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A Peculiar Liapunov Functional for Ternary Reaction-Diffusion Dynamical Systems

Salvatore Rionero

To the memory of Giovanni Prodi.

Abstract. – A Liapunov functional \( W \), depending together with the temporal derivative \( W \) along the solutions - on the eigenvalues via the system coefficients, is found. This functional is “peculiar” in the sense that \( W \) is positive definite and simultaneously \( W \) is negative definite, if and only if all the eigenvalues have negative real part. An application to a general type of ternary system often encountered in the literature, is furnished.

1. Introduction

Let \( \Omega \subset \mathbb{R}^q \), \( q = 1, 2, 3 \), be a smooth bounded domain. This paper is concerned with the reaction-diffusion systems

\[
\frac{\partial u}{\partial t} = Lu + F, \quad \text{in } \Omega \times \mathbb{R}^+,
\]

with \( u = (u_1, u_2, u_3)^T, \ F = (F_1, F_2, F_3)^T \),

\[
L = \begin{pmatrix}
a_{11} + \gamma_1 A & a_{12} & a_{13} \\
a_{21} & a_{22} + \gamma_2 A & a_{23} \\
a_{31} & a_{32} & a_{33} + \gamma_3 A
\end{pmatrix};
\]

\( F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3), \ i = 1, 2, 3 \), being (generally) nonlinear and

\[
\begin{cases}
a_{ij} = \text{const. } \in \mathbb{R}, & \gamma_i = \text{const. } > 0, & \ i, j \in \{1, 2, 3\}, \\
u_i : (x, t) \in \Omega \times \mathbb{R}^+ \to u_i(x, t) \in \mathbb{R}, & \forall i \in \{1, 2, 3\}.
\end{cases}
\]

To (1.1) we append the Robin boundary conditions

\[
\beta u + (1 - \beta)\nabla u \cdot n = 0, \quad \text{on } \partial \Omega \times \mathbb{R}^+,
\]

where \( n \) is the outward unit normal to \( \partial \Omega \),

\[
\begin{cases}
\beta : x \in \partial \Omega \to \beta(x) \in \mathbb{R}, \\
0 \leq \beta \leq 1, & \forall x \in \partial \Omega,
\end{cases}
\]

\( \beta \) being a sufficiently regular function not identically zero.
The nonlinear functions \( F_i = F_i(u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3) \) are assumed to be sufficiently regular and such that
\[
(F_i)_{u_1=u_2=u_3=0} = 0, \quad \forall i \in \{1, 2, 3\}.
\]
Therefore (1.1)-(1.6) admits the zero solution. To the \( L^2 \)-stability of this solution is precisely devoted the present paper.

**Remark 1.1.** — As it is well known, the stability of a non zero solution of a system \( S \) can be reduced to the stability of the zero solution of a system \( S^\ast \) easily linked to \( S \).

We assume that \( \Omega \) is of class \( C^p \) \((p > 2)\) and has the interior cone property. We denote by
- \( \langle \cdot, \cdot \rangle \) the scalar product of \( L^2(\Omega) \);
- \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) the scalar product of \( L^2(\partial \Omega) \);
- \( \| \cdot \| \) the norm of \( L^2(\Omega) \);
- \( \| \cdot \|_{\partial \Omega} \) the norm of \( L^2(\partial \Omega) \);
- \( W^{1,2}(\Omega, \beta) \) the functional space such that
\[
W^{1,2}(\Omega, \beta) = \{ \varphi \in W^{1,2}(\Omega) \cap W^{1,2}(\partial \Omega), \beta \varphi + (1 - \beta) \nabla \varphi \cdot n = 0, \text{ on } \partial \Omega \}.
\]
For \( \beta > 0, \beta \neq 1 \), it follows \{cfr. [1], pp. 92-98 \} that
\[
(1.7) \quad \| \sqrt{\frac{\beta}{1 - \beta}} \varphi \|_{\partial \Omega}^2 + \| \nabla \varphi \|^2 \geq \bar{\alpha} \| \varphi \|^2,
\]
where \( \bar{\alpha} = \bar{\alpha}(\Omega, \beta) = \text{const.} > 0 \), is the smallest eigenvalue of the spectral problem
\[
(1.8) \quad \begin{cases}
\Delta \varphi + \lambda \varphi = 0, & \text{in } \Omega, \\
\beta \varphi + (1 - \beta) \nabla \varphi \cdot n = 0, & \text{on } \partial \Omega,
\end{cases}
\]
i.e. the principal eigenvalue of \( -\Delta \) in \( W^{1,2}(\Omega, \beta) \).

In the sequel we assume that

i) (1.1)-(1.5) has the properties of a dynamical system [2] embedded in \( W^{1,2}(\Omega, \beta) \) and hence
\[
(1.9) \quad u_i \in W^{1,2}(\Omega, \beta);
\]

ii) the functions \( F_i \) are such that
\[
(1.10) \quad \left< \sum_{i=1}^{3} |u_i| |F_j| \right> \leq k_1 (\| u_1 \|^2 + \| u_2 \|^2 + \| u_3 \|^2)^{1+\epsilon_1} +
+ k_2 (\| u_1 \|^2 + \| u_2 \|^2 + \| u_3 \|^2)^{\epsilon_2} (\| \nabla u_1 \|^2 + \| \nabla u_2 \|^2 + \| \nabla u_3 \|^2),
\]
with \( k_i, \epsilon_i, (i = 1, 2) \), non negative constants.
Setting
\begin{equation}
\begin{aligned}
b_{11} &= a_{11} - \bar{\alpha}_{13}, \\
b_{22} &= a_{22} - \bar{\alpha}_{23}, \\
b_{33} &= a_{33} - \bar{\alpha}_{33},
\end{aligned}
\end{equation}
in [3] have been found conditions on \( a_{ij} \), with \( i \neq j \), able to reduce the stability of the zero solution of (1.1)-(1.6) to the stability of the zero solution of the linear system of O.D.Es

\begin{equation}
\frac{du}{dt} = \mathcal{L}u,
\end{equation}

with either

\begin{equation}
\mathcal{L} = \begin{pmatrix}
b_{11} & 0 & 0 \\
0 & b_{22} & a_{23} \\
0 & a_{32} & b_{33}
\end{pmatrix}
\end{equation}
or - when \( a_{ij}a_{ji} > 0, (i,j = 1, 2, 3) \) -

\begin{equation}
\mathcal{L} = \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix}
\end{equation}
b_{11}, b_{22}, b_{33} being given by (1.11) and

\begin{equation}
b_{ij} = b_{ji} = (\text{sign } a_{ij})\sqrt{a_{ij}a_{ji}}.
\end{equation}

In the present paper, in the guideline of [3]-[6], we reconsider the problem aimed to show that:

i) *the local stability* (1) of the zero solution of (1.1)-(1.6) can be reduced always to the stability of the zero solution of the linear system of O.D.Es.

\begin{equation}
\frac{dx}{dt} = \tilde{\mathcal{L}}x,
\end{equation}

with

\begin{equation}
\tilde{\mathcal{L}} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \tilde{\alpha}_{22} & \tilde{\alpha}_{23} \\
0 & \tilde{\alpha}_{32} & \tilde{\alpha}_{33}
\end{pmatrix}
\end{equation}

\( \lambda_1 \) being a real eigenvalue of

\begin{equation}
\tilde{\mathcal{L}} = \begin{pmatrix}
b_{11} & a_{12} & a_{13} \\
a_{21} & b_{22} & a_{23} \\
a_{31} & a_{32} & b_{33}
\end{pmatrix}
\end{equation}

(1) In the context of the Navier-Stokes equations, the concept of local stability was developed extensively by G. Prodi [10]
and $\tilde{a}_{ij}$ real constants, linked in a suitable simple way to $a_{ij}$ \{cfr. Lemma 2.3\} and such that

$$(1.19) \quad I = \tilde{a}_{22} + \tilde{a}_{33} = I_1 - \lambda_1, \quad A = \tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32} = \frac{I_3}{\lambda_1},$$

where $I_1$ and $I_3$ are the invariants of $\tilde{L}$ given by

$$(1.20) \quad \begin{cases} I_1 = b_{11} + b_{22} + b_{33} = \lambda_1 + \lambda_2 + \lambda_3 \\ I_3 = \det \tilde{L} = \lambda_1\lambda_2\lambda_3, \end{cases}$$

$\lambda_i, (i = 1, 2, 3),$ being the eigenvalues of $\tilde{L};$

ii) the function

$$(1.21) \quad W = \frac{1}{2} \left[ x_1^2 + A(x_2^2 + x_3^2) + (\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32})^2 + (\tilde{a}_{23}x_3 - \tilde{a}_{33}x_2)^2 \right],$$

having the temporal derivative along (1.16) given by

$$(1.22) \quad \dot{W} = \lambda_1x_1^2 + IA(x_2^2 + x_3^2),$$

is a “peculiar” Liapunov function for (1.16) in the sense that $-W$ is positive definite and simultaneously $\dot{W}$ is negative definite $-$ if and only if the real part of the eigenvalues $\lambda_i$ are negative,

iii) the functional

$$(1.23) \quad W^* = \frac{1}{2} \left\{ \left( ||\tilde{u}_1||^2 + A(||\tilde{u}_2||^2 + ||\tilde{u}_3||^2) + ||\tilde{a}_{22}\tilde{u}_3 - \tilde{a}_{33}\tilde{u}_2||^2 + ||\tilde{a}_{23}\tilde{u}_3 - \tilde{a}_{33}\tilde{u}_2||^2 \right) \right\}$$

with $\tilde{u}_i$ linked to $u_i$ in a suitable linear way, is a peculiar Liapunov function for (1.1)-(1.6), when (1.10) and some large conditions on $(\gamma_2, \gamma_3)$ hold \{cfr. Lemma 3-2 of [3]\}.

Section 2 is devoted to some preliminary Lemmas concerned with the stability of matrices of systems of O.D.E.s. To the $L^2$-stability of the zero of (1.1)-(1.4) is addressed Section 3 while in Section 4 the results obtained in the previous Sections are applied to a class of systems modeling various phenomena. The paper ends with an appendix in which is recalled a remark concerned with the eigenvalues of (1.1)-(1.4).

2. – Preliminaries

We collect here some Lemmas useful for the sequel

**Lemma 2.1. –** The asymptotic stability of the null solution of
\[
\begin{align*}
\frac{dx_1}{dt} &= \lambda_1 x_1 + f_1(x_1, x_2, x_3) \\
\frac{dx_2}{dt} &= \tilde{a}_{22} x_2 + \tilde{a}_{23} x_3 + f_2(x_1, x_2, x_3) \\
\frac{dx_3}{dt} &= \tilde{a}_{32} x_2 + \tilde{a}_{33} x_3 + f_3(x_1, x_2, x_3)
\end{align*}
\]
(2.1)

with \( f_i(i = 1, 2, 3) \), nonlinear functions such that

\[
\left( \sum_{i=1}^{3} |x_i| \right) \left( \sum_{i=1}^{3} |f_i| \right) \leq k(x_1^2 + x_2^2 + x_3^2)^{1+\varepsilon}
\]

(2.2)

with \( k \) and \( \varepsilon \) positive constants, is guaranteed iff

\[
\lambda_1 < 0, \quad I < 0, \quad A > 0.
\]
(2.3)

**Proof.** – The proof can be obtained either by observing that (2.3) are equivalent to the Routh-Hurwitz necessary and sufficient conditions for all the eigenvalues of \( \mathcal{L} \) have negative real part [7]-[8] (cfr. Remark 2.1) or by introducing the peculiar Liapunov function (1.21) which temporal derivative along the solution of (2.1) is given by [3]

\[
\dot{W} = \lambda_1 x_1^2 + IA(x_2^2 + x_3^2) + \Psi
\]
(2.4)

with

\[
\begin{align*}
\Psi &= (x_1 x_2 - x_2 x_3) f_2 + (x_2 x_3 - x_3 x_2) f_3 + x_1 f_1 \\
x_1 &= A + \tilde{a}_{32}^2 + \tilde{a}_{33}^2, \quad x_2 = A + \tilde{a}_{22}^2 + \tilde{a}_{23}^2, \quad x_3 = \tilde{a}_{22} \tilde{a}_{32} + \tilde{a}_{23} \tilde{a}_{33},
\end{align*}
\]
(2.5)

**Lemma 2.2.** – Let (2.2) hold. Then the asymptotic stability of the null solution of the ternary systems of O.D.E.s.

\[
\begin{align*}
\frac{dx_1}{dt} &= \lambda_1 x_1 + \tilde{a}_{12} x_2 + \tilde{a}_{13} x_3 + f_1 \\
\frac{dx_2}{dt} &= \tilde{a}_{22} x_2 + \tilde{a}_{23} x_3 + f_2 \\
\frac{dx_3}{dt} &= \tilde{a}_{32} x_2 + \tilde{a}_{33} x_3 + f_3
\end{align*}
\]
(2.6)

can be reduced to the stability of the null solution of (2.1).

**Proof.** – Setting

\[
x_i = \mu_i y_i,
\]
(2.7)
with \( \mu_i, \ (i = 1, 2, 3), \) scalings to be chosen suitably later, (2.6) becomes

\[
\begin{aligned}
\frac{dy_1}{dt} &= \lambda_1 y_1 + \tilde{f}_1 \\
\frac{dy_2}{dt} &= \tilde{a}_{22} y_2 + \frac{\mu_3}{\mu_2} \tilde{a}_{23} y_3 + \tilde{f}_2 \\
\frac{dy_3}{dt} &= \frac{\mu_2}{\mu_3} y_2 + \tilde{a}_{33} y_3 + \tilde{f}_3
\end{aligned}
\]  

(2.8)

with

\[
\begin{aligned}
\tilde{f}_1 &= \frac{\mu_2}{\mu_1} \tilde{a}_{12} y_2 + \frac{\mu_3}{\mu_1} \tilde{a}_{13} y_3 + \frac{1}{\mu_1} f_1(\mu_1 y_1, \mu_2 y_2, \mu_3 y_3) \\
\tilde{f}_j &= \frac{1}{\mu_j} f_j(\mu_1 y_1, \mu_2 y_2, \mu_3 y_3), \quad j = 2, 3.
\end{aligned}
\]  

(2.9)

Introducing the functional \( \tilde{W}, \) analogous to (1.21)

\[
\tilde{W} = \frac{1}{2} \left[ y_1^2 + A(y_2^2 + y_3^2) + (\tilde{a}_{22} y_2 - \frac{\mu_2}{\mu_3} \tilde{a}_{23} y_2)^2 + \left( \frac{\mu_3}{\mu_2} \tilde{a}_{23} y_3 - \tilde{a}_{33} y_2 \right)^2 \right]
\]  

(2.10)

and taking into account (2.2), it turns out that

\[
\frac{d\tilde{W}}{dt} \leq \lambda_1 y_1^2 + IA(y_2^2 + y_3^2) + \Phi + k_1 \tilde{W}^{1+\varepsilon_1}
\]  

(2.11)

with \( k_1 \) and \( \varepsilon_1 \) positive constants and \( \Phi \) given by

\[
\Phi = \frac{\mu_2}{\mu_1} \tilde{a}_{12} y_1 y_2 + \tilde{a}_{13} \frac{\mu_3}{\mu_1} y_1 y_3.
\]  

(2.12)

Choosing \( \mu_2 = \mu_3 \) and setting

\[
\mu = \frac{\mu_2}{\mu_1} = \frac{\mu_3}{\mu_1}
\]  

(2.13)

it follows that

\[
\Phi \leq m \mu |y_1|(|y_2| + |y_3|) \leq \frac{m^2 \mu^2 y_1^2}{|IA|} + \frac{1}{2} |IA|(y_2^2 + y_3^2)
\]  

(2.14)

with

\[
m = \max(|\tilde{a}_{12}|, |\tilde{a}_{13}|)
\]  

(2.15)

and hence

\[
\mu^2 = \frac{|\lambda_1 IA|}{2m^2}
\]  

(2.16)
implies
\begin{equation}
\Phi \leq \frac{1}{2} \left[ |\lambda_1| y_1^2 + IA(y_2^2 + y_3^2) \right].
\end{equation}

Then by virtue of (2.11) and (2.17), one obtains
\begin{equation}
\frac{d\bar{W}}{dt} \leq -(k_* - k_1 \bar{W}^{\alpha_1}) \bar{W}
\end{equation}
with \( k_* \) positive constant and hence
\begin{equation}
\bar{W}^{\alpha_1}(0) < \frac{k_*}{k_1} \Rightarrow \bar{W} \leq \bar{W}(0) \exp \left[ -(k_* - k_1 \bar{W}^{\alpha_1}(0)) \right] t.
\end{equation}

**Lemma 2.3.** Let \( \lambda_1 \) be a (real) eigenvalue of \( \bar{L} \) and \( U = (U_1, U_2, U_3) \) to the associated eigenvector with \( U_1 \neq 0 \). Then the transformation
\begin{equation}
X = \bar{L}_1 Z
\end{equation}
with \( X = (X_1, X_2, X_3)^T \), \( Z = (Z_1, Z_2, Z_3)^T \) and
\begin{equation}
\bar{L}_1 = \begin{pmatrix}
U_1 & 0 & 0 \\
0 & U_2 & 1 \\
0 & 0 & U_3
\end{pmatrix}
\end{equation}
reduces the ternary system
\begin{equation}
\begin{cases}
\frac{dX_1}{dt} = b_{11}X_1 + a_{12}X_2 + a_{13}X_3 \\
\frac{dX_2}{dt} = a_{21}X_1 + b_{22}X_2 + b_{23}X_3 \\
\frac{dX_3}{dt} = a_{31}X_1 + a_{32}X_2 + b_{33}X_3
\end{cases}
\end{equation}
with
\begin{equation}
\begin{cases}
\frac{dZ_1}{dt} = \lambda_1 Z_1 + \tilde{a}_{12}Z_2 + \tilde{a}_{13}Z_3 \\
\frac{dZ_2}{dt} = \tilde{a}_{22}Z_2 + \tilde{a}_{23}Z_3 \\
\frac{dZ_3}{dt} = \tilde{a}_{32}Z_2 + \tilde{a}_{33}Z_3
\end{cases}
\end{equation}
where
\begin{equation}
\begin{cases}
\tilde{a}_{12} = \frac{a_{12}}{U_1}, & \tilde{a}_{13} = \frac{a_{13}}{U_1}, \\
\tilde{a}_{22} = b_{22} - U_2\tilde{a}_{12}, & \tilde{a}_{23} = a_{23} - U_2\tilde{a}_{13}, \\
\tilde{a}_{32} = a_{32} - U_3\tilde{a}_{12}, & \tilde{a}_{33} = b_{33} - U_3\tilde{a}_{13}.
\end{cases}
\end{equation}
PROOF. – We apply, in the case (2.20)-(2.21), the procedure given in {[11], pp. 196-197} and begin by observing that

\[
\begin{aligned}
\begin{cases}
 b_{11}U_1 + a_{12}U_2 + a_{13}U_3 = \lambda_1 U_1 \\
 a_{22}U_1 + b_{22}U_2 + a_{23}U_3 = \lambda_1 U_2 \\
 a_{31}U_1 + a_{32}U_2 + b_{33}U_3 = \lambda_1 U_3
\end{cases}
\end{aligned}
\] (2.25)

and

\[
\begin{aligned}
X_1 &= U_1Z_1, \quad X_2 = U_2Z_1 + Z_2, \quad X_3 = U_3Z_1 + Z_3 \\
Z_1 &= \frac{1}{U_1}X_1, \quad Z_2 = X_2 - \frac{U_2}{U_1}X_1, \quad Z_3 = X_3 - \frac{U_3}{U_1}X_1
\end{aligned}
\] (2.26) (2.27)

hold. Then, by virtue of (2.25)-(2.27), it turns out that

\[
\frac{dZ_1}{dt} = \frac{1}{U_1} \frac{dX_1}{dt} = \frac{1}{U_1}(b_{11}X_1 + a_{12}X_2 + a_{13}X_3)
\]

\[
= \frac{1}{U_1}[(b_{11}U_1 + a_{12}U_2 + a_{13}U_3)Z_1 + a_{12}U_2Z_2 + a_{13}U_3Z_3].
\]

i.e.

\[
\frac{dZ_1}{dt} = \lambda_1 Z_1 + \frac{a_{12}}{U_1} U_2Z_2 + \frac{a_{13}}{U_1} U_3Z_3
\] (2.28)

\[
\frac{dZ_2}{dt} = \frac{dX_2}{dt} - U_2 \frac{dZ_1}{dt} = (a_{21}X_1 + b_{22}X_2 + a_{23}X_3) - U_2 \left( \lambda_1 Z_1 + \frac{a_{12}}{U_1} Z_2 + \frac{a_{13}}{U_1} Z_3 \right)
\]

\[
= a_{21}U_1Z_1 + b_{22}(U_2Z_1 + Z_3) + a_{23}(U_3Z_1 + Z_3) - U_2 \left( \lambda_1 Z_1 + \frac{a_{12}}{U_1} Z_2 + \frac{a_{13}}{U_1} Z_3 \right)
\]

\[
= [(a_{21}U_1 + b_{22}U_2 + a_{23}U_3) - \lambda_1 U_1]Z_1 + \left( b_{22} - \frac{U_2}{U_1} a_{12} \right) Z_2 + \left( a_{23} - \frac{U_2}{U_1} a_{13} \right) Z_3;
\]

i.e.

\[
\frac{dZ_2}{dt} = + \left( b_{22} - \frac{U_2}{U_1} a_{12} \right) Z_2 + \left( a_{23} - \frac{U_2}{U_1} a_{13} \right) Z_3
\] (2.29)

\[
\frac{dZ_3}{dt} = \frac{dX_3}{dt} - U_3 \frac{dZ_1}{dt} = (a_{31}X_1 + a_{32}X_2 + b_{33}X_3) - U_3 \left( \lambda_1 Z_1 + \frac{a_{12}}{U_1} Z_2 + \frac{a_{13}}{U_1} Z_3 \right)
\]

\[
= a_{31}U_1Z_1 + a_{32}(U_2Z_1 + Z_2) + b_{33}(U_3Z_1 + Z_3) - U_3 \left( \lambda_1 Z_1 + \frac{a_{12}}{U_1} Z_2 + \frac{a_{13}}{U_1} Z_3 \right)
\]

\[
= [(a_{31}U_1 + a_{32}U_2 + b_{33}U_3) - \lambda_1 U_3]Z_1 + \left( a_{32} - \frac{U_3}{U_1} a_{12} \right) Z_2 + \left( b_{33} - \frac{U_3}{U_1} a_{13} \right) Z_3;
\]
i.e.

\[
\frac{dZ_3}{dt} = \left( a_{32} - \frac{U_3}{U_1} a_{12} \right) Z_2 + \left( b_{33} - \frac{U_3}{U_1} a_{13} \right) Z_3
\]

By virtue of (2.28)-(2.30), (2.23) with the \( \tilde{a}_{ij} \) given by (2.24) immediately follow.

**Remark 2.1.** – Denoting by \( I_2 \) the invariant

\[
I_2 = \lambda_1 (\lambda_2 + \lambda_3) + \lambda_2 \lambda_3 = \begin{vmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{13} \\ a_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{32} & a_{23} \\ b_{32} & b_{33} \end{vmatrix}
\]

of \( \tilde{L} \), the characteristic (eigenvalues) equation of \( \tilde{L} \) is easily found to be

\[
\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0.
\]

Since \( \lambda_1 \) is supposed to be a root of (2.32) as expected, it turns out that

\[
\lambda^2 + (I_1 - \lambda_1)\lambda + \frac{I_3}{\lambda_1} = \lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2 \lambda_3 = 0.
\]

On the other hand \( \lambda_2 \) and \( \lambda_3 \) have to be eigenvalues of \( \tilde{L} \) (cfr. appendix), hence it follows that (1.19) hold.

**Remark 2.2.** – By virtue of Lemma 2.1 with \( f_1 = f_2 = f_3 = 0 \), it follows that

\[
W = \frac{1}{2} \left[ Z_1^2 + A(Z_2^2 + Z_3^2) + (a_{22} Z_3 - \tilde{a}_{32} Z_2)^2 + (a_{23} Z_3 - a_{33} Z_2) \right],
\]

having the temporal derivative along (2.23) given by

\[
\dot{W} = \lambda_1 Z_1^2 + IA(Z_2^2 + Z_3^2)
\]

is a “peculiar” Liapunov function for (2.23) since the conditions (2.3)-equivalent to the Routh-Hurwitz conditions for all the eigenvalues of \( L \) have negative real part - guarantee that \( W \) is positive definite and \( \dot{W} \) is negative definite.

**Remark 2.3.** – In view of (2.27), (2.32) and

\[
\lambda_1 = \sqrt{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt{-\frac{q}{2} - \sqrt{\Delta}}
\]

with

\[
\Delta = \frac{q^2}{4} + \frac{p^3}{27}, \quad p = I_2 + \frac{1}{3} I_1, \quad q = -I_3 + \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3
\]
it follows that the function
\[
W = \frac{1}{2U_1^2} \left[ X_1^2 + \frac{I_3}{\lambda_1} \left[ (U_1X_2 - U_2X_1)^2 + (U_1X_3 - U_3X_1)^2 \right] \right. \\
+ [\tilde{a}_{22}(U_1X_3 - U_3X_1) - \tilde{a}_{32}(U_1X_2 - U_2X_1)]^2 \\
\left. + [\tilde{a}_{23}(U_1X_3 - U_3X_1) - \tilde{a}_{33}(U_1X_2 - U_2X_1)]^2 \right) 
\]
(2.38)
with \( \tilde{a}_{22}, \tilde{a}_{33}, \tilde{a}_{23}, \tilde{a}_{32} \) given by (2.24), has the temporal derivative along (2.22) given by
\[
\dot{W} = \frac{1}{U_1^2} \left[ \lambda_1 X_1^2 + (I_1 - \lambda_1) \frac{I_3}{\lambda_1} \left[ (U_1X_2 - U_2X_1)^2 + (U_1X_3 - U_3X_1)^2 \right] \right] 
\]
(2.39) and is a “peculiar” Liapunov function for (2.22).

**Remark 2.4.** – The system
\[
\frac{dX}{dt} = \tilde{L}X + f 
\]
(2.40)
with \( \tilde{L} \) given by (1.20) and \( f = (f_1, f_2, f_3)^T \), by virtue of (2.26), is reduced to
\[
\begin{cases}
\frac{dZ_1}{dt} = \lambda_1 Z_1 + \tilde{a}_{12} Z_2 + \tilde{a}_{13} Z_3 + f_1^* \\
\frac{dZ_2}{dt} = \tilde{a}_{22} Z_2 + \tilde{a}_{23} Z_3 + f_2^* \\
\frac{dZ_3}{dt} = \tilde{a}_{32} Z_2 + \tilde{a}_{33} Z_3 + f_3^* 
\end{cases} 
\]
(2.41)
with
\[
f_1^* = \frac{1}{U_1} f_1, \quad f_2^* = f_2 - \frac{U_2}{U_1} f_1, \quad f_3^* = f_3 - \frac{U_3}{U_1} f_1. 
\]
(2.42)
Further, when (2.2) holds, it is easily verified that exist two positive constants \( \tilde{\epsilon}, \tilde{k} \) such that
\[
\left( \sum_{i=1}^{3} |Z_i| \right) \left( \sum_{i=1}^{3} |f_i^*| \right) \leq \tilde{k}(z_1^2 + z_2^2 + z_3^2)^{1+\tilde{\epsilon}}. 
\]
(2.43)
Hence, by virtue of Lemma 2.2, it follows that (2.3) guarantee the (local) asymptotic stability of the null solution of (2.41) and that (2.35) is a “peculiar” Liapunov function for (2.41) and hence (2.39) for (2.40).

**Remark 2.5.** – For the sake of completeness we end by recalling that the Routh-Hurwitz conditions for all the eigenvalues of (2.32) have negative real
part are [7]-[8]

\[ I_1 < 0, \quad I_2 > 0, \quad I_3 < 0, \quad I_1I_2 - I_3 < 0. \]

Therefore - in the case of \( \tilde{L} \) being

\[ \begin{cases} 
I_1 = \lambda_1 + I, \quad I_2 = \lambda_1 I + A, \\
I_3 = \lambda_1 A, \quad I_1I_2 - I_3 = \lambda_1 I \left( I + \frac{\lambda_1^2 + A}{\lambda_1} \right) 
\end{cases} \]

it immediately follows that (2.44) are implied by (2.3). Viceversa let (2.44) hold. Then, in view of (2.45)_3 it immediately follows that \( I_3 < 0 \) with \( \lambda_1 > 0 \) and \( A < 0 \) is not admissible. In fact, by virtue of \{I_1 < 0, \lambda_1 > 0\} and (2.45)_1 it follows that \( I > -\lambda_1 < 0 \) and hence \( \lambda_1 I < 0 \) which, together with \( A < 0 \), implies \( I_2 < 0 \). Therefore (2.44)-(2.45) imply \{\lambda_1 < 0, A > 0\}. It remains to obtain \( I < 0 \). Since by virtue of (2.45)_4, (2.44) is equivalent to

\[ (-\lambda_1)I \left( I + \frac{\lambda_1^2 + A}{\lambda_1} \right) > 0 \]

it follows that \( I \notin \left[ 0, -\frac{\lambda_1^2 + A}{\lambda_1} \right] \). By virtue of \( -\frac{\lambda_1^2 + A}{\lambda_1} > -\lambda_1 \), (2.44)_1 does not allow, in view of (2.45)_1, \( I > -\lambda_1 \) hence \( I < 0 \).

**Remark 2.6.** – We remark that the method introduced holds even when the characteristic equation of \( \tilde{L} \) has multiple roots.

3. – \( L^2 \)-stability of the zero solution of (1.1)-(1.4)

In view of (1.11), (1.1) can be written

\[ \frac{\partial u}{\partial t} = \tilde{L}u + F + F^* \]

with

\[ F^* = (F^*_1, F^*_2, F^*_3)^T, \quad F^*_i = \gamma_i(\lambda u_i + \bar{\lambda} u_i), \quad (i = 1, 2, 3). \]

By virtue of Lemma 2.2 the transformation

\[ u = \tilde{L}_1 v \]

i.e.

\[ u_1 = U_1 v_1, \quad u_2 = U_2 v_1 + v_2, \quad u_3 = U_3 v_1 + v_3 \]
(3.1) becomes

$$\begin{align*}
\frac{\partial v_1}{\partial t} &= \lambda_1 v_1 + \bar{a}_{12} v_2 + \bar{a}_{13} v_3 + \gamma_1 (Av_1 + \bar{v}_1) + P_1 \\
\frac{\partial v_2}{\partial t} &= \bar{a}_{22} v_2 + \bar{a}_{23} v_3 + \gamma_2 (Av_2 + \bar{v}_2) + U_2 (\gamma_2 - \gamma_1) (Av_1 + \bar{v}_1) + P_2 \\
\frac{\partial v_3}{\partial t} &= \bar{a}_{32} v_2 + \bar{a}_{33} v_3 + \gamma_3 (Av_3 + \bar{v}_3) + U_3 (\gamma_3 - \gamma_1) (Av_1 + \bar{v}_1) + P_3
\end{align*}$$

with the $\bar{a}_{ij}$ given by (2.24) and

$$\begin{align*}
P_1 &= \frac{1}{U_1} F_1 (\bar{L}_1 v), \\
P_2 &= F_2 (\bar{L}_1 v) - U_2 F_1 (\bar{L}_1 v), \\
P_3 &= F_3 (\bar{L}_1 v) - U_3 F_1 (\bar{L}_1 v).
\end{align*}$$

By virtue of the linearity of (3.4)-(3.6), the boundary conditions

$$\beta v + (1 - \beta) \nabla v \cdot n = 0 \quad \text{on } \partial \Omega \times R^+$$

have to be appended to (3.5).

We remark that, by virtue of the linear transformation (3.4), a cross-diffusion term appears in (3.5)$_2$-(3.5)$_3$. To (3.5)-(3.7) the methodologies introduced in [11]-[12] can be applied. We here confine ourselves to the case $\gamma_1 = \gamma_2 = \gamma_3$. In this case to (3.5)-(3.7) - as it is easily verified - can be applied Theorem 3.1 of [3] which guarantees that the (local) asymptotic stability hold iff

$$\lambda_1 < 0, \quad I = I_1 - \lambda_1 < 0, \quad A = \frac{I_3}{\lambda_1} > 0$$

and

$$W = \frac{1}{2} \left[ ||v_1||^2 + A (||v_2||^2 + ||v_3||^2) + ||\bar{a}_{22} v_2 - \bar{a}_{32} v_3||^2 + ||\bar{a}_{32} v_2 - \bar{a}_{33} v_3||^2 \right]$$

is a peculiar Liapunov function.

4. – An useful application

As shown in {Lemma 4.1 of [3]}, if

$$\begin{align*}
a_{ij} a_{ji} > 0, & \quad i \neq j \\
a_{12} a_{23} a_{31} = a_{13} a_{21} a_{31}
\end{align*}$$

via a suitable scaling of the $u_i$, $L$ can be symmetrized and the zero solution is stable iff the quadratic form associated to the symmetrized matrix is semi-negative definite. Therefore in order to put in evidence the easy applicability of the
procedures introduced in the previous sections, we consider the (non-symmetrizable) system (1.1)-(1.4) with

\[
\begin{cases}
  a_{11} = a_{22} = -a_1, & a_{33} = -a, & \gamma_1 = \gamma_2 = \gamma_3 = \gamma, \\
  a_{12} = a_{21} = 0, & a_{13} = a_{31} = c, & a_{23} = -a_{32} = b,
\end{cases}
\]

\((a, b, c, \gamma\) being positive constants) often encountered in literature \([13]-[14]\). Setting

\[
\begin{align*}
  R &= a + \bar{x} \gamma, \\
  R_1 &= a_1 + \bar{x} \gamma,
\end{align*}
\]

it turns out that

\[
\begin{cases}
  \frac{\partial u_1}{\partial t} = -R_1 u_1 + c u_3 + \gamma (A u_1 + \bar{x} u_1) + F_1, \\
  \frac{\partial u_2}{\partial t} = -R_1 u_2 + b u_3 + \gamma (A u_2 + \bar{x} u_2) + F_2, \\
  \frac{\partial u_3}{\partial t} = c u_1 - b u_2 - R u_3 + \gamma (A u_3 + \bar{x} u_3) + F_3.
\end{cases}
\]

As it is easily verified the matrix

\[
\tilde{L} = \begin{pmatrix}
  -R_1 & 0 & c \\
  0 & -R_1 & b \\
  c & -b & -R
\end{pmatrix}
\]

admits the eigenvalues \(\lambda_1 = -R_1\) with the associated eigenvector, \(U = (U_1 = b, U_2 = c, U_3 = 0)\) and hence

\[
\begin{align*}
  u &= \tilde{L}_1 v, \\
  \tilde{L}_1 &= \begin{pmatrix}
  b & 0 & 0 \\
  c & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\end{align*}
\]

give

\[
\begin{align*}
  u_1 &= b v_1, & u_2 &= c v_1 + v_2, & u_3 &= v_3.
\end{align*}
\]

Then it easily follows that

\[
\begin{cases}
  \frac{\partial v_1}{\partial t} = -R v_1 + \frac{c}{b} v_3 + \gamma (A v_1 + \bar{x} v_1) + \frac{1}{b} F_1(\tilde{L}_1 u), \\
  \frac{\partial v_2}{\partial t} = -R v_2 + \frac{b^2 - c^2}{b} v_3 + \gamma (A v_2 + \bar{x} v_2) + F_2(\tilde{L}_1 u) - \frac{c}{b} F_1(\tilde{L}_1 u), \\
  \frac{\partial v_3}{\partial t} = -b v_2 - R v_3 + \gamma (A v_3 + \bar{x} v_3) + F_3(\tilde{L}_1 u).
\end{cases}
\]
Being

\[(4.9) \quad \lambda_1 = -R, \quad I = -(R + R_1), \quad A = RR_1 + b^2 - c^2\]

with \(R\) and \(R_1\) positive constants, it turns out that the zero solution is stable iff

\[(4.10) \quad C^2 < RR_1 + b^2 = (a + \bar{a}\gamma)(a_1 + \bar{a}_1\gamma) + b^2.\]

5. Appendix

Systems (2.22) and (2.23) can be written respectively

\[(5.1) \quad \frac{dX}{dt} = \hat{L}X,\]
\[(5.2) \quad \frac{dZ}{dt} = \hat{L}_1^{-1}\hat{L}\hat{L}_1 Z.\]

The characteristic equations of \(\hat{L}\) and \(\hat{L}_1^{-1}\hat{L}\hat{L}_1\) are coincident since [9]

\[(5.3) \quad \hat{L}_1^{-1}(\hat{L} - \lambda E)\hat{L}_1 = \hat{L}_1^{-1}\hat{L}\hat{L}_1 - \lambda E\]

with \(E\) unity matrix. Hence

\[(5.4) \quad \det(\hat{L}_1^{-1}\hat{L}\hat{L}_1 - \lambda E) = \det(\hat{L}_1^{-1}\det(\hat{L} - \lambda E) \det \hat{L}_1 = \det(\hat{L} - \lambda E).\]

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