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AUGUSTO VISINTIN

To the memory of Giovanni Prodi

Abstract. – *To any maximal monotone operator* $\alpha : V \to \mathcal{P}(V')$ *(V being a real Banach space), in* [MR 1009594] *S. Fitzpatrick associated a lower semicontinuous and convex function* $f : V \times V' \to \boldsymbol{R} \cup \{+\infty\}$ *such that*

$$(*) \qquad f(v, v') \geq \langle v', v \rangle \quad \forall (v, v'), \qquad f(v, v') = \langle v', v \rangle \quad \Leftrightarrow \quad v' \in \alpha(v).$$

On this basis, in this work two classes of doubly-nonlinear evolutionary equations are formulated as minimization principles:

$$(**) \qquad D_t\alpha(u) - \operatorname{div} \vec{\gamma}(\nabla u) \ni h, \qquad \alpha(D_t u) - \operatorname{div} \vec{\gamma}(\nabla u) \ni h;$$

here α *and* $\vec{\gamma}$ *are maximal monotone mappings, and one of them is assumed to be cyclically monotone. For associated initial- and boundary-value problems, existence of a solution is proved, as well as the stability with respect to variations of the data and of the operators* D_t, ∇, α *and* $\vec{\gamma}$.

Foreword

Giovanni Prodi was widely recognized not only as a leading researcher but also as a distinguished mathematical educator. I met him mainly through two of his books, that from time to time I still revisit; I would like to mention them here, although his contributions go far beyond these works. Unfortunately both volumes are just available in Italian language.

At Pavia in 1971 I had my first impact with mathematical analysis through the course held by Claudio Baiocchi, who used Prodi's newly published textbook: *Analisi matematica* (Boringhieri, 1970). This book introduces notions like mathematical logic, set theory, abstract algebra, metric and topological spaces from the very beginning. This approach was quite at variance with the more traditional teaching of basic analysis of that time. Although just freshmen, we were aware of the chance we had to enter mathematics from the main door, and I got a sort of Lorenz imprinting from that text.

As a teacher, several years later I met another of Prodi's books: *Metodi matematici e statistici per le scienze applicate* (McGraw-Hill, Milano 1992). This is a broad spectrum introduction to the tenets of logic, linear algebra, analysis, geometry, probability, statistics, modeling, up to representing Bayesian inference as an extension of Aristotle's logic. A rich undergraduate text, aimed at integrating different disciplines, enlightened by a vision of the interplay between mathematics and science.

It is with those pages in mind that I devote this work to the memory of Giovanni Prodi, appreciating of having been given this opportunity.

1. – Introduction

Several evolutionary phenomena of mathematical physics may be modeled either by monotone flows or more generally by doubly nonlinear equations of the form

$$D_t\alpha(u) - \nabla \cdot \vec{\gamma}(\nabla u) \ni h, \tag{1.1}$$

$$\alpha(D_t u) - \nabla \cdot \vec{\gamma}(\nabla u) \ni h \tag{1.2}$$

($\nabla\cdot = \text{div}$), with α and $\vec{\gamma}$ maximal monotone operators; see e.g. [56,62]. Maximal monotone relations may be represented variationally. This property is here used to formulate the flows (1.1) and (1.2) as minimization principles, and to show existence of a solution. Our main concern is to prove that these problems are stable with respect to variations of the data and of the operators D_t, ∇, α and $\vec{\gamma}$ *(structural stability).*

1.1 – *The Fitzpatrick Theory*

Let us first recall a classical result of Fenchel [28], that mutually relates a convex and lower semicontinuous function(al) $\varphi : V \to \boldsymbol{R} \cup \{+\infty\}$, its convex conjugate φ^*, and the subdifferential operator $\partial\varphi$, see e.g. [27; Chap. 1]:

$$\varphi(v) + \varphi^*(v') \geq \langle v', v\rangle \qquad \forall (v, v') \in V \times V', \tag{1.3}$$

$$\varphi(v) + \varphi^*(v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \partial\varphi(v). \tag{1.4}$$

In the seminal paper [29] Fitzpatrick extended this system to more general monotone operators $\alpha : V \to \mathcal{P}(V')$. First he introduced the convex and lower semicontinuous function (afterwards named the Fitzpatrick function of α)

$$\begin{aligned} f_\alpha(v, v') &:= \langle v', v\rangle + \sup\left\{\langle v' - v'_0, v_0 - v\rangle : v'_0 \in \alpha(v_0)\right\} \\ &= \sup\left\{\langle v', v_0\rangle - \langle v'_0, v_0 - v\rangle : v'_0 \in \alpha(v_0)\right\} \qquad \forall (v, v') \in V \times V'; \end{aligned} \tag{1.5}$$

he then proved that whenever α is maximal monotone (in the sense e.g. of [11, 15])

$$f_\alpha(v,v') \geq \langle v', v\rangle \qquad \forall (v,v') \in V\times V', \tag{1.6}$$

$$f_\alpha(v,v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{1.7}$$

After several years, this result was independently rediscovered by Martinez-Legaz and Théra [41] and by Burachik and Svaiter [18]. This started an intense research about relations between monotone operators and convex functions; see e.g. [19, 34, 42, 43, 53, 54], just to quote few contributions. See also the related notion of bipotential [17].

The above result of Fitzpatrick has been further extended. Nowadays one says that a convex and lower semicontinuous function $f : V\times V' \to \boldsymbol{R} \cup \{+\infty\}$ (variationally) *represents* an operator $\alpha : V \to \mathcal{P}(V')$ whenever it fulfills the system (1.6), (1.7). Accordingly, we shall say that f is a representative of α, and that α is representable; we shall denote the class of these functions by $\mathcal{F}(V)$. So e.g. the Fenchel function $f(v,v') := \varphi(v) + \varphi^*(v')$ represents the operator $\partial\varphi$. But representable operators need not be either cyclically monotone or maximal monotone, although they are necessarily monotone. Some results of this theory are briefly reviewed in the parallel work [67; Sect. 2].

1.2 – *Variational Formulation of Monotone Flows.*

Prior to Fitzpatrick's [29], Brezis and Ekeland [16] and Nayroles [51] independently proposed a first example of variational formulation of a nonlinear evolutionary P.D.E. of first order. Let V be a (real) Banach space and H a Hilbert space such that

$$V \subset H = H' \subset V' \ \text{ with continuous and dense injections.} \tag{1.8}$$

They associated the functional

$$\Phi : X_{u^0} := \left\{ v \in L^2(0,T;V) \cap H^1(0,T;V') : v(0) = u^0 \right\} \to \boldsymbol{R} \cup \{+\infty\}, \tag{1.9}$$

$$\Phi(v) := \int_0^T \left[\varphi(v) + \varphi^*(h - D_t v) - \langle h, v\rangle\right] dt + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 \tag{1.10}$$

to the Cauchy problem

$$\begin{cases} D_t u + \partial\varphi(u) \ni h & \text{in } V',\ \text{a.e. in }]0,T[\\ u(0) = u^0. \end{cases} \tag{1.11}$$

They pointed out that $\Phi \geq 0$, and that this Cauchy problem is equivalent to

$\Phi(u) = 0$, namely

$$\Phi(u) = \inf \Phi = 0 \qquad \text{(a } \textit{null-minimization principle}\text{)}. \tag{1.12}$$

The solution u may thus be computed by means of a descent procedure, provided that it has already been established that a null-minimizer does exist, namely, $\Phi^{-1}(0) \neq \emptyset$. The latter property is nontrivial; this issue was settled by Auchmuty [8], and was then extensively studied by Ghoussoub in a series of works; in particular see e.g. [34, 36], the monograph [35], and references therein.

The Fitzpatrick theorem allows one to extend the formulation (1.12) to non-cyclically-monotone operators as follows. Whenever an operator $\alpha : V \to \mathcal{P}(V')$ is (variationally) represented by a function f, for any $h \in L^{p'}(0, T; V')$ the Cauchy problem (1.11) is equivalent to (1.12), provided that in the definition (1.4) the term $\varphi(v) + \varphi^*(h - D_t v)$ is replaced by $f(v, h - D_t v)$; see [63].

This variational approach allows one to apply a different viewpoint: instead of prescribing h, one may deal with the family of all pairs (u, h) that fulfill the Cauchy problem (1.11). One may then use variational techniques, e.g. De Giorgi's theory of Γ-convergence (see e.g. [25, 24]), to study the qualitative properties of the dependence of the solution on the data and on the operator, viz., $\partial\varphi$ or more generally α. In this work this point of view is applied to two classes of nonlinear flows.

1.3 – *Doubly Nonlinear Flows*

The equation (1.1) occurs in several models. For instance, it may represent the entropy balance in diffusion phenomena, as in the Eckart theory of irreversible thermodynamics; see e.g. [47; Chap. 8], [62; Sects. V.4, V.5]. The equation (1.2) also arises in thermodynamics, with α equal to the subdifferential of a *dissipation potential.* In the last years much attention has been paid to rate-independent evolution, as it occurs in hysteresis phenomena; this may be represented by (1.2) with α homogeneous of zero degree. For the latter equation an alternative approach, which has become known as the *energetic formulation,* has been introduced by Mielke and coworkers [45, 46], and then applied to a multitude of physical models; see e.g. the survey [44] and references therein.

For both the equations (1.1) and (1.2) existence of a solution, its large time behavior, and other results have already been proved in a number of works: see [1, 2, 4, 5, 6, 10, 12, 13, 14, 22, 23, 26, 30, 32, 37, 38, 52] for (1.1), [3, 9, 22, 23, 33, 55, 58, 57, 60] for (1.2), and also the monographs [31; Chap. V], [56; Chap. 11], [62; Chap. III] for both classes.

In particular in [55] Rossi, Mielke and Savaré provided a detailed analysis of the equation (1.2) in the case of α homogeneous of degree zero in the more general environment of metric spaces. In [60] Stefanelli extended the Brezis-

Ekeland-Nayroles principle to (1.2), including the case of α homogeneous of degree zero. Assuming that both α and $\vec{\gamma}$ are cyclically maximal monotone and using the corresponding Fenchel functions, he reformulated the equation (1.2) as a minimization principle, introducing an expedient that here we also use in formulating Problems 3.2 and 5.2 as the minimization of a single functional. [60] also addressed the dependence of the solution on the (potentials of) the operators α and $\vec{\gamma}$, mainly using Mosco-convergence. The present work deals with a weaker notion of convergence, in the spirit of [67]; see Theorems 4.1 and 6.1 ahead.

The uniqueness of the solution of (1.1) was proved by Carrillo [20] via the notion of entropic solution and the use of L^1-contractions, see also [14, 21, 39]; on the other hand, for (1.2) uniqueness is still an open question. The homogenization of (1.1) was also addressed in [40, 49, 50].

Here we provide a variational formulation of the equations (1.1) and (1.2) via the Fitzpatrick approach. For both equations, we retrieve existence of a weak solution via an approximation procedure. We then prove the structural stability with respect to variations of the data and of the operators D_t, ∇, α and $\vec{\gamma}$, in the following sense.

Let us first denote by $\mathcal{D}$ the set of the admissible data (e.g., the initial datum and the source term); by $\mathcal{O}$ the set of the operators D_t, ∇, and those associated to the mappings α and $\vec{\gamma}$; by $\mathcal{S}$ the set of the admissible solutions. These three sets must be equipped with appropriate notions of convergence. By the existence theorem, there exists a (possibly multivalued) *resolution operator*, $\mathcal{R} : \mathcal{D} \times \mathcal{O} \to \mathcal{S}$. We shall show that this operator is sequentially closed; that is, for any sequence $\{(d_n, o_n)\}$ that converges to some (d, o), and for any corresponding sequence of solutions $\{s_n \in \mathcal{R}(d_n, o_n)\}$, the following occurs: (i) there exists s, such that $s_n \to s$, up to subsequences; (ii) this entails that $s \in \mathcal{R}(d, o)$. This also applies to the discretization of the operators D_t and ∇, e.g. by finite difference or by finite element.

In this paper we proceed as follows. In Sect. 2 we display some examples of representative functions of monotone operators (more may be found e.g. in [67]). In Sect. 3 we formulate equation (1.1) as a minimization principle, and prove existence of a solution. In Sect. 4 we show that the solution is stable with respect to variations of the mappings $\alpha, \vec{\gamma}$, and illustrate how this stability may easily be extended to allow for variations of the differential operators ∇ and D_t and of the domain Ω. In Sects. 5 and 6 we proceed similarly for equation (1.2).

The results of this work are not conclusive, and some issues seem worth of further consideration. These include an analysis of the hypotheses that are here assumed for the stability theorems, see (4.1)-(4.5) and (6.1)-(6.5); questions of this sort are addressed in [67] for evolutionary equations with a single nonlinearity. Another issue of interest is the variational representation of the equation (1.2) for α homogeneous of degree zero, which is pursued in the parallel work [68]. The application of null-minimization principles to specific physical models looks a fertile field, too.

In the last five years, the representation of nonlinear flows as null-minimization has also taken this author to apply variational techniques to the analysis of homogenization, see e.g. [64, 65, 66] and references therein.

2. – Some Examples of Representative Functions

In this section we exhibit some representative functions of maximal monotone operators. We shall still denote a real Banach space by V.

EXAMPLE 2.1. – We already pointed out that for any proper, convex and lower semicontinuous function $\varphi : V \to \boldsymbol{R} \cup \{+\infty\}$ the subdifferential operator $\partial\varphi : V \to \mathcal{P}(V')$ is represented by the *Fenchel function*

$$f(v, v') = \varphi(v) + \varphi^*(v') \qquad \forall (v, v') \in V \times V'. \tag{2.1}$$

EXAMPLE 2.2. – Let $L : V \to V'$ be a bounded skew-symmetric (hence monotone) operator. For instance, this holds for $H^1_0(\Omega)^2 \to H^{-1}(\Omega)^2$: $(v_1, v_2) \mapsto (-\Delta v_2, \Delta v_1)$; here Ω is a domain of $\boldsymbol{R}^N$ and $N \geq 1$. Let us denote by I_L the indicator function of the graph of L, that is

$$I_L(v, v') := 0 \quad \text{if } v' = Lv, \qquad I_L(v, v') :=:= +\infty \quad \text{otherwise.}$$

It is easily seen that the operator L is represented e.g. by the functions

$$f_L(v, v') = I_L(v, v') \qquad \forall (v, v') \in V \times V', \tag{2.2}$$

$$f^*_L(v', v) = I_{L^*}(v, v') = I_{-L}(v, v') \qquad \forall (v, v') \in V \times V'. \tag{2.3}$$

EXAMPLE 2.3. – Let $L : V \to V'$ be a monotone, bounded, linear operator, that we write as the sum of its symmetric and skew-symmetric parts: $L = L_s + L_{ss}$. Let us then set

$$\varphi(v) := \frac{1}{2} \langle L_s v, v \rangle \ \Big(= \frac{1}{2} \langle Lv, v \rangle\Big) \quad \forall v \in V, \qquad f_{L_{ss}} := I_{L_{ss}} \quad \text{in } V \times V'. \tag{2.4}$$

A direct computation shows that the operator L may be represented for instance either by

$$f_1(v, v') = \varphi(v) + \varphi^*(v' - L_{ss} v) \qquad \forall (v, v') \in V \times V', \tag{2.5}$$

or, if L_{ss} is invertible, by

$$f_2(v, v') = \varphi(v + (L_{ss})^{-1} v') + \varphi^*(v') \qquad \forall (v, v') \in V \times V'. \tag{2.6}$$

EXAMPLE 2.4. – Let Ω be a bounded domain of $\boldsymbol{R}^N$ ($N > 1$), $p \in\,]1, +\infty[$, and set $V := W^{1,p}_0(\Omega)$. Let a maximal monotone mapping $\vec{\gamma} : \boldsymbol{R}^N \to \mathcal{P}(\boldsymbol{R}^N)$ be represented by a function $f \in \mathcal{F}(\boldsymbol{R}^N)$. If

$$\exists a_1, a_2 \in \boldsymbol{R}^+ : \forall \vec{w} \in \boldsymbol{R}^N, \forall \vec{z} \in \vec{\gamma}(\vec{w}), \qquad |\vec{z}| \le a_1 |\vec{w}|^{p'} + a_2, \tag{2.7}$$

then the operator $\widehat{\beta} : V \to \mathcal{P}(V') : v \mapsto -\nabla \cdot \vec{\gamma}(\nabla v)$ is maximal monotone. This includes e.g. the case of the p-Laplacian: $-\nabla \cdot \vec{\gamma}(\nabla v) = -\nabla \cdot (|\nabla v|^{p-2} \nabla v)$.

We claim that $\widehat{\beta}$ may be represented by the following function $\varphi \in \mathcal{F}(V)$: for any $(v, v') \in V \times V'$,

$$\varphi(v, v') = \int_\Omega f(\nabla v, \vec{\xi}_{v'})\, dx \quad \text{with} \quad \begin{cases} \vec{\xi}_{v'} \in \nabla H^1_0(\Omega)^N, \\ -\nabla \cdot \vec{\xi}_{v'} = v' \quad \text{in } \mathcal{D}'(\Omega) \end{cases} \tag{2.8}$$

($\vec{\xi}_{v'}$ is thus determined by finding $\theta \in H^1_0(\Omega)^N$ such that $-\Delta\theta = v'$ in $\mathcal{D}'(\Omega)$, and then setting $\vec{\xi}_{v'} = \nabla\theta$). Indeed,

$$\varphi(v, v') = \int_\Omega f(\nabla v, \vec{\xi}_{v'})\, dx \overset{f \in \mathcal{F}(\boldsymbol{R}^N)}{\ge} \int_\Omega \nabla v \cdot \vec{\xi}_{v'}\, dx = -\langle v, \nabla \cdot \vec{\xi}_{v'} \rangle \overset{(2.8)}{=} \langle v, v' \rangle. \tag{2.9}$$

Moreover, as $f(\nabla v, \vec{\xi}_{v'}) \ge \nabla v \cdot \vec{\xi}_{v'}$ pointwise, in (2.9) equality holds if and only if $f(\nabla v, \vec{\xi}_{v'}) = \nabla v \cdot \vec{\xi}_{v'}$ a.e. in Ω. As f represents $\vec{\gamma}$, this equality is equivalent to $\vec{\xi}_{v'} \in \vec{\gamma}(\nabla v)$ a.e. in Ω, namely by (2.8)

$$v' \in -\nabla \cdot \vec{\gamma}(\nabla v) \qquad \text{in } \mathcal{D}'(\Omega). \tag{2.10}$$

Notice that the curl-free $\vec{\xi}_{v'}$ is just one of the many selections $\vec{\eta} \in \vec{\gamma}(\nabla v)$ such that $\vec{\eta} \in L^2(\Omega)^N$ and $-\nabla \cdot \vec{\eta} = v'$ in $\mathcal{D}'(\Omega)$.

EXAMPLE 2.5. – Let $L : V \to V'$ be a monotone, bounded, linear operator. If another operator $\alpha : V \to \mathcal{P}(V')$ is represented by a function $f_\alpha \in \mathcal{F}(V)$, then the operator $\alpha + L$ is represented for instance by the function

$$f(v, v') = f_\alpha(v, v' - Lv) + \langle Lv, v \rangle \qquad \forall (v, v') \in V \times V'. \tag{2.11}$$

In some cases L is a first-order operator, as in the next example.

EXAMPLE 2.6. – Let us fix a triplet of Banach spaces as in (1.8), any $T > 0$, any $p \in]1, +\infty[$, and set

$$X^p_0 := \{ v \in L^p(0, T; V) \cap W^{1,p'}(0, T; V') : v(0) = 0 \}. \tag{2.12}$$

As $\int_0^T \langle D_t v, v \rangle\, dt = \frac{1}{2} \|v(T)\|^2_H \ge 0$ for any $v \in X^p_0$, the linear operator $\alpha : X^p_0 \to L^{p'}(0, T; V') : v \mapsto D_t v$ is maximal monotone. Its Fitzpatrick function (see (1.5)) reads

$$f_\alpha(v,v') = \begin{cases} \dfrac{1}{2}\|v(T)\|_H^2 & \text{if } v' = D_t v \\ +\infty & \text{if } v' \neq D_t v \end{cases} \qquad \forall (v,v') \in X_0^p \times (X_0^p)'. \tag{2.13}$$

One may also deal with the periodicity condition $v(T) = v(0)$.

EXAMPLE 2.7. – If an operator $\beta : L^p(0,T;V) \to \mathcal{P}(L^{p'}(0,T;V'))$ is represented by a function f_β, then the operator $\gamma : D_t + \beta : X_0^p \to \mathcal{P}(L^{p'}(0,T;V')) \subset \mathcal{P}((X_0^p)')$ may be represented by the function

$$f_\gamma(v,v') = \begin{cases} f_\beta(v, v' - D_t v) + \dfrac{1}{2}\|v(T)\|_H^2 & \text{if } v' \in L^{p'}(0,T;V') \\ +\infty & \text{otherwise} \end{cases} \tag{2.14}$$

for any $(v,v') \in X_0^p \times (X_0^p)'$. This applies to a number of quasilinear parabolic equations of applicative interest. These include e.g. the operator $v \mapsto D_t - \Delta\alpha(v)$, that occurs in the weak formulation of the classical Stefan problem, see e.g. [62].

3. – A First Class of Doubly-Nonlinear Parabolic Equations

In this section we formulate a minimization principle that is equivalent to an initial- and boundary-value problem for a doubly-nonlinear parabolic equation of the form

$$D_t\alpha(u) - \nabla \cdot \vec{\gamma}(\nabla u) \ni h \qquad \text{in } Q := \Omega \times]0,T[. \tag{3.1}$$

Here Ω is a bounded domain of $\boldsymbol{R}^N$ of Lipschitz class, $\alpha : \boldsymbol{R} \to \mathcal{P}(\boldsymbol{R})$ and $\vec{\gamma} : \boldsymbol{R}^N \to \mathcal{P}(\boldsymbol{R}^N)$ are maximal monotone mappings such that

$$\begin{aligned} &\exists c_1, c_2 \in \boldsymbol{R}^+ : \forall v \in \boldsymbol{R}, \quad |\alpha(v)| \leq c_1 + c_2|v|, \\ &\exists c_3, c_4 \in \boldsymbol{R}^+ : \forall \vec{v} \in \boldsymbol{R}^N, \quad |\vec{\gamma}(\vec{v})| \leq c_3 + c_4|\vec{v}|, \end{aligned} \tag{3.2}$$

and the source field h is prescribed. The inclusion (3.1) also reads

$$D_t w - \nabla \cdot \vec{z} = h \qquad \text{in } Q, \tag{3.3}$$

$$w \in \alpha(u) \qquad \text{in } Q, \tag{3.4}$$

$$\vec{z} \in \vec{\gamma}(\nabla u) \qquad \text{in } Q; \tag{3.5}$$

each of these two inclusions is tantamount to a variational inequality.

We shall couple this system with appropriate initial- and boundary-conditions, prove existence of a weak solution, and show its stability with respect to variations of the data $h, \alpha, \vec{\gamma}$ and of the operators D_t, ∇.

3.1 – *Weak Formulation*

For the sake of simplicity, we prescribe the homogeneous Dirichlet condition for u on $(\partial\Omega) \times]0, T[$. We set

$$V := H^1_0(\Omega) \subset H := L^2(\Omega) = L^2(\Omega)' \subset V' = H^{-1}(\Omega), \tag{3.6}$$

assume that

$$w^0 \in H, \qquad h \in L^2(0, T; V'), \tag{3.7}$$

and introduce a weak formulation of the Cauchy problem for the system (3.3)-(3.5).

PROBLEM 3.1. – *Find $u \in L^2(0, T; V)$, $w \in L^2(0, T; H)$ and $\vec{z} \in L^2(Q)^N$ such that*

$$\iint_Q (-wv_t + \vec{z}\cdot\nabla v)\, dxdt = \int_0^T {}_{V'}\langle h, v\rangle_V \, dt + {}_{V'}\langle w^0, v(\cdot, 0)\rangle_V \tag{3.8}$$

$$\forall v \in H^1(0, T; V), v(\cdot, T) = 0,$$

$$w \in \alpha(u) \qquad \textit{a.e. in } Q, \tag{3.9}$$

$$\vec{z} \in \vec{\gamma}(\nabla u) \qquad \textit{a.e. in } Q. \tag{3.10}$$

The equation (3.8) yields (3.3) in $V' = H^{-1}(\Omega)$ a.e. in $]0, T[$. By comparing the terms of (3.3), we see that $w \in H^1(0, T; V')$; from (3.3) and (3.8) we then retrieve the initial condition

$$w|_{t=0} = w^0. \tag{3.11}$$

Let us now select a representative functional $\varphi \in \mathcal{F}(V)$ of the maximal monotone operator $V \to \mathcal{P}(V') : v \mapsto -\nabla \cdot \vec{\gamma}(\nabla v)$, see Example 2.4 of Sect. 2. Thus φ is lower semicontinuous and convex, and (cf. (1.6) and (1.7))

$$\varphi(v, v') \geq \langle v', v\rangle \qquad \forall (v, v') \in V \times V', \tag{3.12}$$

$$\varphi(v, v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in -\nabla \cdot \vec{\gamma}(\nabla v) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.13}$$

We claim that the functional $(v, v') \mapsto \int_0^T \varphi(v, v')\, dt$ is then an element of $\mathcal{F}(L^2(0, T; V))$, and represents the operator

$$\widehat{\vec{\gamma}} : L^2(0, T; V) \to \mathcal{P}(L^2(0, T; V')) : v \mapsto -\nabla \cdot \vec{\gamma}(\nabla v),$$

that is,

$$\int_0^T \varphi(v, v')\, dt \geq \int_0^T \langle v', v\rangle \, dt \qquad \forall (v, v') \in L^2(0, T; V) \times L^2(0, T; V'), \tag{3.14}$$

$$\int_0^T \varphi(v, v')\, dt = \int_0^T \langle v', v\rangle \, dt \quad \Leftrightarrow \quad v' \in -\nabla \cdot \vec{\gamma}(\nabla v) \text{ in } \mathcal{D}'(\Omega), \text{ a.e. in }]0, T[. \tag{3.15}$$

The inequality (3.14) and the implication "⇐" of (3.15) directly follow from (3.12) and (3.13). Because of (3.12),

$$\int_0^T \varphi(v,v')\,dt = \int_0^T \langle v',v\rangle\,dt \quad \Rightarrow \quad \varphi(v,v') = \langle v',v\rangle \quad \text{a.e. in }]0,T[;$$

by (3.13) the latter equality yields $v' \in -\nabla\cdot\vec{\gamma}(\nabla v)$ in $\mathcal{D}'(\Omega)$, a.e. in $]0,T[$. The implication "⇒" of (3.15) is thus established, too.

As the variable u is scalar, the maximal monotone operator α is cyclically monotone; indeed,

(3.16)
$$g : \boldsymbol{R} \to \boldsymbol{R} \cup \{+\infty\} : v \to \int_0^v \alpha(s)\,ds$$
is convex and lower semicontinuous, and $\partial g = \alpha$.

Denoting by g^* the convex conjugate function of g, we have $\alpha = \partial g$ and $\alpha^{-1} = \partial g^*$; see e.g. [27; Chap. 1]. By $(3.2)_1$ the growth of g is at most quadratic; hence

$$\exists L > 0, \exists M \in \boldsymbol{R} : \forall v \in \boldsymbol{R}, \quad g^*(v) \geq L|v|^2 - M. \tag{3.17}$$

Let us define the integral functional

$$R : H \to \boldsymbol{R} \cup \{+\infty\},\ R(v) = \int_\Omega g(v(x))\,dx \quad \Big(\text{whence } R^*(v) = \int_\Omega g^*(v(x))\,dx\Big). \tag{3.18}$$

3.2 – *Null-Minimization.*

Next we reformulate our problem.

PROBLEM 3.2. – *Find $u \in L^2(0,T;V)$ and $w \in L^2(0,T;H) \cap H^1(0,T;V')$ such that $w|_{t=0} = w^0$ and*

$$\begin{aligned}\Phi(u,w) :=& \int_0^T \big[R(u) + R^*(w)\big]\,dt - \iint_Q wu\,dxdt \,+ \\ & \left\{\int_0^T \big[\varphi(u, h - D_t w) - \langle h,u\rangle\big]\,dt + R^*(w(\cdot,T)) - R^*(w^0)\right\}^+ \leq 0.\end{aligned} \tag{3.19}$$

(The idea of using the positive part in a functional like this was introduced by U. Stefanelli in [60].) The functional $\Phi : L^2(0,T;V) \times \big(L^2(0,T;H) \cap H^1(0,T;V')\big) \to \boldsymbol{R}$ is convex and is finite on the whole space. As $g(u) + g^*(w) \geq wu$ for any u and w, the difference between the first two integrals is nonnegative. The inequality (3.19) is thus a null-minimization principle, cf. (1.12), and is equivalent to the system

$$\int_0^T \big[R(u) + R^*(w)\big]\,dt \le \iint_Q wu\,dxdt, \tag{3.20}$$

$$\int_0^T \varphi(u, h - D_t w)\,dt + R^*(w(\cdot,T)) \le R^*(w^0) + \int_0^T \langle h, u\rangle\,dt. \tag{3.21}$$

Each of these inequalities may also be rewritten as a null-minimization principle.

PROPOSITION 3.1. – *If the triplet* $(u, w, \vec{z})$ *solves Problem* 3.1, *then the pair* (u, w) *solves Problem* 3.2. *Conversely, if* (u, w) *solves Problem* 3.2, *then there exists* $\vec{z} \in L^2(Q)^N$ *such that* $(u, w, \vec{z})$ *solves Problem* 3.1.

PROOF. – By the Fenchel system (1.6), (1.7), the inequality (3.20) is equivalent to the inclusion (3.9) a.e. in Q. Let us next recall the chain-rule

$$\begin{aligned} &\forall u \in L^2(0,T;V), \forall w \in L^2(0,T;H) \cap H^1(0,T;V'),\\ &\text{if } w \in \alpha(u) \text{ a.e. in } Q, \text{ then}\\ &R^*(w) \in W^{1,1}(0,T), \quad \text{and} \quad D_t R^*(w) = \langle D_t w, u\rangle \quad \text{a.e. in }]0,T[. \end{aligned} \tag{3.22}$$

By (3.9) and (3.11), the inequality (3.21) is then equivalent to

$$\int_0^T \varphi(u, h - D_t w)\,dt \le \int_0^T \langle h - D_t w, u\rangle\,dt;$$

as φ represents $\vec{\gamma}$, this is tantamount to (3.3) and (3.10). As we saw, (3.8) is equivalent to (3.3) an (3.11). The desired equivalence is thus established. □

THEOREM 3.2. – *(Existence) Let us define g as in* (3.16), *assume that* (3.17) *is satisfied, and that*

$$w^0 \in H, \qquad h \in L^2(0,T;V'), \tag{3.23}$$

$$\exists a > 0, \exists b \in \boldsymbol{R} : \forall(\vec{v}, \vec{z}) \in \operatorname{graph}(\vec{\gamma}), \quad \vec{z}\cdot\vec{v} \ge a|\vec{z}|^2 + a|\vec{v}|^2 - b. \tag{3.24}$$

Then Problem 3.2 *has at least one solution such that* $w \in L^\infty(0,T;H)$.

The function w may then be identified to a weakly continuous mapping $[0,T] \to H$.

PROOF. – The existence of a solution of the equivalent Problem 3.1 was already proved e.g. in [26]. Here we use an argument that refers to the null-minimization Problem 3.2; this will also pave the way to the proof of the stability

Theorem 3.3. We shall approximate Problem 3.1 by an implicit time-discretization scheme, reformulate it as a minimization principle analogous to Problem 3.2, derive a priori estimates, and pass to the limit in this variational formulation.

(i) Approximation. Let us fix any $m \in \boldsymbol{N}$, set $k := T/m$, and

$$w_m^0 := w^0, \qquad h_m^n := \frac{1}{k} \int_{(n-1)k}^{nk} h(\tau) d\tau \qquad \text{in } V', \text{ for } n = 1, \dots, m. \tag{3.25}$$

PROBLEM 3.1_{m}. – *Find $u_m^n \in V, w_m^n \in L^2(\Omega), \vec{z}_m^{\,n} \in L^2(\Omega)^N$ for $n = 1, \dots, m$, such that*

$$w_m^n - k\nabla \cdot \vec{z}_m^{\,n} = w_m^{n-1} + kh_m^n \qquad \text{in } V', \tag{3.26}$$

$$w_m^n \in \alpha(u_m^n) \qquad \text{a.e. in } \Omega, \tag{3.27}$$

$$\vec{z}_m^{\,n} \in \vec{\gamma}(\nabla u_m^n) \qquad \text{a.e. in } \Omega. \tag{3.28}$$

Existence of a solution of this problem can be proved step by step. Defining the operator

$$\mathcal{B} : V \to \mathcal{P}(V') : v \mapsto \alpha(v) - k\nabla \cdot \vec{\gamma}(\nabla v), \tag{3.29}$$

the equation (3.26) reads $\mathcal{B}(u_m^n) \ni w_m^{n-1} + kh_m^n$ in V' for $n = 1, \dots, m$. By (3.24), the operator $\mathcal{B}$ is monotone and coercive on V; it is then maximal monotone and surjective. Problem 3.1_m thus has a solution.

In view of providing a time-continuous reformulation of Problem 3.1_m, let us set

$$w_m(x, \cdot) := \text{ linear time-interpolate of } \{w_m(x, nk) := w_m^n(x)\}_{n=0,\dots,m}, \tag{3.30}$$

$$\bar{w}_m(x, t) := w_m^n(x) \qquad \text{if } (n-1)k < t \le nk, \text{ for } n = 1, \dots, m, \tag{3.31}$$

for a.e. $x \in \Omega$, and define $\bar{u}_m, \bar{h}_m$ similarly. The system (3.26)-(3.28) thus also reads

$$D_t w_m - \nabla \cdot \bar{\vec{z}}_m = \bar{h}_m \qquad \text{in } V', \text{ a.e. in }]0, T[, \tag{3.32}$$

$$\bar{w}_m \in \alpha(\bar{u}_m) \qquad \text{a.e. in } Q, \tag{3.33}$$

$$\bar{\vec{z}}_m \in \vec{\gamma}(\nabla \bar{u}_m) \qquad \text{a.e. in } Q. \tag{3.34}$$

(ii) Reformulation and a Priori Estimates. Similarly to what we saw for Problems 3.1 and 3.2, the system (3.32)-(3.34) is equivalent to the inequality

$$\begin{aligned} &\Phi_m(\bar{u}_m, \bar{w}_m) := \int_0^T \left[R(\bar{u}_m) + R^*(\bar{w}_m)\right] dt - \iint_Q \bar{w}_m \bar{u}_m \, dxdt + \\ &\left\{ \int_0^T \left[\varphi(\bar{u}_m, \bar{h}_m - D_t w_m) - \langle \bar{h}_m, \bar{u}_m \rangle\right] dt + R^*(w_m(\cdot, T)) - R^*(w^0) \right\}^+ \le 0. \end{aligned} \tag{3.35}$$

In passing note that this inequality is equivalent to the separate null-minimization of two functionals: the difference of the first two integrals, and the bracketed term. Note that, denoting the N-dimensional measure of Ω by $|\Omega|$,

$$R^*(w_m(\cdot,T)) \overset{(3.17)}{\geq} L\int_\Omega |w_m(\cdot,T)|^2\,dx - M|\Omega|,$$

$$\int_0^T \varphi(\bar{u}_m, \bar{h}_m - D_t w_m)\,dt \overset{(3.12)}{\geq} \int_0^T \langle \bar{u}_m, \bar{h}_m - \nabla\cdot\bar{\vec{z}}_m\rangle\,dt,$$

$$\begin{aligned}\int_0^T \langle \bar{u}_m, -\nabla\cdot\bar{\vec{z}}_m\rangle\,dt &= \int_\Omega \nabla\bar{u}_m\cdot\bar{\vec{z}}_m\,dx \\ &\overset{(3.24),(3.34)}{\geq} a\|\nabla\bar{u}_m\|^2_{L^2(\Omega)^N} + a\|\bar{\vec{z}}_m\|^2_{L^2(\Omega)^N} - b|\Omega|.\end{aligned} \tag{3.36}$$

Moreover, in the last four formulas T might be replaced by any $\widetilde{T} \in\,]0,T]$. By inserting the three latter formulas into (3.35) and recalling (3.23), a simple calculation then yields

$$\|\bar{u}_m\|_{L^2(0,T;V)}, \|\bar{w}_m\|_{L^\infty(0,T;H)}, \|\bar{\vec{z}}_m\|_{L^2(\Omega)^N} \leq \text{Constant (independent of } m). \tag{3.37}$$

By comparing the terms of (3.32), we then get

$$\|\bar{w}_m\|_{H^1(0,T;V')} \leq \text{Constant (independent of } m). \tag{3.38}$$

(iii) Passage to the Limit. By these estimates there exist u, w such that, possibly taking $m \to \infty$ along a subsequence, (1)

$$u_m, \bar{u}_m \rightharpoonup u \qquad \text{in } L^2(0,T;V), \tag{3.39}$$

$$\bar{w}_m \overset{*}{\rightharpoonup} w \qquad \text{in } L^\infty(0,T;H)\cap H^1(0,T;V'), \tag{3.40}$$

$$w_m(\cdot,T) \rightharpoonup w(\cdot,T) \qquad \text{in } H. \tag{3.41}$$

By the regularity of Ω, the injection $V \to H$ is compact; the same then applies to the injection $H \to V'$ (by Schauder's theorem). By the classical Lions-Aubin compactness lemma (see e.g. [59]), (3.40) then entails that

$$w_m, \bar{w}_m \to w \qquad \text{in } L^2(0,T;V'); \tag{3.42}$$

(1) We denote the strong, weak, and weak star convergence respectively by $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$.

joined with (3.39), this yields

$$\iint_Q \bar{w}_m \bar{u}_m \, dxdt = \int_0^T \langle \bar{w}_m, \bar{u}_m \rangle \, dt \to \int_0^T \langle w, u \rangle \, dt = \iint_Q wu \, dxdt. \tag{3.43}$$

By passing to the inferior limit in (3.35), we then get (3.19). □

4. – Structural Stability of Problem 3.2

In this section we deal with the dependence of the solution $\mathcal{S} := (u, w)$ of Problem 3.1 on the data $\mathcal{D} := (w^0, h)$ and on the operators $\nabla, \widehat{\alpha}, \widehat{\vec{\gamma}}$ (this might also be extended to include dependence on the operator D_t). Here by

$$\widehat{\alpha} : H \to H, \qquad \widehat{\vec{\gamma}} : L^2(\Omega)^N \to L^2(\Omega)^N$$

we denote the operators that are respectively associated to the mappings α and $\vec{\gamma}$, respectively. The hypotheses on $\widehat{\alpha}$ may also be expressed in terms of its potential R, see (3.16)-(3.18). We shall show the structural stability of Problem 3.2, in the sense that we illustrated in the introduction. First we deal with variations of the two nonlinear operators above.

THEOREM 4.1. – *Let* $\{\alpha_m\}$, $\{\vec{\gamma}_m\}$, $\{h_m\}$, $\{w^0_m\}$ *be sequences such that:*

(i) *For any* m, $\alpha_m = \partial g_m$ *with* g_m *as in* (3.16), *and the mapping* $\vec{\gamma}_m : \boldsymbol{R}^N \to \mathcal{P}(\boldsymbol{R}^N)$ *is maximal monotone; similar properties are assumed for* α, $\vec{\gamma}$ *and* g.

(ii) *The sequences* $\{g_m\}$ *and* $\{\vec{\gamma}_m\}$ *fulfill* (3.17) *and* (3.24) *uniformly with respect to* m, *and*

$$w^0_m \to w^0 \quad in\ H, \qquad h_m \to h \quad in\ L^2(0, T; V'). \tag{4.1}$$

(iii) *There exists a lower semicontinuous and convex functional* $R : H \to \boldsymbol{R} \cup \{+\infty\}$ *such that (defining* R_m *as in* (3.18))

$$\forall\ sequence\ \{v_m\}, \quad v_m \to v\ in\ H \quad \Rightarrow \quad R^*_m(v_m) \to R^*(v), \tag{4.2}$$

$$\begin{gathered} \forall\ sequence\ \{v_m\},\ if\ v_m \rightharpoonup v\ in\ L^2(0, T; V), \\ then\ \liminf_{m\to\infty} \int_0^T R_m(v_m)\, dt \geq \int_0^T R(v)\, dt, \end{gathered} \tag{4.3}$$

$$\begin{gathered} \forall\ sequence\ \{v'_m\},\ if\ v'_m \to v'\ in\ L^2(0, T; V'), \\ then\ \liminf_{m\to\infty} \int_0^T R^*_m(v'_m)\, dt \geq \int_0^T R^*(v')\, dt. \end{gathered} \tag{4.4}$$

(iv) *For any m, let* $\varphi_m \in \mathcal{F}(V)$ *represent the operator* $v \mapsto -\nabla \cdot \vec{\gamma}_m(\nabla v)$, *cf.* (2.8); *define* φ *similarly with* $\vec{\gamma}$ *in place of* $\vec{\gamma}_m$, *and assume that*

$$\begin{aligned}
&\forall \text{ sequence } \{(v_m, v'_m)\}, \text{ if } (v_m, v'_m) \rightharpoonup (v, v') \text{ in } L^2(0,T; V \times V') \\
&\text{and } \limsup_{m\to\infty} \int_0^T \langle v'_m, v_m\rangle \, dt \le \int_0^T \langle v', v\rangle \, dt, \qquad (4.5)\\
&\text{then } \limsup_{m\to\infty} \int_0^T \varphi_m(v_m, v'_m)\, dt \ge \int_0^T \varphi(v, v')\, dt.
\end{aligned}$$

For any m, let (u_m, w_m) *be a solution of Problem* 3.2_m *corresponding to the data* α_m, $\vec{\gamma}_m$, h_m, w^0_m. *Then there exists a pair* (u, w) *such that, up to subsequences,*

$$u_m \rightharpoonup u \qquad \text{in } L^2(0,T;V), \tag{4.6}$$

$$w_m \overset{*}{\rightharpoonup} w \qquad \text{in } L^\infty(0,T;H) \cap H^1(0,T;V'), \tag{4.7}$$

$$w_m(\cdot, T) \rightharpoonup w(\cdot, T) \qquad \text{in } H. \tag{4.8}$$

Moreover, this entails that (u, w) *is a solution of Problem* 3.2 *corresponding to the data* α, $\vec{\gamma}$, h, w^0.

Note that (4.5) is fulfilled whenever the functional $\int_0^T \varphi_m \, dt$ Γ-converges to $\int_0^T \varphi \, dt$ with respect to the product of the weak topology of V by the weak star topology of V'. But (4.5) is a weaker hypothesis, because of the condition that is written in the second line of (4.5). This will allow us to take advantage of the specific form of the P.D.E., see (4.11) below.

PROOF. – Uniform estimates like (3.37) and (3.38) may be derived by mimicking the procedure of part (ii) of the proof of Theorem 3.2; these details are here left to the interested reader. The convergences (4.6)-(4.8) then hold up to subsequences.

Let us next label by the index m any equation written in terms of $\alpha_m, \vec{\gamma}_m$, and so on. As above, by the Lions-Aubin compactness lemma, (4.7) entails (3.42) (here written without the bars); (3.43) then follows as above. By (3.43) and (4.4), we then get

$$\begin{aligned}
&\int_0^T \big[R(u) + R^*(w)\big]\, dt - \iint_Q wu \, dxdt \\
&\le \liminf_{m\to\infty} \int_0^T \big[R_m(u_m) + R^*_m(w_m)\big]\, dt - \lim_{m\to\infty} \iint_Q w_m u_m \, dxdt \qquad (4.9)\\
&\overset{(3.43)}{=} \liminf_{m\to\infty} \left\{ \int_0^T \big[R_m(u_m) + R^*_m(w_m)\big]\, dt - \iint_Q w_m u_m \, dxdt \right\} \overset{(3.35)}{\le} 0,
\end{aligned}$$

namely, (3.20). On the other hand, as $w_m(\cdot, T) \to w(\cdot, T)$ in V' because of the compactness of the injection $H \to V'$,

$$\liminf_{m\to\infty} R^*_m(w_m(\cdot, T)) \overset{(4.4),(4.8)}{\geq} R^*(w(\cdot, T)); \tag{4.10}$$

moreover,

$$\begin{aligned} &\limsup_{m\to\infty} \int_0^T \langle h_m - D_t w_m, u_m\rangle \, dt \\ &\overset{(3.4)_m(3.22)}{=} \lim_{m\to\infty} \int_0^T \langle h_m, u_m\rangle \, dt - \liminf_{m\to\infty} R^*_m(w_m(\cdot, T)) + \lim_{m\to\infty} R^*_m(w^0_m) \\ &\overset{(4.1),(4.2),(4.10)}{\leq} \int_0^T \langle h, u\rangle \, dt - R^*(w(\cdot, T)) + R^*(w^0) \overset{(3.22)}{=} \int_0^T \langle h - D_t w, u\rangle \, dt. \end{aligned} \tag{4.11}$$

We may then apply (4.5) to the sequence $\{(u_m, h_m - D_t w_m)\}$, and thus get

$$\limsup_{m\to\infty} \int_0^T \varphi_m(u_m, h_m - D_t w_m) \, dt \geq \int_0^T \varphi(u, h - D_t w) \, dt. \tag{4.12}$$

By passing to the upper limit in $(3.21)_m$, we finally get

$$\begin{aligned} &\int_0^T \varphi(u, h - D_t w) \, dt + R^*(w(\cdot, T)) \\ &\overset{(4.10),(4.12)}{\leq} \limsup_{m\to\infty} \int_0^T \varphi_m(u_m, h_m - D_t w_m) \, dt + \liminf_{m\to\infty} R^*_m(w_m(\cdot, T)) \\ &\leq \limsup_{m\to\infty} \left\{ \int_0^T \varphi_m(u_m, h_m - D_t w_m) \, dt + R^*_m(w_m(\cdot, T)) \right\} \\ &\overset{(3.21)_m}{\leq} \limsup_{m\to\infty} R^*_m(w^0_m) + \int_0^T \langle h_m, u_m\rangle \, dt \overset{(4.1),(4.6)}{=} R^*(w^0) + \int_0^T \langle h, u\rangle \, dt. \end{aligned} \tag{4.13}$$

The inequality (3.21) is thus established. □

4.1 – *Variations of the Differential Operators and of the Domain*

Variations of the operator ∇ are accounted for by variations of the potential φ, see (4.5). This may also include approximations of interest for the numerical analysis of the problem, e.g. by finite difference or by finite element.

The operator D_t may also be varied; for instance, for any $m \in \boldsymbol{N}$ one may set

$$D_m v(t) := m[v(t) - v(t - T/m)], \tag{4.14}$$

and replace the exact time-derivative D_t by the discretized derivative D_m in $(3.19)_m$ (after extending the function u for $t<0$).

The domain Ω may be replaced by a sequence of Lipschitz domains $\{\Omega_m\}$. In this case we extend any functions $\Omega_m \to \boldsymbol{R}$ to the whole $\boldsymbol{R}^N$ with vanishing value, thus preserving the H^1_0-regularity. Denoting the indicator function of V_m by I_{V_m}, we may thus reformulate (3.1) in Ω_m as

$$D_t\alpha(u) - \nabla\cdot\vec{\gamma}(\nabla u) + \partial I_{V_m}(u) \ni h \qquad \text{in } \mathcal{D}'(\boldsymbol{R}^N). \tag{4.15}$$

The extension of the associated Problem 3.2 is obvious.

REMARKS. – (i) If $\vec{\alpha} : \boldsymbol{R}^M \to \mathcal{P}(\boldsymbol{R}^M)$ and $\gamma : \boldsymbol{R}^{M\times N} \to \mathcal{P}(\boldsymbol{R}^{M\times N})$, the above analysis may easily be extended. In this case $\vec{\alpha}$ must explicitly be assumed to be cyclically monotone.

(ii) The results of this sections might easily be extended to several other equations. These include the case of time-dependent α and $\vec{\gamma}$, and abstract equations of the form

$$D_t\alpha(u) + \beta(u) \ni h \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{4.16}$$

for a maximal monotone operator $\beta : V \to \mathcal{P}(V')$, which might also explicitly depend on time.

For instance, taking $V = L^2(\Omega) \subset H := H^{-1}(\Omega) = H' \subset V' = H^{-2}(\Omega)$, (4.16) also encompasses the equation

$$D_t u - \Delta\alpha(u) = f \qquad \text{in } H^{-2}(\Omega), \text{ a.e. in }]0,T[, \tag{4.17}$$

that may represent the weak formulation of the classical Stefan problem, see e.g. [62].

(iii) As (4.5) involves weak convergences, it does not seem obvious to retrieve this condition from properties either of the corresponding time-independent operators $v \mapsto -\nabla\cdot\vec{\gamma}_m(\nabla v)$ or of the mappings $\vec{\gamma}_m$s. Similarly, it is not clear how (4.2)-(4.4) might be reduced to properties of the mappings α_ms.

(iv) The above analysis may also be extended if the elliptic term is of the form $-\nabla\cdot\vec{\gamma}_u(\nabla u)$, where the multivalued mapping $V \to \mathcal{P}(L^2(\Omega)^N) : u \mapsto \vec{\gamma}_u(\vec{v})$ is weakly closed for any $\vec{v} \in L^2(\Omega)^N$.

(v) The use of the Murat and Tartar notion of *compensated compactness* (see e.g. [48,61]) allows one also to deal with equations in which the injection $V \to H$ is not compact. This includes the following system, that occurs by coupling the Maxwell equations of electromagnetism with two nonlinear constitutive relations:

$$
\begin{cases} D_t\vec{B} + \nabla \times \vec{E} = \vec{0}, \\ \vec{J} = \nabla \times \vec{H}, \\ \vec{E} \in \vec{\gamma}(\vec{J}) + \vec{g}, \\ \vec{B} \in \vec{\alpha}(\vec{H}). \end{cases} \tag{4.18}
$$

Here $\vec{B}$ is the magnetic induction, $\vec{H}$ is the magnetic field, $\vec{E}$ is the electric field, and $\vec{J}$ is the density of electric current. $\vec{\gamma}$ accounts for a nonlinear Ohm's conduction law, and a multivalued $\vec{\alpha}$ may represent the ferromagnetic behavior.

(vi) If the mapping $\vec{\gamma}$ is cyclically monotone, one may prove the further regularity

$$
u \in H^1(0,T;H) \cap L^\infty(0,T;V). \tag{4.19}
$$

A uniform estimate for u_m in this space may indeed be derived by multiplying the approximate equation (3.32) by $D_t u_m$, provided that the mapping α is strongly monotone, and under stronger regularity hypotheses for the data. When dealing with a vector field $\vec{u}$, this applies also if $\vec{\alpha}$ is not cyclically monotone; in this case in the minimization principle (3.19) the Fenchel function $g(u) + g^*(w)$ must be replaced by any representative function of $\vec{\alpha}$. □

5. – Another Class of Doubly-Nonlinear Parabolic Equations

In this section we study an initial- and boundary-value problem for a doubly-nonlinear parabolic equation of the form

$$
\alpha(D_t u) - \nabla\cdot\vec{\gamma}(\nabla u) \ni h \qquad \text{in } Q. \tag{5.1}
$$

As in the previous section, $\alpha : \boldsymbol{R} \to \mathcal{P}(\boldsymbol{R})$ and $\vec{\gamma} : \boldsymbol{R}^N \to \mathcal{P}(\boldsymbol{R}^N)$ are maximal monotone mappings, Ω is a bounded domain of Lipschitz class, and h is a prescribed field. We reformulate this inclusion as a null-minimization problem, prove existence of a variational solution and its structural stability with respect to variations of the data and of the operators D_t, ∇, α and $\vec{\gamma}$. In spite of several analogies, the analysis of this problem will differ from that of the previous section. The structure of the two equations is indeed different, and here we deal with a stronger notion of solution: $D_t u \in L^p(Q)$.

The inclusion (5.1) is tantamount to the following system:

$$
w - \nabla\cdot\vec{z} = h \qquad \text{in } Q, \tag{5.2}
$$

$$
w \in \alpha(D_t u) \qquad \text{in } Q, \tag{5.3}
$$

$$
\vec{z} \in \vec{\gamma}(\nabla u) \qquad \text{in } Q. \tag{5.4}
$$

5.1 – *Weak Formulation*

We define V, H, V' as in (3.6), and fix any $p \in [2,6[$. Thus, setting $p' = p/(p-1) \in]6/5, 2]$,

$$V = H^1_0(\Omega) \subset L^p(\Omega), \qquad L^{p'}(Q) \subset V' = H^{-1}(\Omega).$$

We assume that

$$u^0 \in V, \qquad h \in L^{p'}(Q), \tag{5.5}$$

$$\exists c_5, c_6 \in \boldsymbol{R}^+ : \forall v \in \boldsymbol{R}, \qquad |\alpha(v)| \le c_5 + c_6 |v|^{p-1}, \tag{5.6}$$

$$\exists c_7, c_8 \in \boldsymbol{R}^+ : \forall \vec{v} \in \boldsymbol{R}^N, \quad |\vec{\gamma}(\vec{v})| \le c_7 + c_8 |\vec{v}|, \tag{5.7}$$

and introduce a weak formulation of the Cauchy problem for the system (5.2)-(5.4).

PROBLEM 5.1. – *Find* $u \in W^{1,p}(0,T; L^p(\Omega)) \cap L^2(0,T;V)$, $w \in L^{p'}(Q)$ *and* $\vec{z} \in L^2(Q)^N$ *such that*

$$\iint_Q (wv + \vec{z} \cdot \nabla v)\, dxdt = \iint_Q hv\, dxdt \qquad \forall v \in L^2(0,T;V) \cap L^p(Q), \tag{5.8}$$

$$w \in \alpha(D_t u) \qquad a.e.\ in\ Q, \tag{5.9}$$

$$\vec{z} \in \vec{\gamma}(\nabla u) \qquad a.e.\ in\ Q, \tag{5.10}$$

$$u(\cdot, 0) = u^0 \qquad a.e.\ in\ \Omega. \tag{5.11}$$

Note that (5.8) is equivalent to (5.2) in $L^2(0,T;V') + L^{p'}(Q)$. Let $\psi \in \mathcal{F}(\boldsymbol{R}^N)$ represent the maximal monotone mapping α. In analogy with (3.12)-(3.15), one easily sees that $(v, v') \mapsto \int_0^T \psi(v, v')\, dt$ then represents the maximal monotone operator $\widehat{\alpha} : L^p(\Omega) \to \mathcal{P}(L^{p'}(\Omega))$, namely,

$$\iint_Q \psi(v, v')\, dxdt \ge \iint_Q v'v\, dxdt \qquad \forall (v, v') \in L^p(Q) \times L^{p'}(Q), \tag{5.12}$$

$$\iint_Q \psi(v, v')\, dxdt = \iint_Q v'v\, dxdt \quad \Leftrightarrow \quad v' \in \alpha(v) \text{ a.e. in } Q. \tag{5.13}$$

We shall assume that $\vec{\gamma}$ is cyclically monotone, i.e., $\vec{\gamma} = \partial r$, the subdifferential of a lower semicontinuous convex function $r : \boldsymbol{R}^N \to \boldsymbol{R} \cup \{+\infty\}$. We also suppose that

$$\begin{aligned} &\exists c_9 > 0, \exists c_{10} \in \boldsymbol{R} : \forall \vec{v} \in \boldsymbol{R}^N, \quad r(\vec{v}) \ge c_9 |\vec{v}|^2 - c_{10}, \\ &r(\nabla u^0) < +\infty \qquad \text{a.e. in } \Omega. \end{aligned} \tag{5.14}$$

5.2 – *Null-Minimization*

Next we reformulate Problem 5.1.

PROBLEM 5.2. – *Find* $u \in W^{1,p}(0,T;L^p(\Omega)) \cap L^2(0,T;V), \vec{z} \in L^2(Q)^N$ *such that* $\nabla \cdot \vec{z} \in L^{p'}(Q)$, (5.11) *is fulfilled, and*

$$\Psi(u,\vec{z}) := \iint_Q \left[r(\nabla u) + r^*(\vec{z}) - \vec{z}\cdot\nabla u\right] dxdt + \left\{\iint_Q \left[\psi(D_t u, h + \nabla\cdot\vec{z}) - hD_t u\right] dxdt + \int_\Omega \left[r(\nabla u(\cdot,T)) - r(\nabla u^0)\right] dx\right\}^+ \leq 0. \tag{5.15}$$

Note that the functional Ψ is convex. As the first integral is nonnegative for any pair of functions u and $\vec{z}$, (5.15) is a null-minimization principle, cf. (1.12).

As $r(\nabla u) + r^*(\vec{z}) \geq \vec{z}\cdot\nabla u$ for any u and $\vec{z}$, the inequality (5.15) is equivalent to the system

$$\iint_Q \left[r(\nabla u) + r^*(\vec{z}) - \vec{z}\cdot\nabla u\right] dxdt \leq 0, \tag{5.16}$$

$$\iint_Q \left[\psi(D_t u, h + \nabla\cdot\vec{z}) - hD_t u\right] dxdt + \int_\Omega \left[r(\nabla u(\cdot,T)) - r(\nabla u^0)\right] dx \leq 0. \tag{5.17}$$

PROPOSITION 5.1. – *If the triplet* $(u, w, \vec{z})$ *is a solution of Problem* 5.1, *then* $\nabla \cdot \vec{z} \in L^{p'}(Q)$ *and the pair* $(u,\vec{z})$ *solves Problem* 5.2. *Conversely, if* $(u,\vec{z})$ *is a solution of Problem* 5.2, *then there exists* $w \in L^{p'}(Q)$ *such that* $(u, w, \vec{z})$ *solves Problem* 5.1.

PROOF. – First note that (5.8) entails (5.2) in the sense of distributions, and a comparison of the terms of this equation yields $\nabla \cdot \vec{z} \in L^{p'}(Q)$.

We already pointed out that (5.16) is equivalent to the inclusion (5.10). It then suffices to show that, defining $\nabla \cdot \vec{z}$ via (5.2) and assuming (5.10) and (5.11), the inclusion (5.9) is equivalent to (5.17).

For any $u \in W^{1,p}(0,T;L^p(\Omega))$ and any $\vec{z} \in L^2(Q)^N$ such that $\nabla \cdot \vec{z} \in L^{p'}(Q)$, by (5.12)

$$\iint_Q \psi(D_t u, h + \nabla\cdot\vec{z})\, dxdt \geq \iint_Q \left[hD_t u + (\nabla\cdot\vec{z})D_t u\right] dxdt; \tag{5.18}$$

the opposite inequality is equivalent to (5.9). By the chain rule, cf. (3.22),

$$\int_\Omega r(\nabla u)\, dx \in W^{1,1}(0,T),$$

$$\frac{d}{dt}\int_\Omega r(\nabla u)\, dx \overset{(5.10)}{=} \int_\Omega \vec{z}\cdot D_t\nabla u\, dx = -\int_\Omega (\nabla\cdot\vec{z})D_t u\, dx \qquad \text{a.e. in }]0,T[.$$

By (5.11), the inequality (5.18) then also reads

$$(5.19)\qquad \iint_Q \big[\psi(D_t u, h + \nabla\cdot\vec{z}) - hD_t u\big]\, dxdt + \int_\Omega \big[r(\nabla u(\cdot,T)) - r(\nabla u^0)\big]\, dx \geq 0;$$

assuming (5.10) and (5.11), the opposite inequality is equivalent to (5.9). □

THEOREM 5.2. – *(Existence) Assume that* (5.5)-(5.7) *and* (5.14) *are satisfied, and that*

$$(5.20)\qquad \exists c_{11}, c_{12} > 0 : \forall (u,w) \in \text{graph}\,(\alpha) \quad wu \geq c_{11}|u|^p + c_{11}|w|^{p'} - c_{12}.$$

Then Problem 5.2 *has at least one solution such that* $u \in L^\infty(0,T;V)$ *and* $\vec{z} \in L^\infty(0,T;L^2(\Omega)^N)$.

The function u may then be identified to a weakly continuous mapping $[0,T] \to V$.

PROOF. – The existence of a solution of the equivalent Problem 5.1 was already proved e.g. in [23]; for Problem 5.2 here we use a different argument, that rests on the approximation of this null-minimization principle. We shall meet a similar procedure in the stability Theorem 6.1. Let us first notice that, by (5.12) and (5.20),

$$(5.21)\qquad \iint_Q \psi(v,v')dxdt \geq \iint_Q \big(c_{11}|v|^p + c_{11}|v'|^{p'} - c_{12}\big)dxdt \quad \forall (v,v') \in L^p(\Omega)\times L^{p'}(\Omega).$$

Next we fix any $m \in \boldsymbol{N}$, set $k := T/m$, $u_m^0 := u^0$, define h_m^n as in (3.25), and approximate Problem 5.1 by an implicit time-discretization scheme.

PROBLEM $5.1_{\rm m}$. – *Find* $u_m^n \in V$, $w_m^n \in L^{p'}(\Omega)$ *and* $\vec{z}_m^{\,n} \in L^2(\Omega)^N$ *for* $n = 1, \ldots, m$, *such that for any* n

$$(5.22)\qquad w_m^n - \nabla\cdot\vec{z}_m^{\,n} = h_m^n \qquad \textit{in } V',$$

$$(5.23)\qquad w_m^n \in \alpha((u_m^n - u_m^{n-1})/k) \qquad \textit{a.e. in } \Omega,$$

$$(5.24)\qquad \vec{z}_m^{\,n} \in \vec{\gamma}(\nabla u_m^n) \qquad \textit{a.e. in } \Omega.$$

Existence of a solution of this problem can be proved step by step, as we did for Problem 3.1_m. Defining the operator

$$(5.25)\qquad \mathcal{C}_m^n : V \to \mathcal{P}(V') : v \mapsto \alpha((v - u_m^{n-1})/k) - \nabla\cdot\vec{\gamma}(\nabla v),$$

the system (5.22)-(5.24) reads $\mathcal{C}_m^n(u_m^n) \ni h_m^n$ in V' for any n. By (5.14) and (5.20), the operator $\mathcal{C}_m^n$ is monotone and coercive, hence also maximal monotone. Problem 5.1$_m$ then has a solution. In view of reformulating Problem 5.1$_m$, let us define time-interpolate functions as in (3.30) and (3.31), so that the system (5.22)-(5.24) also reads

$$\bar{w}_m - \nabla\cdot\bar{\vec{z}}_m = \bar{h}_m \qquad \text{in } L^2(0,T;V') + L^{p'}(Q), \tag{5.26}$$

$$\bar{w}_m \in \alpha(D_t u_m) \qquad \text{a.e. in } Q, \tag{5.27}$$

$$\bar{\vec{z}}_m \in \vec{\gamma}(\nabla \bar{u}_m) \qquad \text{a.e. in } Q. \tag{5.28}$$

Similarly to what we saw for Problem 5.1, this system is equivalent to the following inequality

$$\begin{aligned} \Psi(\bar{u}_m, \bar{\vec{z}}_m) :=& \iint_Q \left[r(\nabla \bar{u}_m) + r^*(\bar{\vec{z}}_m) - \bar{\vec{z}}_m \cdot \nabla \bar{u}_m \right] dxdt + \\ & \left\{ \iint_Q \left[\psi(D_t u_m, \bar{h}_m + \nabla\cdot\bar{\vec{z}}_m) - \bar{h}_m D_t u_m \right] dxdt \right. \\ & \left. + \int_\Omega \left[r(\nabla \bar{u}_m(\cdot,T)) - r(\nabla u^0) \right] dx \right\}^+ \le 0. \end{aligned} \tag{5.29}$$

Notice that here T might be replaced by any $\widetilde{T} \in \,]0,T]$. By (5.5) and by the coerciveness properties (5.14) and (5.21), we then get

$$\|D_t u_m\|_{L^p(Q)}, \|\nabla\cdot\bar{\vec{z}}_m\|_{L^{p'}(Q)}, \|\bar{u}_m\|_{L^\infty(0,T;V)} \le \text{Constant (independent of } m). \tag{5.30}$$

By (5.5) and (5.24), then

$$\|\bar{\vec{z}}_m\|_{L^\infty(0,T;L^2(\Omega)^N)} \le \text{ Constant (independent of } m). \tag{5.31}$$

By these estimates, there exist $u, \vec{z}$ such that, possibly taking $m \to \infty$ along a subsequence,

$$u_m \overset{*}{\rightharpoonup} u \qquad \text{in } W^{1,p}(0,T;L^p(\Omega)) \cap L^\infty(0,T;V), \tag{5.32}$$

$$\bar{\vec{z}}_m \overset{*}{\rightharpoonup} \vec{z} \qquad \text{in } L^\infty(0,T;L^2(\Omega)^N), \tag{5.33}$$

$$\nabla\cdot\bar{\vec{z}}_m \rightharpoonup \nabla\cdot\vec{z} \qquad \text{in } L^{p'}(Q), \tag{5.34}$$

$$u_m(\cdot,T) \rightharpoonup u(\cdot,T) \qquad \text{in } V. \tag{5.35}$$

As $p \in [2,6[$ and by the regularity of $\Omega \subset \boldsymbol{R}^3$, $V = H_0^1(\Omega) \subset L^p(\Omega)$ with compact injection. By the classical Lions-Aubin lemma (see e.g. [59]), (5.32) then yields

$$\bar{u}_m \to u \qquad \text{in } L^p(Q), \tag{5.36}$$

so that by (5.34)

$$\iint_Q \bar{\vec{z}}_m \cdot \nabla \bar{u}_m \, dxdt = -\iint_Q (\nabla \cdot \bar{\vec{z}}_m)\bar{u}_m \, dxdt \tag{5.37}$$
$$\to -\iint_Q (\nabla \cdot \vec{z})u \, dxdt = \iint_Q \vec{z} \cdot \nabla u \, dxdt.$$

By passing to the inferior limit in (5.29), we then get (5.15). □

6. – Structural Stability of Problem 5.2

In this section we prove that Problem 5.2 is structurally stable, in the sense that we illustrated in the introduction, and along the lines of Sect. 4.

THEOREM 6.1. – *Let* $\{\alpha_m\}$, $\{\vec{\gamma}_m\}$, $\{h_m\}$, $\{u_m^0\}$ *be sequences such that:*

(i) *For any* m, $\alpha_m : \boldsymbol{R} \to \mathcal{P}(\boldsymbol{R})$ *is maximal monotone and* $\vec{\gamma}_m = \partial r_m$, *with* $r_m : \boldsymbol{R}^N \to \boldsymbol{R} \cup \{+\infty\}$ *lower semicontinuous and convex; similar properties are assumed for* α, $\vec{\gamma} = \partial r$ *and* r.

(ii) *Let* $p \in [2, 6[$, *and the sequences* $\{\alpha_m\}$ *and* $\{\vec{\gamma}_m\}$ *fulfill* (5.5)-(5.7), (5.14), (5.20) *uniformly with respect to* m. *Assume that*

$$u_m^0 \to u^0 \quad \text{in } V, \qquad h_m \to h \quad \text{in } L^{p'}(Q), \tag{6.1}$$

$$\forall \text{ sequence } \{(\vec{v}_m, \vec{v}\,'_m)\}, \text{ if } (\vec{v}_m, \vec{v}\,'_m) \rightharpoonup (\vec{v}, \vec{v}\,') \text{ in } \left(L^2(Q)^N\right)^2 \tag{6.2}$$
$$\text{and } \iint_Q \vec{v}_m \cdot \vec{v}\,'_m \, dxdt \to \iint_Q \vec{v} \cdot \vec{v}\,' \, dxdt, \text{ then}$$
$$\liminf_{m\to\infty} \iint_Q [r_m(\vec{v}_m) + r_m^*(\vec{v}\,'_m)] \, dxdt \geq \iint_Q [r(\vec{v}) + r^*(\vec{v}\,')] \, dxdt,$$

$$\forall \text{ sequence } \{\vec{v}_m\}, \quad \vec{v}_m \to \vec{v} \text{ in } L^2(\Omega)^N \Rightarrow \int_\Omega r_m(\vec{v}_m) \, dx \to \int_\Omega r(\vec{v}) \, dx, \tag{6.3}$$

$$\forall \text{ sequence } \{\vec{v}_m\}, \quad \vec{v}_m \rightharpoonup \vec{v} \text{ in } L^2(\Omega)^N \Rightarrow \liminf_{m\to\infty} \int_\Omega r_m(\vec{v}_m) \, dx \geq \int_\Omega r(\vec{v}) \, dx. \tag{6.4}$$

(iii) *For any* m, *let* ψ_m *(*ψ*, resp.)* $\in \mathcal{F}(L^p(\Omega))$ *represent the operator that is associated to the mapping* α_m *(*α*, resp.), and assume that*

$$
(6.5)\qquad
\begin{aligned}
&\forall \textit{ sequence } \{(v_m, v'_m)\},\ \textit{if } (v_m, v'_m) \rightharpoonup (v, v) \textit{ in } L^p(Q)\times L^{p'}(Q)\\
&\textit{and } \limsup_{m\to\infty} \int_0^T \langle v'_m, v_m\rangle\, dt \le \int_0^T \langle v', v\rangle\, dt,\\
&\textit{then } \limsup_{m\to\infty} \iint_Q \psi_m(v_m, v'_m)\, dxdt \ge \iint_Q \psi(v, v')\, dxdt.
\end{aligned}
$$

For any m, let $(u_m, \vec{z}_m)$ be a solution of the corresponding Problem 5.2_m. Then there exists a pair $(u, \vec{z})$ such that, up to subsequences,

$$
(6.6)\qquad u_m \overset{*}{\rightharpoonup} u \qquad \textit{in } W^{1,p}(0,T;L^p(\Omega)) \cap L^\infty(0,T;V),
$$

$$
(6.7)\qquad \vec{z}_m \overset{*}{\rightharpoonup} \vec{z} \qquad \textit{in } L^\infty(0,T;L^2(\Omega)^N),
$$

$$
(6.8)\qquad \nabla\cdot\vec{z}_m \rightharpoonup \nabla\cdot\vec{z} \qquad \textit{in } L^{p'}(Q),
$$

$$
(6.9)\qquad u_m(\cdot,T) \rightharpoonup u(\cdot,T) \qquad \textit{in } V.
$$

Finally, this entails that $(u, \vec{z})$ is a solution of Problem 5.2 corresponding to the data α, $\vec{\gamma}$, h, u^0.

PROOF. – Uniform estimates like (5.30) and (5.31) may be derived mimicking the above procedure; this is here left to the interested reader. The convergences (6.6)-(6.9) then hold up to subsequences.

Let us label by the index m any equation written in terms of $\alpha_m, \vec{\gamma}_m$ and so on. Problem 5.2_m thus consists in coupling the inequality $(5.29)_m$ (here rewritten without the bars) with the initial condition

$$
(6.10)\qquad u(\cdot,0) = u^0_m \qquad \text{a.e. in } \Omega.
$$

By the Lions-Aubin lemma, (6.6) yields (5.36) (this one also without the bar). By (6.2), (6.6) and (6.7),

$$
(6.11)\qquad \liminf_{m\to\infty} \iint_Q [r_m(\nabla\bar{u}_m) + r^*_m(\bar{\vec{z}}_m)]\, dxdt \ge \iint_Q [r(\nabla u) + r^*(\vec{z})]\, dxdt.
$$

By passing to the lower limit in $(5.16)_m$ and recalling (5.37), we then get (5.16). Moreover,

$$
(6.12)\qquad
\begin{aligned}
&\limsup_{m\to\infty} \int_0^T \langle h_m + \nabla\cdot\vec{z}_m, D_t u_m\rangle\, dt\\
&= \lim_{m\to\infty} \int_0^T \langle h_m, D_t u_m\rangle\, dt - \liminf_{m\to\infty} \iint_Q \vec{z}_m \cdot D_t \nabla u_m\, dxdt\\
&\overset{(5.10)_m}{=} \lim_{m\to\infty} \int_0^T \langle h_m, D_t u_m\rangle\, dt - \liminf_{m\to\infty} \int_\Omega [r_m(\nabla u_m(x,T)) - r_m(\nabla u^0_m)]\, dx\\
&\overset{(6.1),(6.3),(6.4)}{\le} \int_0^T \langle h, D_t u\rangle\, dt - \int_\Omega [r(\nabla u(x,T)) - r(\nabla u^0)]\, dx \overset{(5.10)}{=} \int_0^T \langle h + \nabla\vec{z}, D_t u\rangle\, dt.
\end{aligned}
$$

By (6.5), then

$$\limsup_{m\to\infty} \iint_Q \psi_m(D_t u_m, h_m + \nabla\cdot \vec{z}_m)\, dxdt \geq \iint_Q \psi(D_t u, h + \nabla\cdot\vec{z})\, dxdt. \tag{6.13}$$

By passing to the upper limit in $(5.17)_m$, we finally get

$$\begin{aligned} &\iint_Q \psi(D_t u, h + \nabla\cdot\vec{z})\, dxdt + \int_\Omega r(\nabla u(\cdot,T))\, dx \\ &\overset{(6.2),(6.13)}{\leq} \limsup_{m\to\infty} \iint_Q \psi_m(D_t u_m, h_m + \nabla\cdot\vec{z}_m)\, dxdt + \liminf_{m\to\infty} \int_\Omega r_m(\nabla u_m(\cdot,T))\, dx \\ &\overset{(5.17)_m}{=} \limsup_{m\to\infty} \int_\Omega r_m(\nabla u_m^0)\, dx + \lim_{m\to\infty} \iint_Q h_m D_t u_m\, dxdt \\ &\overset{(6.1),(6.3)}{=} \int_\Omega r(\nabla u^0)\, dx + \iint_Q h D_t u\, dxdt. \end{aligned} \tag{6.14}$$

The inequality (5.17) is thus established. □

REMARKS. – (i) Variations of the operators ∇, D_t and of the domain Ω may be dealt with as we pointed out in Sect. 4, see (4.14) and (4.15).

(ii) The first three remarks at the end of Sect. 4 may easily be extended to this section.

(iii) The above analysis may also be extended if the parabolic term is of the form $\alpha_u(D_t u)$, where the multivalued mapping $V \to \mathcal{P}(L^2(\Omega)) : u \mapsto \alpha_u(v)$ is weakly closed for any $v \in L^2(\Omega)$.

(iv) One may deal with the equation (5.1) also if the maximal monotone mapping $\vec{\gamma}$ is not cyclically monotone. In this case an alternative variational formulation is easily given, using any representative function of $\vec{\gamma}$. If $\vec{\gamma}$ is strongly monotone, i.e.

$$\exists c >: \forall(\vec{v}_i, \vec{z}_i) \in \text{graph}\,(\vec{\gamma})\ (i = 1, 2), \quad (\vec{z}_1 - \vec{z}_2)\cdot(\vec{v}_1 - \vec{v}_2) \geq c|\vec{v}_1 - \vec{v}_2|^2, \tag{6.15}$$

then the further regularity $u \in H^1(0, T; V)$ may also be proved. In this case, the cyclical monotonicity of α plays a role, see (3.16).

This is based on an estimate procedure, that we briefly outline operating on the exact equation *formally*. By differentiating (5.1) in time, we have

$$D_t\theta - \nabla\cdot D_t\vec{z} = D_t h \qquad \text{with} \qquad \theta \in \alpha(D_t u). \tag{6.16}$$

Notice that

$$-\int_\Omega (\nabla\cdot D_t\vec{z}) D_t u\, dx = \int_\Omega D_t\vec{z}\cdot D_t\nabla u\, dx \overset{(6.15)}{\geq} c\int_\Omega |D_t\nabla u|^2\, dx.$$

Moreover, as by (3.16) $D_t u \in \alpha^{-1}(\theta) = \partial g^*(\theta)$, we have $D_t\theta D_t u = D_t g^*(\theta)$. By multiplying the equation (6.16) by $D_t u$, one then gets the desired estimate on $\iint_Q |D_t \nabla u|^2 \, dxdt$.

This takes over if u is replaced by a vector field $\vec{u} : Q \to \boldsymbol{R}^M$ with $M > 1$, provided that $\vec{\alpha} : \boldsymbol{R}^M \to \mathcal{P}(\boldsymbol{R}^M)$ is cyclically monotone; namely, $\alpha = \partial g$, with g as in (3.16). In the physical literature g is known as the *dissipation potential.*

(v) If α is homogeneous of zero degree, that is, $\alpha(\lambda v) = \alpha(v)$ for any $v \in \boldsymbol{R}$ and any $\lambda > 0$, then the equation (5.1) corresponds to a rate-independent flow, and may thus represent hysteresis phenomena. The variational formulation of this equation and its structural stability are addressed in [68], where a non-cyclically-monotone mapping $\vec{\gamma}$ is also considered, and the estimate procedure outlined in the latter remark is used. □

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