{c_0}(A_0)}, |D^m u|_{L^{p_1}_{c_1}(A_1)}).$$

$W^{(m)}_{c_0,c_1}$ is a Banach space.

In what follows we shall set $M_i(v) = |v|_{L^{p_i}_{c_i}(A_i)}$, $i = 0, 1$.

Now, we define

$$\overset{0}{W}{}^{(m)}[p_0, c_0, A_0; p_1, c_1, A_1] = \overset{0}{W}{}^{(m)}_{c_0,c_1},$$

the subspace of $W^{(m)}_{c_0,c_1}$:

$$\overset{0}{W}{}^{(m)}_{c_0,c_1} = \left\{u;\ u \in W^{(m)}_{c_0,c_1},\ D^j u(0) = 0,\ j = 0, 1, \ldots, m-1\right\} \tag{3.3}$$

where the dérivatives are ever understood in the sense of distribution on R^+ with values in Y.

Equipped with the relative topology, $\overset{0}{W}{}^{(m)}_{c_0,c_1}$ is a Banach space which, sometimes, coincide with $W^{(m)}$.

REMARK 3.1. – i) From the the first part of (3.2), $c_0 \notin L^{p_0}(0,1;R^+)$ is authorized and then, one has necessarily: $\underset{t\longrightarrow 0}{lim}\, u(t) = 0\ in\ Y$. Indeed, if we have $\underset{t\longrightarrow 0}{lim}\, u(t) = u(0) = a \neq 0\ in\ Y$, we can find $\varepsilon > 0$ such that $|u(t)|_Y \geq \beta > 0$ for all $t \in (0, \varepsilon)$, then

$$\int_0^{\varepsilon} |c_0 u|^{p_0}_{A_0} dt \geq \int_0^{\varepsilon} |c_0 u|^{p_0}_{Y} dt \geq \beta^{p_0} \int_0^{\varepsilon} c_0(t) dt = +\infty,$$

which is a contradiction.

ii) Moreover if $t^k c_0 \notin L^{p_0}(0,1;R^+)$ for $0 \leq k \leq j < m$, then $\underset{t\longrightarrow 0}{lim}\, D^k u(t) = u^{(k)}(0) = 0$ in Y. Similarly, if $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $t \in (0, \eta)$ implies $t^{-k}|u(t)|_Y \geq \varepsilon$, then

$$\int_0^{\eta} |c_0 u|^{p_0} dt \geq \varepsilon^{p_0} \int_0^{\eta} \left(t^k c_0\right)^{p_0} dt = +\infty,$$

which is a contradiction.

Therefore, we can find a sequence $\{t_q\}$ with $\underset{q\longrightarrow +\infty}{lim}\, t_q = 0$ such that $\underset{q\longrightarrow +\infty}{lim}\, (t_q^{-k}|u(t_q)|_Y) = 0$, and that, is in contradiction with $\underset{t\longrightarrow 0}{lim}\, D^k u(t) = a_k \neq 0$ in Y. The result follows by induction.

iii) A necessary and sufficient condition for $D^j u$ to have a trace at $t = 0$ is due to J. Poulsen [23] only for u such that $c_1 D^m u \in L^{p_1}(R^+; B)$ (where B is a Banach space). It is: $\dfrac{t^{m-j-1}}{c_1} \in L^{p_1'}(0,1;R^+)$ and if $\dfrac{t^{m-j-1}}{c_1} \notin L^{p_1'}(0,1;R^+)$, $u^{(j)}(0) = 0$.

Thus, if the last property occurs for all j, $0 \leq j \leq m-1$, we have $W^{(m)}_{c_0,c_1} = \overset{0}{W}{}^{(m)}_{c_0,c_1}$.

REMARK 3.2. – When $c_i(t) = t^{\alpha_i}$, $(i = 0, 1)$, the corresponding spaces are mainly studied by J. L. Lions [16] which assume the condition

$$\theta_i = \alpha_i + \frac{1}{p_i} \in\ ]0,1[,\ \ i = 0, 1. \tag{3.4}$$

The condition $\theta_i > 0$ implies $c_i \in L^{p_i}(0,T)\ \forall T > 0$ and $\theta_i < 1$, involves not only that $\dfrac{1}{c_i} \in L^{p'_i}(0,T)$, $\forall T > 0$, but also that $c_i \in \mathcal{H}(p_i)$.

Conditon (3.2) involves also that in the case $c_i(t) = t^{\alpha_i}$

$$1 \leq \mu_t(c_i)(\mu_t(1/c_i))^{p_i/p'_i} \leq K(p_i,\alpha_i) = p_i\left(\frac{p_i}{p_i-1}\right)^{p_i-1}\theta_i(1-\theta_i)^{p_i-1},\ \mu_t(f) = \frac{1}{t}\int_0^t f(\tau)d\tau.$$

That condition is of the B. Muckenhoupt type $\mathcal{A}(p)$ for intervals $I = (0,t)$ ([21]) which implies that $c_i \in \mathcal{H}(p_i)$ (see section 5).

In what follows some proofs need the condition $c_i \in \mathcal{H}(p_i)$ either for $i = 0$ or $i = 1$, (or both), assumptions which are implicitely assumed in the case where $c_i(t) = t^{\alpha_i}$.

3.2 – *An inclusion*

A first consequence of Proposition 2.11 gives the following result, which is of interest in itself, but also in some applications to partial differential equations (see for example [3]).

Before stating the proposition ant its proof, we recall some definitions closed with interpolation theory.

An *intermediate space* "between" A_0 and A_1 is a topologic space V, with

$$X \subset V \subset Y,$$

so that A_0 and A_1 are themselves intermediate spaces.

Complex and real interpolation methods (see [7], [15], [19], [22], [27]) allow to built intermediate spaces between A_0, A_1, with the following property: *every linear mapping from Y into itself, which is continuous from A_i ($i = 0$ or 1) into itself is automatically continuous from V into itself.* According to complex method we consider here the space $(A_0, A_1)_\theta$, $\theta \in (0,1)$ which is obtained like the image in Y of $f(\theta)$, where f belongs to the space of real analytic functions with values in Y, defined on the strip $0 < \mathcal{R}(z) < 1$, continuous on $0 \leq \mathcal{R}(z) \leq 1$, with

$$\underset{y}{sup}\,|f(iy)|_{A_0},\ \underset{y}{sup}\,|f(1+iy)|_{A_1} < +\infty,$$

and equipped with the norm

$$\|f\| = max(\underset{y}{sup}\,|f(iy)|_{A_0}, \underset{y}{sup}\,|f(1+iy)|_{A_1}),$$

so that

$$(A_0, A_1)_\theta = \{a \in Y;\ a = f(\theta),\ \text{equipped with the norm } \|a\|_\theta = \underset{f:\,f(\theta)=a}{inf}\ \|f\|\}.$$

For example, if $A_0 = L^{p_0}_{\omega_0}(B)$, $A_1 = L^{p_1}_{\omega_1}(B)$, then $(A_0,\ A_1)_\theta = L^{p_\theta}_{\omega_\theta}(B)$, where one has $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, $\omega_\theta = \omega_0^{1-\theta}\omega_1^\theta$ (see. [15]).

We state

PROPOSITION 3.3. – *Let $c_1 \in \mathcal{H}(p_1)$. Then*

$$\overset{0}{W}{}^{(m)}_{c_0,c_1} \subset \overset{0}{W}{}^{(m)}[p_\theta, c_\theta, (A_0, A_1)_\theta; p_1 c_1 A_1], \tag{3.5}$$

where $\theta \in (0,1)$, $c_\theta = c_0^{1-\theta}\, c_1^\theta\, t^{-m\theta}$, $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$, the inclusion is algebrical and topological.

PROOF. – From Proposition 2.11, tacking $B = Y$, one can writes $u(t) = \dfrac{1}{(m-1)!}\int\limits_0^t (t-\tau)^{m-1} D^m u(\tau)d\tau$, then like $c_1 D^m u \in L^{p_1}(A_1)$, one has

$$u \in L^{p_0}_{c_0}(A_0),\ \text{and}\ u \in L^{p_1}_{q_m}(A_1),\ q_m = \frac{c_1}{t^m}.$$

By complex interpolation ([7], [15]), we obtain:

$$\Big(L^{p_0}_{c_0}(A_0), L^{p_1}_{q_m}(A_1)\Big)_\theta = L^{p_\theta}_{c_\theta}[(A_0, A_1)_\theta],\ c_\theta = c_0^{1-\theta} q_m^\theta, \tag{3.6}$$

hence the result follows.

REMARK 3.4. – We can apply also the real interpolation method of [19] (see also [22], [27]) where we can replace $[A_0, A_m]_\theta$ by $[A_0, A_m]_{\theta,p}$, $0<\theta<1$, $1<p \leq +\infty$.

3.3 – *Approximation properties*

We assume that the couple (A_0, A_1) has the following *approximation property* $(\mathcal{P})$ (1):

"for every $n \in N$, there is $P_n \in \mathcal{L}((A_i,\ X)$, $(i = 0,1)$ such that

$$P_n a_i \longrightarrow a_i \text{ in } A_i \text{ as } n \longrightarrow +\infty,\ (i = 0,1).\text{"} \tag{3.7}$$

The main consequence of (3.7) is the following result

(1) Introduced in [15] by J. L. Lions.In particular the property is true for domains of Semi-Group operators.

LEMMA 3.5. – *The space*

$$W^{(m)}_{c_0,c_1}(X) \text{ is dense in } W^{(m)}_{c_0,c_1}.$$

PROOF. – From (3.7), $|P_n|_{\mathcal{L}(A_i,X)} \leq k_i$, $i = 0, 1$, so that *if* $u \in W^{(m)}_{c_0,c_m}$ letting $v_m = D^m u$, one has $P_n u(t) \longrightarrow u(t)$ a.e. in A_0, $P_n v_m \longrightarrow v_m$ *a.e. in* A_1, therefore the Lebesgue theorem of dominated convergence gives the result.

Now, consider

$$W^{(m)}_K(X) = \{u;\ u \in\ W^{(m)}_{c_0,c_1}(X),\ \text{with compact support in } R^+ =]0, +\infty[\}$$

one has

LEMMA 3.6. – *If* $c_1 \in \mathcal{H}(p_1)$, *then* $W^{(m)}_K(X)$ *is dense in* $\overset{0}{W}{}^{(m)}_{c_0,c_1}$.

PROOF.

Step 1 : From Lemma 3.5 obviousely, $\overset{0}{W}{}^{(m)}_{c_0,c_1}$ is dense in $\overset{0}{W}{}^{(m)}_{c_0,c_1}$.

Step 2 : It remains to prove that $W^{(m)}_K(X)$ is dense in $\overset{0}{W}{}^{(m)}_{c_0,c_2}(X)$.

Let $\theta \in \mathcal{D}(R^+)$, $\theta(t) = 0$ if $t \geq 2$, $\theta = 1$ if $t \in [0, 1]$, and $\theta_n(t) = \theta\left(\dfrac{t}{n}\right)$, $\rho_n \in C^\infty_c(R^+)$, $\rho_n(t) = 0$, if $t \in \left[0, \dfrac{1}{n}\right)$, $\rho_n(t) = 1$, if $t \geq \dfrac{2}{n}$, with $\rho^{(k)}_n(t) \leq cn^k$, $k = 0, 1, \ldots$.

If $u \in \overset{0}{W}{}^{(m)}_{c_0,c_1}(X)$, we introduce $u_n = \chi_n u$, $\chi_n = \rho_n \theta_n$, then $u_n \in W^{(m)}_K(X)$. To prove the claimed result, we have to show:

$$u_n = \chi_n u \longrightarrow u,\ \text{in } L^{p_0}_{c_0}(R^+; X),\quad D^m u_n \longrightarrow D^m u,\ \text{in } L^{p_1}_{c_1}(R^+; X).$$

It is obvious that

$$\chi_n D^i u \longrightarrow D^i u \text{ in } X_i = L^{p_i}_{c_i}(R^+; X),\ i = 0, 1\ (D^0 = I). \tag{3.8}$$

Since

$$D^m u_n = \chi_n D^m u + \sum_{q+r+s=m, s\neq m} \alpha(q, r, s) D^q \rho_n D^r \theta_n D^s u,$$

and because one has $D^q \rho_n D^r \theta_n = 0$, if $q \neq r$, the sum on the right-hand side in the last formula reads

$$\sum_{s=0}^{m-1} \alpha(0, m-s, s) \rho_n D^{m-s} \theta_n D^s u + \sum_{s=0}^{m-1} \alpha(m-s, 0, s) \theta_n D^{m-s} \rho_n D^s u,$$

so that, it is sufficient to check

$$i)\ \rho_n D^{m-s} \theta_n D^s u \longrightarrow 0, \tag{3.9}$$

$$ii)\ \theta_n D^{m-s} \rho_n D^s u \longrightarrow 0,\ in\ X_1 = L^{p_1}_{c_1}(R^+; X),\ as\ n \longrightarrow +\infty, 0 \leq s \leq m-1 \tag{3.10}$$

In order to do that, is easy to see, first, that

$$|\rho_n D^{m-s}\theta_n D^s u|_{X_1}^{p_1} \leq \frac{k_\theta}{n^{(m-s)p_1}} \int_n^{2n} c_1^{p_1} |D^s u(t)|_X^{p_1} dt.$$

Now, according to Proposition 2.11: $\dfrac{D^s u(t)}{t^{m-s}} \in X_1$, and one has

$$\int_n^{2n} c_1 |D^s u(t)|_X^{p_1} dt \leq k_\theta 2^{(m-s)p_1} n^{(m-s)p_1} \int_n^{2n} \left(\frac{c_1}{t^{m-s}}\right)^{p_1} |D^s u(t)|_X^{p_1} dt,$$

so that

$$(3.11)\quad |\rho_n D^{m-s}\theta_n D^s u|_{X_1}^{p_1} \leq k_\theta 2^{(m-s)p_1} \int_n^{2n} \left(\frac{c_1}{t^{m-s}}\right)^{p_1} |D^s u(t)|_X^{p_1} dt \longrightarrow 0 \ as \ n \longrightarrow +\infty$$

and (3.9) holds.

Similarly, we obtain

$$(3.12)\quad |\theta_n D^{m-s}\rho_n D^s u|_{X_1}^{p_1} \leq k_\rho n^{(m-s)p_1} \int_{1/n}^{2/n} c_1^{p_1} |D^s u(t)|_X^{p_1} dt$$

$$\leq k_\rho 2^{(m-s)p_1} \int_{1/n}^{2/n} \left(\frac{c_1}{t^{m-s}}\right)^{p_1} |D^s u(t)|_X^{p_1} dt, \longrightarrow 0 \ as \ n \longrightarrow +\infty$$

and (3.10) also holds.

Now consider the case of $W^{(m)}_{c_0,c_1}$, where here $W^{(m)}_{\tilde{K}} = \{u;\ u \in W^{(m)}_{c_0,c_1}$ *with compact support in* $[0,+\infty[\}$

LEMMA 3.7. – *Assume* $c_1 \in \mathcal{H}(p_1)$, *then* $W^{(m)}_{\tilde{K}}$ *is dense in* $W^{(m)}_{c_0,c_1}$.

PROOF. – According to Lemma 3.5 it remains to prove the density of $W^{(m)}_{\tilde{K}}(X)$ in $W^{(m)}_{c_0,c_1}(X)$. Let $u \in W^{(m)}_{c_0,c_m}(X)$ which is (m-1)- time continuously differentiable with values in X. Define a function $v\ (t) = \sum_{i=0}^{i=m-1} \alpha(t) \frac{t^i}{i!} u^{(i)}(0)$, where

$$\alpha \in \mathcal{D}([0,+\infty]),\ \alpha(t) = 1,\ if\ t \in [0,R],\ \alpha(t) = 0,\ if\ t \geq 2R.$$

Actually from the left condition (3.2) we easily see that $v \in W^{(m)}_{\tilde{K}}(X)$. Then $w = v - u \in \overset{0}{W}{}^{(m)}_{c_0,c_1}(X)$, so that, since $u = v + w$ the result follows from Lemma 3.5.

In order to symplify the notations, we let: $W^{(m)}_{c_0,c_1} = W^{(m)}$, $W^{(m)}_{c_0,c_1}(X) = W^{(m)}(X)$.

3.4 – *Duality*

We assume now, that the spaces A_i $(i=0,1)$ are reflexive. The dual or antidual of A_i is denoted by A'_i and the scalar product in the duality (A'_i, A_i) is denoted by $\langle .,. \rangle_i$.

Like X is dense in A_i, $i=0,1$, then we can identify A'_i with a subspace of X' so that

$$A'_i \subset X', \quad i=0,1. \tag{3.13}$$

Indeed if i_k is the injective mapping of X into A_k $(k=0,1)$ its range is then dense in A_k and this implies that the adjoint i_k^* is a continuous injective mapping from A'_k into X' whose the range is dense. Thus $A'_0 \cap A'_1$ and $A'_0 + A'_1$ are well defined. Notice that in what follows $(A'_0, A'_1,\ X')$ will play there the part that (A_0, A_1, Y) played before.

Now, from the structure of the space $W^{(m)}$, it is straightforward to see that

Proposition 3.8. – *Every linear form $u \longrightarrow L(u)$ continuous on $W^{(m)}_{c_0,c_1} = W^{(m)}$ can be defined by*

$$L(u) = \int_0^{+\infty} \langle l_0(t), c_0(t)u(t)\rangle_0 dt + \int_0^{+\infty} \langle l_1(t), c_1(t)D^m u(t)\rangle_1 dt, \tag{3.14}$$

where $u \in W^{(m)}$ and

$$l_0 \in L^{p'_0}(R^+; A'_0),\ l_1 \in L^{p'_1}(R^+; A'_1),\ \frac{1}{p_i} + \frac{1}{p'_i} = 1,\ i=0,1. \tag{3.15}$$

Hereafter we denote by $\tilde{W}$ the closure of $\mathcal{D}(]0,+\infty[, X)$ in $W^{(m)}(X)$; therefore if $\tilde{W}^o$ denotes the polar set (2) of $\tilde{W}$ in $\big(W^{(m)}(X)\big)'$, one has

Lemma 3.9. – *Let L with* (3.14), (3.15); *then $L \in \tilde{W}^0$ if and only if we can write*

$$L(u) = \int_0^{+\infty} \langle \lambda, D^m u\rangle_1 dt + (-1)^{m+1} \int_0^{+\infty} \langle D^m\lambda, u\rangle_0 dt, \tag{3.16}$$

where $u \in W^{(m)}(X)$ and

$$\frac{\lambda}{c_1} \in L^{p'_1}(A'_1),\ \frac{D^m\lambda}{c_0} \in L^{p'_0}(A'_0), \tag{3.17}$$

(*that is* $\lambda \in W^{(m)}(p'_1, 1/c_1,\ A'_;\ p'_0, 1/c_0,\ A'_0) = \mathcal{W}$)

(2) that is the orthogonal set in the sense of the duality

PROOF. – We have

$$\tilde{W}^0 = \{L \in (W^{(m)}(X))';\ \langle L, u\rangle = 0,\ \forall u \in \mathcal{D}(R^+; X)\}$$

$$\tilde{W} = \{u \in W^{(m)}(X);\ \langle L, u\rangle = 0, \forall L \in \tilde{W}^0\}$$

Since $L \in (W^{(m)}(X))'$, is of the form (3.14) and like $c_i \in L^{p_i}_{loc}(R^+)$, $c_i l_i \in L^1_{loc}(R^+; A'_i)$, $(i = 0, 1)$, then $\lambda_i = c_i l_i$ defines a distribution on R^+ such that $\lambda_i \in L^{p'_i}_{1/c_i}(A'_i)$. Let $a \in X$, we define $u \in \mathcal{D}(R^+; X)$ by $u(t) = \phi(t)a$, where $\phi \in \mathcal{D}(R^+)$ and $\langle L, u\rangle = 0$ gives first (in the sense of scalar distributions) $\langle \lambda_0, a\rangle_0 + (-1)^{m+1} D^m \langle \lambda_1, a\rangle_1 = 0$, then $\langle \lambda_0 + (-1)^{m+1} D^m \lambda_1, a\rangle = 0,\ \forall a \in X$. Thus $\lambda_0(t) = (-1)^{m+1} D^m \lambda_1(t)$, a.e. in X'. Letting $\lambda = \lambda_1$, one get (3.16)-(3.17).

Conversely if $\lambda \in \mathcal{W}$, then λ is $(m-1)$-time differentiable with values in X' and from remark (3.2) the traces $\lambda^{(j)}(o) = \zeta_j$ for $0 \leq j \leq m-1$ exist with values in X'. Since L defined by $\lambda \in \mathcal{W}$ is continuous on $W^{(m)}(X)$, in order to evaluate $L(u)$ when $u \in W^{(m)}(X)$ it is enough from Lemma (3.7) to do that when $u \in W^{(m)}_{\tilde{K}}$. Using the injective mapping i_k and its transposed i^t_k we see that we can write

$$L(u) = \int_0^{+\infty} \langle \lambda, D^m u\rangle dt + (-1)^{m+1} \int_0^{+\infty} \langle D^m \lambda, u\rangle dt, \tag{3.18}$$

where $\langle .,.\rangle$ means the product in the duality $\langle X', X\rangle$.

Since $u \in W^{(m)}_{\tilde{K}}$ is $(m-1)$-time differentiable with values in X, we can integrate by part in (3.18) and we obtain

$$L(u) = \sum_{i=0}^{m-1} (-1)^{m-i} \langle \zeta_{m-i-1},\ D^i u(0)\rangle \tag{3.19}$$

which stays valid by density when $u \in W^{(m)}(X)$. Now $L(u) = 0$ imply $D^i u(0) = 0$, $0 \leq i \leq m-1$, therefore $u \in \overset{0}{W}{}^{(m)}(X) \equiv \tilde{W}$ (from the definition).

Now, from Lemma 3.5 we deduce

PROPOSITION 3.10. – *Assume* $c_1 \in \mathcal{H}(p_1)$, *then* $\mathcal{D}(R^+; X)$ *is dense in* $\overset{0}{W}{}^{(m)}_{c_0,c_1}$.

If we use for $u \in W^{(m)}(X)$, the same decomposition of the proof of Lemma 3.7: $u = v + w$ where $v \in \mathcal{D}(\overline{R}^+; X),\ w \in \overset{0}{W}{}^{(m)}(X)$, shows that from Proposition 3.10 and Lemma 3.5, we have

COROLLARY 3.11. – *Assume* $c_1 \in \mathcal{H}(p_1)$, *then* $\mathcal{D}(\overline{R}^+, X)$ *is dense in* $W^{(m)}_{c_0,c_1}$.

4. – Spaces of traces.

4.1 – *The space* $T_j^{(m)}(p_0, c_0, A_0\ ; p_1, c_1, A_1)$.

If we assume

$$t^j c_0 \in L^{p_0}(0,1), \tag{4.1}$$

$$\frac{t^{m-j-1}}{c_1} \in L^{p_1'}(0,1),\ \ 0 \le j \le m-1, \tag{4.2}$$

then we can define $D^j u(0) = u^{(j)}(0) = a_j$ *in* $A_0 + A_1 = Y$.

LEMMA 4.1. – *Assume* $c_0 \in L^{p_0}(0,1)$; *then* (4.2) *is a necessary and sufficient condition for* $\lim\limits_{t \longrightarrow +0} D^j u(t)$ *exists in* Y.

PROOF. – 1) *Necessity:*

Let $W^{(m)}_{(0,1)}(X)$ the restriction of $W^{(m)}(X)$ on $(0,1)$. In order to prove the result, it is enought to find a function $u \in W^{(m)}_{(0,1)}(X)$ such that $\left|u^{(j)}(t)\right|_Y \longrightarrow +\infty$ as $t \longrightarrow +0$.

Assume $t^{m-k-1} c_1^{-1} \in L^{p_1'}(0,1)$ for $0 \le k < j$. Therefore if $t^{m-j-1} c_1^{-1} \notin L^{p_1'}(0,1)$, there is $\chi \in L^{p_1}(0,1)$ such that $\int\limits_0^1 \chi(t) \dfrac{t^{m-j-1}}{c_1(t)}\, dt = +\infty$.

Consider for $t > 0$, $\phi(t) = \gamma_m \int\limits_t^1 (\tau - t)^{m-1} c_1^{-1}(\tau) \chi(\tau) d\tau$, $\gamma_m = (-1)^{m-1}/(m-1)!$. Obviously, we have from assumptions on c_0, c_1 and Hölder inequality: $c_0 \phi \in L^{p_0}(0,1)$ and like $D^m \phi = C_1^{-1} \chi$, if $a \ne 0$, $a \in X$, then we can define $u \in W^{(m)}_{(0,1)}(X)$ by $u(t) = \phi(t) a$. Now the wanted result for $\left|u^{(j)}(t)\right|_Y$ follows from

$$D^j \phi(t) = \gamma_{j,m} \int_t^1 (\tau - t)^{m-j-1} c_1^{-1}(\tau) \chi(\tau) d\tau \longrightarrow +\infty,\ \textit{if}\ t \longrightarrow +0,$$

where $\gamma_{j,m} = (-1)^{m-j-1}/(m-j-1)!$.

2) (4.2) *is a sufficient condition.*

If (4.2) holds, Hölder inequality gives

$$t^{m-j-1} D^m u \in L^1(0,1,Y). \tag{4.3}$$

Then if $m-j-1 = 0$, u is absolutely continuous on $[0,1]$ with values in Y.

If $\mu = m - j - 1 \geq 1$, we have $\left|D^{m-1}u(t)\right|_Y \leq \left|D^{m-1}u(1)\right|_Y + \int_t^1 |D^m u(\tau)d\tau|_Y$. We deduce, after integration by parts,

$$\int_\varepsilon^1 t^{\mu-1}\left|D^{m-1}u(t)\right|_Y dt \leq \left|D^{m-1}u(1)\right|_Y \left(\int_\varepsilon^1 t^{\mu-1}dt\right) + \frac{1}{\mu}\int_\varepsilon^1 (\tau^\mu - \varepsilon^\mu)|D^m u(\tau)|_Y d\tau$$

so that, since $\mu - 1 \geq 0$ and thanks to (4.3), one has: $t^{m-j-2}D^{m-1}u \in L^1(0,1;Y)$. Step by step, we obtain $D^{j+1}u \in L^1(0,1;Y)$ and $D^j u(0)$ exists.

REMARK 4.2. – One has that if $t^{m-j-1}c_1^{-1} \notin L^{p_1'}(0,1)$, for all $j \in \{0,1,...,m-1\}$, then $W^{(m)}_{c_0,c_1} = \overset{0}{W}{}^{(m)}_{c_0,c_1}$.

Thus we can set

DEFINITION 4.3. – *If* (4.1), (4.2) *hold, we denote by* $T_j^{(m)}(c_0,p_0,A_0;c_1,p_1,A_1) = T_j^{(m)}$ *the space spanned in* Y *by* $u^{(j)}(0)$ *when* u *spans the space* $W^{(m)}_{c_0,c_1}$.

Equipped with the norm

$$\|a\|_{T_j^m} = \inf_{u^{(j)}(0)=a} \|u\|_{W^m} \tag{4.4}$$

we obtain a Banach space. The spaces $T_j^{(m)}$ *are called spaces of traces.*

REMARK 4.4. – The definition of the norm of $T_j^{(m)}$ shows that the space can be interpreted like the quotient space $W^{(m)}/W_j^{(m)}$ where

$$W_j^{(m)} = \{u: \ u \in W^{(m)}, \ u^{(j)}(0) = 0\}.$$

The first property is given by the

LEMMA 4.5. – *Assume* $c_i \in L^{p_i}(0,T)$ *for all* $T > 0$. *Then* X *is dense in* $T_j^{(m)}$.

PROOF. – First, actually

$$X \subset T_j^{(m)}, \tag{4.5}$$

because if $a \neq 0$, $a \in X$ and $\phi \in \mathcal{D}(\overline{R^+})$, with $\phi^{(j)}(0) = 1$, therefore $u(t) = \phi(t)a$ belongs to $W^{(m)}$and $u^{(j)}(0) = a$.

Thus we have the algebrical inclusion and like $\|u\|_{W^m} \leq CMax(|a|_{A_0}, |a|_{A_1})$, (4.4) implies that the injective mapping in (4.5) is continuous.

Now, let $a \in T_j^{(m)}$, with $a = u^{(j)}(0)$ where $u \in W^{(m)}$. The claimed result is a consequence of Lemma 3.5 : Indeed, with the notations of the proof of Lemma 3.5, we know that $v_n = P_n u \longrightarrow u$ in $W^{(m)}$, which implies $v^{(j)}(0) \longrightarrow a$ in $T_j^{(m)}$.

Since $P_n \in \mathcal{L}(A_i, X)$ $(i = 0, 1)$, we have $v_n \in W^{(m)}(X)$, so that $v^{(j)}(0) \in X$ and Lemma 4.5 is proved.

4.2 – *Some other properties of spaces of traces.*

First, one has

PROPOSITION 4.6. – *Let $u \in W^{(m)}$ with $u^{(j)}(0) = a_j$ then*

$$(4.6) \quad |a_j|_{T_j^m} = \inf_u \, (M_0(u)^{1-\gamma_{j,m}} M_1(D^m u)^{\gamma_{j,m}}),$$

$$\textit{with } \gamma_{j,m} = \frac{j + 1/p_0}{m + 1/p_0 - 1/p_1}, \; 0 \leq j \leq m - 1.$$

PROOF. – Let $\lambda > 0$ and $f_\lambda(t) = f(\lambda t)$, then if $v_{(\lambda)}(t) = \lambda^{-j} u_\lambda(t)$ then $v_{(\lambda)} \in W^{(m)}_{c_{0\lambda}, c_{1\lambda}}$ and since $v^{(j)}_{(\lambda)}(0) = u^{(j)}(o)$ for all u, then the space of traces associated to that space is $T_j^{(m)}$ and from (4.4) we deduce

$$|a_j|_{T_j^m} \leq \inf_{\lambda > 0} (max[\lambda^{-j-1/p_0} M_0(u), \lambda^{m-j-1/p_1} M_1(D^m u)]);$$

Choosing λ such that $\lambda^{-j-1/p_0} M_0(u) = \lambda^{m-j-1/p_1} M_1(D^m u)$ we obtain (4.6).

COROLLARY 4.7. – *Assume that $a \in X$. Then there is a constant $k = k(c_0, p_0; c_1, p_1)$ such that*

$$(4.7) \qquad |a|_{T_j^m} \leq k |a|_{A_0}^{1-\gamma_{j,m}} |a|_{A_1}^{\gamma_{j,m}}.$$

PROOF. – Take $u(t) = \phi(t) a$ like in the proof of Lemma 4.5 and apply (4.6).

Next the main result is the *property of interpolation*: let $(B_0, B_1, \mathcal{B})$ be a family of three spaces with analogous properties like $(A_0, A_1, \mathcal{A})$. We shall say that

$$\pi \in \mathcal{L}(A_0; B_0) \cap \mathcal{L}(A_1; B_1)$$

if, assuming $X = A_0 \cap A_1$ is dense in A_i, $(i = 0, 1)$, π is a linear mapping of X into $B_0 \cap B_1$ which can be extended by continuity to an element of $\mathcal{L}(A_i; B_i)$, $(i = 0, 1)$. If,

$$(4.8) \qquad |\pi a|_{B_i} \leq \tilde{m}_i |a|_{A_i}, \quad a \in X, \quad i = 0, 1$$

then it is easily checked that $\pi \in \mathcal{L}(Y; B_0 + B_1)$. Indeed if $a \in A_0 + A_1$ then, $\pi a \in B_0 + B_1$ and $|\pi a|_{B_0 + B_1} \leq max(\tilde{m}_0, \tilde{m}_1) |a|_X$.

Let A and B be two Banach spaces, $A \subset Y$, $B \subset B_0 + B_1$. We say that (A, B) is an *interpolation couple* if, for every π satisfaying (4.8), one has $\pi \in \mathcal{L}(A; B)$.

THEOREM 4.8. – *Assume* c_i, $i = 0, 1$, *satisfies* (4.1), (4.2) *and that* π *satisfies* (4.8). *Then for* $1 \le p_i \le +\infty$, $i = 0, 1$, *and* $j \in \{0, 1, \ldots, m-1\}$, *one has*

$$\pi \in \mathcal{L}(T_j^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1); T_j^{(m)}(c_0, p_0, B_0, c_1 p_1 B_1) \tag{4.9}$$

the norm of π *in that space is bounded by* $\tilde{m}_0^{1-\gamma_{j,m}} \tilde{m}_1^{\gamma_{j,m}}$, *where* $\gamma_{j,m}$*is given by* (4.6).

PROOF. – Like $T_j^{(m)} \subset Y$, πa is defined if $a \in T_j^{(m)}$. Let $a \in T_j^{(m)}$ and $u^{\cdot} \in W^{(m)}/W_j^{(m)}$ such that $u^{(j)}(0) = a$ for all $u \in u^{\cdot}$, the mapping $a \longrightarrow u^{\cdot}$ being continuous. Since $\pi u \in W^{(m)}(c_0, p_0, B_0; c_1, p_1, B_1)$ then $\pi u^{(j)}(0)(= \pi a)$ belongs to $T_j^{(m)}(c_0, p_0, B_0; c_1, p_1, B_1)$ and is continuously depending on a in this space. Thus (4.9) holds true.

Moreover, using the index B for the norms related to the spaces built on the spaces denoted by B, (4.6) gives

$$|a| T_{jB}^{(m)} \le \inf_u M_{0B}(u)^{1-\gamma_{j,m}} M_{1B}(D^m u)^{\gamma_{j,m}} \le \inf_u \tilde{m}_0^{1-\gamma_{j,m}} \tilde{m}_1^{\gamma_{j,m}} M_0(u)^{1-\gamma_{j,m}} M_1(D^m u)^{\gamma_{j,m}};$$

that is $|a|_{T_{jB}^{(m)}} \le \tilde{m}_0^{1-\gamma_{j,m}} \tilde{m}_1^{\gamma_{j,m}} |a|_{T_j^{(m)}}$.

Now we denote by $T(c_0, p_0, A_0; c_1, p_1, A_1) = T_0^{(1)}(c_0, p_0, A_0; c_1, p_1, A_1)$ and we assume

$$c_i \in \mathcal{H}(p_i); \quad i = 0, 1 \tag{4.10}$$

then, one has

THEOREM 4.9. – *Assume that* (4.1), (4.2) *and* (4.7) *hold and* $1 \le p_i \le +\infty$, $i = 0, 1$, *therefore*

$$T_j^{(m)}(c_0, p_0, A_0; c_1, p_1 A_1) = T(t^j c_0, p_0, A_0; t^{j-m+1} c_1, p_1, A_1)$$

with equivalent norms.

PROOF OF THEOREM 4.9. – The proof is an obvious consequence of the following lemmas

LEMMA 4.10. – *We assume* (4.1), (4.2) *hold for* $j = m - 1$ *and that* (4.7) *holds with* $m - 2 \ge 0$. *Then*

$$T_{(m-1)}^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1) = T_{(m-2)}^{((m-1)}(t c_0, p_0, A_0; c_1, p_1, A_1) \tag{4.11}$$

with equivalent norms

LEMMA 4.11. – *Assumptions are those of theorem* 4.9 *and* $m - 1 > j$. *Then*

$$T_j^{(m)}(c_0, p_0, A_0; c_1, p_1 A_1) = T_j^{(m-1)}(c_0, p_0, A_0; t^{-1} c_1, p_1, A_1) \tag{4.12}$$

with equivalent norms.

For the proof of both lemmas we note

REMARK 4.12. – From the proof of (4.2) (sufficient condition) it results that if $l > 0$, is sufficiently large, then $\int_0^t \tau^l u^{(j)}(\tau)d\tau < +\infty,\ \forall t > 0, \forall j \in \{0, 1 \dots m-1\}$.

In what follows we set γ to be some constant which can be distinct from a formula to another.

4.2.1 – Proof of Lemma 4.10.

a) In order to prove

$$T^{(m)}_{m-1}(c_0, p_0 A_0; c_1, p_1, A_1) \subset T^{(m-1)}_{m-2}(tc_0, p_0, A_0; c_1, p_1, A_1)$$

with continuous injective mapping, consider: $\mathcal{F}:\ u(t) \longrightarrow v(t) = (m+l)\mathcal{H}_{l+1}(u')(t)$, for l sufficiently large, which makes a sense in Y. We have to show that $\mathcal{F}$ is linear continuous from $W^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1)$ into $W^{(m-1)}(tc_0, p_0, A_0; c_1, p_1, A_1)$ with $v^{(m-2)}(0) = u^{(m-1)}(0)$.

From (4.10) and Proposition 2.9-i), since $\mathcal{H}_{l+1}(u')(t) = \mathcal{H}'_l(u)(t) = \dfrac{u(t)}{t} - \dfrac{l+1}{t}\mathcal{H}_l(u)(t) \in L^{p_0}_{tc_0}(A_0)$, we obtain $|v|_{L^{p_0}_{tc_0}(A_0)} \leq \gamma M_0(u)$.

From Remark 2.10, one has in Y, $\mathcal{H}^{(j)}_l(u)(t) = \mathcal{H}_{l+j}(u^{(j)})(t)$ for $j \in \{0, 1, \dots, m\}$, which implies $\mathcal{H}^{(m-1)}(u)(0) = \dfrac{1}{l+m}u^{(m-1)}(0)$, so that $v^{(m-2)}(0) = u^{(m-1)}(0)$ and from (4.10) and Proposition 2.9-i) then $\mathcal{H}^{(m)}_l(u) \in L^{p_1}_{c_1}(A_1)$; we obtain $v^{(m-1)} \in L^{p_1}_{c_1}(A_1)$ with $\left|v^{(m-1)}\right|_{L^{p_1}_{c_1}(A_1)} \leq \gamma M_1(u^{(m)})$.

b) To prove the reciprocal inclusion, we consider now, always for l sufficiently large, the mapping $\mathcal{F}_1 : u \longrightarrow v = \dfrac{l+m-1}{m-1}w$, with $w(t) = \dfrac{1}{t^l}\int_0^t \tau^l u(\tau)d\tau$, which is linear and continuous from $W^{(m-1)}(tc_0, p_0, A_0; c_1, p_1, A_1)$ into $W^{(m)}(c_0, p_0, A_0; c_1, p_1 A_1)$. Indeed, from Proposition 2.9-i) it is obvious that $M_0(v) \leq \gamma |u|_{L^{p_0}_{tc_0}(A_0)}$.

Then observing that $w'(t) = u(t) - l\mathcal{H}_l(u)(t)$ in Y, we deduce that $w^{(j)} = u^{(j-1)} - l\mathcal{H}_{l+j-1}(u^{(j-1)})$ with values in Y for $j \in \{1, 2, \dots, m\}$.

Therefore for $j = m-1$, we obtain first that: $W^{(m-1)}(0) = \dfrac{m-1}{l+m-1}u^{(m-2)}(0)$, as $t \longrightarrow 0$, then $v^{(m-1)}(0) = u^{(m-2)}(0)$ and from (4.10) and Proposition 2.9- i) $M_1(v^{(m)}) \leq \gamma M_1(u^{(m-1)})$.

4.2.2 – Proof of Lemma 4.11.

a) In order to prove that $T_j^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1) \subset T_j^{(m-1)}(c_0, p_0, A_0; t^{-1}c_1, p_1, A_1)$ with injective mapping, we consider here the mapping $\tilde{\mathcal{F}}$ defined by $u \longrightarrow v$ such that, for l large $v(t) = \lambda_{l,m,j} w(t)$, where $w(t) = u(t) - l\mathcal{H}_{l-m}(u)(t)$ and we have to check that for l large $\tilde{\mathcal{F}}$ is continuous from $W^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1)$ into $W^{(m-1)}(c_0, p_0, A_0; t^{-1}c_1, p_1, A_1)$ with $v^{(j)}(0) = u^{(j)}(0)$.

Actually $c_0 \in \mathcal{H}(p_0)$ implies $w \in L^{p_0}_{c_0}(A_0)$ with $M_0(w) \le \gamma M_0(u)$, then as in the proof of the part b) of Lemma 4.10, we have $w^{(j)}(t) = u^{(j)}(t) - l\mathcal{H}_{l-m+j}(u^{(j)})(t)$ (that implies $w^{(j)}(0) = \dfrac{j+1-m}{l-m+j+1} u^{(j)}(0)$ so that $v^{(j)}(0) = u^{(j)}(0)$ from a choice of $\lambda_{l,m,j}$ and after tacking $j = m-1$ and integrating by parts (3), one obtain $w^{(m-1)}(t) = t\mathcal{H}_l(u^{(m)})(t)$ which obviously gives

$$\left|v^{(m-1)}\right|_{L^{p_1}_{t^{-1}c_1}(A_1)} \le \gamma M_1(u^{(m)}).$$

b) In order to prove the reciprocal inclusion, we consider the mapping:

$$\tilde{\mathcal{F}}_1 \ : \ u \longrightarrow v = (l+j+1)w,$$

with $w(t) = \mathcal{H}_l(u)(t)$, for l large.

Next we obtain, follows that: $w \in L^{p_0}_{c_0}(A_0)$, $M_0(v) \le \gamma M_0(u)$, and also $w^{(j)}(t) = \mathcal{H}_{l+j}(u^{(j)})(t)$ in Y, as before. Thus $w^{(j)}(0) = \dfrac{u^{(j)}(0)}{l+j+1}$ and $v^{(j)}(0) = u^{(j)}(0)$ follows. Now it remains to prove that $w^{(m)} \in L^{p_1}_{c_1}(A_1)$.

Since one has $w^{(m-1)}(t) = \mathcal{H}_{l+m-1}(u^{(m-1)})(t)$ in Y, we deduce $w^{(m)}(t) = \frac{u^{(m-1)}(t)}{t} - \frac{l}{t^{l+m}} \cdot \int\limits_0^t \tau^{l+l-1} u^{(m-1)}(\tau) d\tau$. Like $u^{(m-1)} \in L^{p_1}_{t^{-1}c_1}(A_1)$ and since $t^{-1}c_1 \in \mathcal{H}(p_1)$, we arrive to $M_1(v^{(m)}) \le \left|u^{(m-1)}\right|_{L^{p_1}_{t^{-1}c_1}(A_1)}$. Thus the map $\mathcal{F}_1$ is linear continuous from $W^{(m-1)}(c_0, p_0, A_0; t^{-1}c_1, p_1, A_1)$ into $W^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1)$ and the Lemma is proved.

4.3 – *The space* $T(c_0, p_0, A_0; c_1, p_1, A_1)$

Theorem 4.9 reduces the study of $T_j^{(m)}$, $0 \le j \le m-1$ to $T_0^{(1)} = T(c_0, p_0, A_0; c_1, p_1, A_1)$, then first, the condition on the weights to have existence of the trace $u(0)$ are

$$\forall T \in \,]0, +\infty[, \ c_0 \in L^{p_0}(0,T), \quad c_1^{-1} \in L^{p_1'}(0,T), \tag{4.13}$$

(3) It holds because l is large.

next, recalling Proposition 4.5, we note that for $u \in W^{(1)}$ with $u(0) = a$, one has:

$$|a|_T \leq M_0(u)^{1-\theta} M_1(Du)^\theta, \ \textit{where } \theta = \gamma_{0,1} = \frac{1/p_0}{1 + 1/p_0 - 1/p_1}. \tag{4.14}$$

Now, we want to prove a result of "symmetry" adapted from the case where $c_i(t) = t^{\alpha_i}$ studied in [17]. Following [17] we need the

LEMMA 4.13. – *Let B be a Banach space. Assuming $c_0 \in \mathcal{H}(p_0)$ and $u \in W^{(1)}(B)$ we have in B equipped with the strong topology:*

$$u(0) = \lim_{h \longrightarrow \infty} \int_0^h \frac{ds}{s^2} \int_0^s [u(s) - u(\sigma)] d\sigma. \tag{4.15}$$

PROOF. – We use Ill'in identity [13]

$$u(0) = \mathcal{H}(u)(h) \ - \int_0^h \frac{ds}{s^2} \int_0^s [u(s) - u(\sigma)] d\sigma \tag{4.16}$$

and it is sufficient to prove that

$$\mathcal{H}(u)(h) \longrightarrow 0 \textit{ in } B \textit{ strongly as } h \longrightarrow +\infty.$$

Like $u \in L^{p_0}_{c_0}(B)$, applying the Hölder inequality, one obtains

$$|\mathcal{H}(u)(h)|_B \leq |u|_{L^{p_0}_{c_0}(B)} C(h), \ C(h) = \frac{1}{h} \left[\int_0^h \frac{ds}{c_0(s)^{p'_0}} \right]^{1/p'_0}$$

and we have to prove that $C(h) \longrightarrow 0$ as $h \longrightarrow +\infty$.

Since $c_0 \in \mathcal{H}(p_0)$,

$$\left[\int_0^h \frac{ds}{c_0(s)^{p'_0}} \right]^{1/p'_0} \leq K \left[\int_h^{+\infty} \left(\frac{c_0(s)}{s} \right)^{p_0} \right]^{-1/p_0},$$

we deduce that

$$C(h) \leq K \left[\int_h^{+\infty} \left(\frac{h}{s} \right)^{p_0} c_0(s)^{p_0} ds \right]^{-1/p_0} \leq K \left[\int_h^{+\infty} c_0(s)^{p_0} ds \right]^{-1/p_0} = 0$$

because we have only $c_0 \in L^{p_0}_{loc}(\overline{R}^+)$.

Setting $c_i^*(t) = c_i\left(\frac{1}{t}\right)$, one has

THEOREM 4.14. – *Assuming* $c_i \in \mathcal{H}(p_i)$, $1<p_1 \leq +\infty$, $(i=0,1)$, *then*

$$T(c_0,p_0,A_0;c_1,p_1,A_1) = T(t^{1-2/p_1}c_1^*,p_1,A_1;t^{1-2/p_0}c_0^*,p_0,A_0) \tag{4.17}$$

with equivalent norms.

PROOF. – Let a be an element of $T(c_0,p_0,A_0;c_1,p_1,A_1) = T$ and let u be an arbitrary element of $W^{(1)}$ satisfying $u(0)=a$. We introduce

$$v(t) = -\int_0^{1/t} \frac{ds}{s^2}\int_0^s [u(s)-u(\sigma)]d\sigma$$

and we set $v^*(t) = v\left(\frac{1}{t}\right)$. An integration by parts gives

$$v^*(t) = -\int_0^t \left(\frac{1}{s^2}\int_0^s \tau u'(\tau)d\tau\right)ds = -\int_0^t \mathcal{H}_1(u')(s)ds.$$

We know, first that $\mathcal{H}_1(u') \in L^{p_1}_{c_1t^{-1}}(A_1)$, if $c_1t^{-1} \in \mathcal{H}_1(p_1)$ (that is true there) and next, since $v^*(t) = -t\mathcal{H}\circ\mathcal{H}_1(u')(t)$, one has

$$t^{-1}c_1v^* \in L^{p_1}(A_1) \ \simeq t^{1-2/p_1}c_1^*v \in L^{p_1}(A_1). \tag{4.18}$$

On the other hand, from the definition of $v^*(t)$ we obtain $v^*(t) = \frac{1}{t}(\mathcal{H}(u)(t)-u(t))$, so that

$$tc_0v^* \in L^{p_0}(A_0) \ \simeq \ t^{1-2/p_0}c_0^*v \in L^{p_0}(A_0) \tag{4.19}$$

Fom (4.18) and (4.19), we deduce

$$v \in W^{(1)}(t^{1-2/p_1}c_1^*,p_1,A_1;t^{1-2/p_0}c_0^*,p_0,A_0) = \tilde{W}^{(1)}$$

the mapping $u \longrightarrow v$ being continuous from $W^{(1)}$ to $\tilde{W}^{(1)}$.

Then

$$v(0) \in T(t^{1-2/p_1}c_1^*,p_1,A_1;t^{1-2/p_0}c_0^*,p_0,A_0) = \tilde{T}$$

and we have

$$|v(0)|_{\tilde{T}} \leq \gamma_1|a|_T, (\gamma_1 = constant).$$

From lemma 4.13, with $B=Y$ and like $u \in W^{(1)}$ implies $u \in W^{(1)}(Y)$, we see that $v(t) \longrightarrow a$, as $t \longrightarrow 0$ in the strong topology of Y.Therefore $v(0)=a$ and $|a|_{\tilde{T}} \leq \gamma_1|a|_T$. That is

$$T(c_0,p_0,A_0;c_1,p_1,A_1) \subset T(t^{1-2/p_1}c_1^*,p_1,A_1;t^{1-2/p_0}c_0^*,p_0,A_0),$$

the identity mapping being continuous.

Now exchanging p_0 and p_1, c_0 and $t^{1-2/p_1}c*_1$, c_1 and $t^{1-2/p_0}c*_0$, A_0 and A_1, one has the inverse inclusion, hence Theorem 4.14 follows.

4.4 – *Duality.*

In this subsection:

– Assumptions on (A_0, A_1) are those of Subsections 3.1 and 3.4.
– Assumptions on the weights are (4.1), (4.2) and $c_i \in \mathcal{H}(p_i)$.

If we set $\gamma_j : u^{\cdot} \longrightarrow \gamma_j(u^{\cdot}) = u^{(j)}(0)$, γ_j is an isomorphism from $W^{(m)}/W_j^{(m)}$ onto $T_j^{(m)}$ and its transposed γ_j^t is an isomorphism from $\left(T_j^{(m)}\right)'$ onto W_j^0 the polar set of $W_j^{(m)}$ in $\mathcal{W}$.

Like $W_j^0 = \{L \in \mathcal{W},\ \langle L, u\rangle = 0\ \forall u \in W_j^{(m)}\}$, with analogous arguments as in Subsection 3.4, we obtain

LEMMA 4.15. – *Let $L \in \mathcal{W}$. Then $L \in W_j^0$ if and only if we can find $\lambda \in \mathcal{W} = W^{(m)}(c_1^{-1}, p_1', A_1'; c_0^{-1}, p_0', A_0')$ such that*

$$L(u) = \int_0^{+\infty} \langle \lambda, D^m u\rangle_1 dt + (-1)^{m+1} \int_0^{+\infty} \langle D^m \lambda, u\rangle_0 dt \tag{4.20}$$

where $u \in W^{(m)}$ and

$$D^i \lambda(0) = 0 \textit{ for } i \neq m-j-1,\ 0 \leq j \leq m-1. \tag{4.21}$$

REMARK 4.16. – Note that the trace $D^{m-j-1}\lambda(0)$ exists and belongs to $T_{m-j-1}^{(m)}(c_1^{-1}, p_1', A_1'; c_0^{-1}, p_0', A_0') = \tilde{T}_{m-j-1}^{(m)}$ for $\lambda \in \mathcal{W}$. Indeed if we apply the analogous of [(4.1) , (4.2)] to $\mathcal{W}$, we find again globally [(4.1), (4,2)] that is (4.1) and (4.2) are exchanged.

Now from (4.19), (4.20), according to the density of $W_{\tilde{K}}^{(m)}(X)$ in $W^{(m)}$, integration by parts gives

$$L(u) = (-1)^{m-j} \langle D^{m-j-1}\lambda(0), D^j u(0)\rangle. \tag{4.22}$$

On the other hand if $b \in \tilde{T}_{m-j-1}^{(m)}$ there is a function $l_b \in \mathcal{W}$ with $D^{m-j-1} l_b(0) = b$ and using a straightforward proceeding of Babitch (see [15] for instance) we can define $\lambda_b(t) = \sum_{k=1}^{m} \gamma_k l_b(kt)$, where the γ_k are defined by $\sum_{k=1}^{m} k^i \gamma_k = 0, i \neq m-j-1, 0 \leq i \leq m-1$, and $\sum_{k=1}^{m} k^{m-j-1}\gamma_k = 1$. Then $\lambda_b \in \mathcal{W}$ and satisfies (4.20), (4.21) and $D^{m-j-1}\lambda_b(0) = b$, the mapping $b \longrightarrow \lambda_b$ being continuous from $\tilde{T}_{m-j-1}^{(m)} \longrightarrow \mathcal{W}$. Therefore following the outline of [15] we obtain the

THEOREM 4.17. – *We assume that hypothesis of Subsections* 3.1 *and* 3.4 *hold true, that c_i satisfies* [(4.1), (4.2)] *and belongs to $\mathcal{H}(p_i)$ with $1 < p_i < +\infty, i = 0, 1$.*

Then

$$[T_j^{(m)}(c_0, p_0, A_0; c_1, p_1, A_1)]' = T_{m-j-1}^{(m)}(c_1^{-1}, p_1', A_1'; c_0^{-1}, p_0', A_0'). \tag{4.23}$$

If $u \in W^{(m)},\ \lambda \in \mathcal{W}$, *one has*

$$\langle \lambda, u\rangle = \sum_{j=o}^{m-1} (-1)^{m-j} \langle D^{m-j-1}\lambda(0), D^j u(0)\rangle_j, \tag{4.24}$$

where $\langle .,. \rangle_j$ *means here the scalar product between* $\tilde{T}_{m-j-1}^{(m)} = \big[T_j^{(m)}\big]'$ *and* $T_j^{(m)}$.

5. – Intermediate derivatives and spaces of traces.

The section is devoted to a particular case of the following problem:

"Find an intermediate space V_j *between* A_0 *and* A_1 *such that for* $1 \le j \le m-1$, *the mapping* $u \longrightarrow D^j u$ *is continuous from* $W^{(m)}$ *into an intermediate space* $L_{\omega_j}^{p_j}(V_j)$ *between* $L_{\omega_0}^{p_0}(A_0)$ *and* $L_{\omega_1}^{p_1}(A_1)$."

Some results of that type for spaces without weights are given in [4], [9], [17], (see also the bilbiography therein) and for hilbert or Banach weighted spaces in [2], [3], [5].

Here we want to extend a result of [17] to the weighted space $W^{(m)}$ using properties of Hardy-Littlewood functions and the class $\mathcal{A}(p)$ of B. Muckenhoupt [21] (see also [5]).

5.1 – *The Class* $\mathcal{A}(p)$.

DEFINITION 5.1. – *Let J be a fixed interval of R. We shall said that a weight* $\omega \in \mathcal{A}(p)$, $1 < p < +\infty$, *if and only if*

$$\left(\int_I [\omega(t)]dt\right)\left(\int_I [\omega(t)]^{-p'+1}dt\right)^{p-1} \le K^p |I|^p,\ \frac{1}{p} + \frac{1}{p'} = 1, \tag{5.1}$$

where I is any subinterval of J, $|I|$ *denotes the length of I, and K is a constant independent of I.*

REMARK 5.2. – Set $\omega = c^p$. Then (5.1) becomes symmetrical:

$$\left[\frac{1}{|I|}\int_I c(t)^p dt\right]^{1/p} \left[\frac{1}{|I|}\int_I c^{-p'}(t)dt\right]^{1/p'} \le K. \tag{5.2}$$

We still shall write $c \in \mathcal{A}(p)$, with such a representation of ω, if (5.2) holds.

Some properties for $\omega \in \mathcal{A}(p)$ are summarized in the

LEMMA 5.3. – i) *If* $\omega \in \mathcal{A}(p)$ *with the constant* K, *then* $\omega \in \mathcal{A}(r)$, $r \geq p$ *with the same constant* K.

ii) *If* $\omega \in \mathcal{A}(p)$ *with the constant* K, *then* $\omega^{-p'+1} \in \mathcal{A}(p')$ *with the constant* $K^{p'-1}, \dfrac{1}{p} + \dfrac{1}{p'} = 1$.

iii) *If* $1 < p < +\infty$ *and* $\omega \in \mathcal{A}(p)$ *with the constant* K, *then there is* r, $1 < r < p$, *and a constant* $\tilde{K}(p, K)$, *such that* $\omega \in \mathcal{A}(r)$ *with the constant* $\tilde{K}(p, K)$.

For the proof we send to [21].

Now, taking $J = R^+$, we claim

PROPOSITION 5.4. – *If* $\omega \in \mathcal{A}(p)$, $1 < p < +\infty$, *then* $\omega \in \mathcal{H}(p)$.

PROOF. – Let $\omega \in \mathcal{A}(p)$, $1 < p < +\infty$, with the constant K. From Lemma 5.3 part iii), there is $r \in]1, p[$, such that $\omega \in \mathcal{A}(r)$, with a constant $\tilde{K}$ and one has (a fortiori from (5.1) with $p = r$, $I = (0, s)$)

$$\int_t^s \omega(\tau)d\tau \left[\leq \int_0^t \omega^{-r'+1}(\tau)d\tau \right]^{r-1} \leq \tilde{K}^r s^r \text{ for } s > t > 0. \tag{5.3}$$

By Hölder inequality $t^r \leq \int_0^t \omega(\tau)d\tau \Big[\int_0^t \omega(\tau)^{-r'+1} d\tau \Big]^{r-1}$, so that from (5.3) we get

$$s^{-p-1} \int_t^s \omega(\tau)d\tau \leq \frac{\tilde{K}^r}{t^r} s^{r-p-1} \int_0^t \omega(\tau)d\tau. \tag{5.4}$$

Because $\int_t^{+\infty} s^{r-p-1} ds = \dfrac{t^{r-p}}{p-r}$, we can integrate (5.4) on $(0, +\infty)$ and one has

$$\int_t^{+\infty} \left[s^{-p-1} \int_t^s \omega(\tau)d\tau \right] ds \leq \frac{\tilde{K}^r t^{-p}}{p-r} \int_0^t \omega(\tau)d\tau. \tag{5.5}$$

Exchanging the order of integrations, the left hand- side of (5.5) reads

$$\int_t^{+\infty} [\omega(\tau) \int_\tau^{+\infty} s^{-p-1} ds] d\tau = \frac{1}{p} \int_t^{+\infty} \frac{\omega(\tau)}{\tau^p} d\tau$$

then

$$\int_t^{+\infty} \frac{\omega(\tau)}{\tau^p} d\tau \leq \frac{1}{t^p} \frac{p\tilde{K}^r}{p-r} \int_0^t \omega(\tau)d\tau. \tag{5.6}$$

Like $\omega \in \mathcal{A}(p)$, from (5.6) one deduces

$$\left[\int_0^t \omega(\tau)^{-p'+1} d\tau\right]^{p-1} \int_t^{+\infty} \frac{\omega(\tau)}{\tau^p} d\tau \leq \frac{p\tilde{K}^r K^p}{p-r}, \tag{5.7}$$

that is $\omega \in \mathcal{H}(p)$.

REMARK 5.5. – Obviously a non increasing weight belongs to $\mathcal{H}(p)$. But if $c(t) = e^{-t},\ t \geq 0$ then $c \notin \mathcal{A}(p)$. Indeed the left side of (5.2) reads

$$\frac{e^t}{t}\left(\frac{1-e^{-pt}}{p}\right)^{1/p}\left(\frac{1-e^{-p't}}{p'}\right)^{1/p'}$$

which is unbounded as $t \longrightarrow +\infty$.

Now we introduce the maximal Hardy- Littlewood function f^*, defined for f with values in some Banach space B by

$$f^*(t) = \sup_{\tau \neq t} \frac{1}{\tau - t} \int_t^\tau |f(\sigma)|_B d\sigma. \tag{5.8}$$

Therefore

THEOREM 5.6. – *The mapping $f \longrightarrow f^*$ is continuous from $L^p_\omega(B)$ into itself, for $1 < p \leq +\infty$, if and only if, $\omega \in \mathcal{A}(p)$.*

PROOF. – The theorem was proved in [21] by B. Muckenhoupt for f defined on R (resp. R^N) with values in R or C. The case where B is a Banach space is obtained modulo a vectorial extension of inequalities of Calderón-Zygmund [8] and of the Marcinkiewicz theorem by the methods of J. Schwartz [24].

5.2 – *Intermediate derivatives*

We set $W^{(m)}_{\infty,\infty} = W^{(m)}(1,\infty,A_0;1,\infty,A_1)$. We know that $\Phi \in W^{(m)}_{\infty,\infty}$ is $(m-1)$-time continuously differentiable with values in Y so that $D^j\Phi(0)$ is well defined. Set $\mathcal{T}^m_j = T^{(m)}_j(1,\infty,A_0;1,\infty,A_1)$, then (4.6) becomes

$$\left|D^j\Phi(0)\right|_{\mathcal{T}^m_j} \leq M_0(\Phi)^{1-j/m} M_1(D^m\Phi)^{j/m}. \tag{5.9}$$

THEOREM 5.7. – *Assume $1 < p_i \leq +\infty$ and $\omega_i = c_i^{p_i} \in \mathcal{A}(p_i), i = 0,1$. Then for all integer j, such that $0 < j \leq m-1$, the mapping $u \longrightarrow D^j u$ is continuous from*

$W^{(m)}$ into $L^{p_j}_{\omega_j}[\mathcal{T}_j^{(m)}] = Z_j$, where

$$\omega_j = c_j^{p_j},\ c_j = c_0^{1-j/m} c_1^{j/m},\ \frac{1}{p_j} = \frac{1-j/m}{p_0} + \frac{j/m}{p_1}. \tag{5.10}$$

Moreover, there is a constant $k_j > 0$ such that, for all $u \in W^{(m)}$, one has the logarithmic convexity inequality

$$\left|D^j u\right|_{Z_j} \leq k_j M_0(u)^{1-j/m} M_1^{j/m}(D^m u) \tag{5.11}$$

PROOF (see also [5]). – Since $\mathcal{D}(\overline{R}^+; X)$ is dense in $W^{(m)}$ it is sufficient to prove the theorem when $u \in \mathcal{D}(\overline{R}^+; X)$. Then, we can associate to u the function v defined on $R^+ \times R^+$ by

$$v(x,t) = \frac{1}{t} \int_x^{x+t} u(y) dy \tag{5.12}$$

and note that for $0 \leq j \leq m$:

$$D_t^j v(x,t) = \frac{1}{t^{j+1}} \int_x^{x+t} (y-x)^j D^j u(y) dy. \tag{5.13}$$

Therefore

$$|v(x,t)|_{A_0} \leq v_0^*(x) = \sup_{t \geq 0} \frac{1}{t} \int_x^{x+t} |u(y)|_{A_0} dy,$$

$$\left|D_t^m v(x,t)\right|_{A_1} \leq v_m^*(x) = \sup_{t \geq 0} \frac{1}{t} \int_x^{x+t} |D^m u(y)|_{A_1} dy.$$

Thus, from Theorem 5.6, one has $v_0^* \in L^{p_0}_{c_0}(A_0)$, $v_m^* \in L^{p_1}_{c_1}(A_1)$ and

$$M_0(v_0^*) \leq k_0 M_0(u), \quad M_1(v_m^*) \leq k_1 M_1(D^m u). \tag{5.14}$$

One deduces that, for almost every x, the function $t \longrightarrow v(x,t)$ belongs to $W^{(m)}_{\infty,\infty}$, so that $D_t^j v(x,0) \in \mathcal{T}_j^{(m)}$.

Now from (5.13) we obtain $D_t^j v(x,0) = \dfrac{1}{j+1} D^j u(x)$ and from (5.9) we have "à fortiori"

$$\left|D_t^j v(x,0)\right|_{\mathcal{T}_j^m} \leq v_0^*(x)^{1-j/m} v_m^*(x)^{j/m},$$

so that from (5.14) and Hölder's inequality, we obtain the Theorem 5.7 for $u \in \mathcal{D}\overline{R}^+; X)$ with the choice of p_j, c_j, given by (5.10). The proof follows by density.

REMARK 5.8. – i) Since $c_i(t) = t^{\alpha_i} \in \mathcal{A}(p)$, $\alpha_i + \dfrac{1}{p_i} \in]0,1[$, $i = 0,1$, theorem 5.7 holds true in the framework of [16], [19].

ii) In [2], [3], a result of intermediate derivatives is obtained in the framework of weighted Hilbert spaces with non increasing weights which belongs to $\mathcal{H}(p)$ and are not in $\mathcal{A}(p)$. Then the assumption on the weights, in the previous theorem, is only sufficient and depends on the method used. Nevertheless, it seems that taking weights belonging to $\mathcal{A}(p_i)$ should be a good basic assumption to justify all the previous results.

iii) A set of conditions, both on the spaces and the weights, is given in [5] in the framework of complex interpolation, using the operator $D^{i\eta}$, $\eta > 0$, defined in the introduction. A more complete study will be done in a forthcoming paper [6].

REFERENCES

[1] M. ARTOLA, *Perturbations des équations d'évolution. Application à des problèmes de retard*, Ann. Ecole Norm. Sup. Paris (1969).

[2] M. ARTOLA, *Dérivées intermédiaires dans les espaces de Hilbert pondérés*, C.R.A.S. Paris, **264** (1967), 693-695.

[3] M. ARTOLA, *Dérivées intermédiaires dans les espaces de Hilbert pondérés. Application au comportement à l'infini des solutions d'équations d'évolution*. Rend. Sem. Padova, **43** (1970), 177-202.

[4] M. ARTOLA, *Sur un théorème d'interpolation*, Math. Anal. and Applications, **41**, n. 1 (1973), 148-163.

[5] M. ARTOLA, *Sur un théorème d'interpolation dans des espaces de Banach pondérés. Articles dédiés à Jacques-Louis Lions*. Gauthier-Villars, (1998) 35-50.

[6] M. ARTOLA, *On complex order derivatives in some weighted Banach spaces* (to appear).

[7] A. P. CALDERÓN, *Intermediate spaces and interpolation; the complex method*, Studia. Math., **246** (1964), 85-139.

[8] A. P. CALDERÓN - A. ZYGMUND, *On the existence of certain singular integrals*, Acta. Math., **88** (1952), 85-133.

[9] E. GAGLIARDO, *Ulteriori proprietà di alguna classi di funzioni in più variabili*, Ric. Math., **8** (1959), 24-51.

[10] P. GRISVARD, *Commutativité de deux foncteurs d'interpolation et applications*, Journal de Mathématiques Pures et Appliquées, **45** (1966) I) 143-206, II) 207-290.

[11] P. GRISVARD, *Identités entre espaces de traces*. Math. Scand. **13** (1963) 70-74.

[12] G. H. HARDY - J. E. LITTLEWOOD - G. POLYÀ, *Inequalities*, Cambridge press, London (1934).

[13] V. P. ILL'IN, *Sur un théorème de Hardy-Littlewood*, Trudy Mat. Inst. Steklova, **53** (1959) 128-144.

[14] A. KUFNER - L. MALINGRADA - L. E. PERSSON, *The Hardy Inequality. About its history and some related results*. Pilsen (2007).

[15] J. L. LIONS, *Une construction d'espaces d'interpolation*, C.R.A.S. Paris, **251** (1961),1853-1855.

[16] J. L. LIONS, *Sur les espaces d'interpolation. Dualité*, Math. Scand., **9** (1961), 147-177.

[17] J. L. LIONS, *Dérivées intermédiaires et espaces intermédiaires*, C.R.A.S. Paris, **56** (1963), 4343-4345.

[18] J. L. LIONS, *Properties of some interpolation spaces*, Journal of Mathematics and Mechanics, **11**, n. 6 (1962), 969-977.

[19] J. L. LIONS - J. PEETRE, *Sur une classe d'espaces d'interpolation*, Pub. Math. de l'IHES, **19** (1964), 5-68.
[20] B. MUCKENHOUPT, *Hardy's inequalities with weights*, Studia Math., **44** (1972), 207-226.
[21] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal functions*, Trans. Am. Math. Soc., **165** (1972), 207-226.
[22] J. PEETRE, *A new approach in interpolation spaces*, Studia Math., **XXXIV** (1970), 23-41.
[23] E. T. POULSEN, *Boundary values in function spaces*, Math. Scand., **X** (1962), 45-52.
[24] J. SCHWARTZ, *A remark on inequalities of Caldero'n-Zygmund type for vectorial valued functions*, Comm. Pure Applied Maths., **14** (1961), 785-799.
[25] L. SCHWARTZ, *Distributions à valeurs vectorielles, I, II.* Ann. de L'inst. Fourier, **7** (1957), 8 (1958).
[26] G. TALENTI, *Osservationi sopra una classe di disguaglianze*, Rend. Sem. Mat. e Fis. Milano, **39** (1969), 171-185.
[27] L. TARTAR, *An introduction to Sobolev spaces and interpolation spaces*. Lectures notes of the Unione Matematica Italiana. **6**, Springer, Berlin (2007).
[28] G. A. TOMASELLI, *A class of Inequalities*, Boll. Un.Mat.Ital., **2** (1969), 622-631.

Michel Artola Emeritus Professor
27, Avenue du Ribeyrot, 33610 CESTAS, France
E-mail: michel.artola@neuf.fr

Received April 23, 2011 and in revised form September 5, 2011