Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.3, p. 347–361.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_3_347_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 347-361

Weakly Subharmonic Function

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Dedicated to the memory of Giovanni Prodi

We introduce a class of functions $u : \Omega \to \mathbb{R}_e$ closely related to and generalizing the concept of subharmonic function; here Ω denotes a domain in $\mathbb{R}^n, n \geq 1$ and $\mathbb{R}_e$ the extended reals $[-\infty, +\infty]$. Functions in this class, which can be called *weakly subharmonic*, have many of the properties of subharmonic functions, and moreover, as we shall show, are equivalent to subharmonic functions on the natural subclass of upper semi-continuous functions whose Lebesgue set is all of Ω.

In Section 1 we recall various properties of subharmonic functions, and in Section 2 define the class of weakly subharmonic functions. In Sections 3 and 4 we show that weakly subharmonic functions share many of the properties of subharmonic functions. In final remarks we discuss the possibility of future investigation.

1. – Subharmonic Functions

By a subharmonic function we mean (see Hayman and Kennedy [2], or Helms [3]) an upper semi-continuous function $u : \Omega \to R_e$ with the property that

(i) u is not identically $-\infty$ on Ω.
(ii) for each $x_0 \in \Omega$ and all sufficiently small $\delta > 0$, we have

$$L(u; x_0, \delta) \geq u(x_0),$$

where $L(u : x_0, \delta)$ denotes the average of u over the sphere $S_\delta : \{|y - x_0| = \delta\}$ centered at x_0 with radius δ.

Clearly $L(u; x, \delta)$ is well-defined for all $\delta < \mathrm{dist}(x, \partial\Omega)$ since u is measurable and bounded above on S_δ (u is upper semi-continuous and hence attains its maximum on any compact subset of Ω). It is of course possible that both $L(u; x_0, \delta)$ and $u(x_0)$ are $-\infty$ ([1]).

([1]) Kellogg [4] considers only continuous functions as candidates for being subharmonic, making the entire theory somewhat simpler, but it seems best to maintain generality here.

Obviously any harmonic function is subharmonic, in view of the Gauss mean value theorem. Here we recall several of the principal properties of subharmonic functions.

THEOREM 1.1. – *If $u \in C^2(\Omega)$, then u is subharmonic in Ω if and only if $\Delta u \geq 0$, where Δ denotes the Laplace operator*

$$\Delta = \frac{\partial}{\partial x_1^2} + \cdots + \frac{\partial}{\partial x_n^2}.$$

See Hayman and Kennedy, p. 41 or Helms, Theorem 4.8.

[Helms considers only superharmonic functions in his results, but it is routine to rephrase them for subharmonic functions.]

THEOREM 1.2. – *Let u be subharmonic in Ω. Given any domain Ω' with compact support in Ω, and a function $h(x)$ harmonic in Ω' and such that*

$$u(x) \leq h(x)$$

on $\partial\Omega'$, then also $u(x) \leq h(x)$ in Ω'. See Hayman and Kennedy, Th. 2.4, or Helms, p. 60.

[Here by $u(x) \leq v(x)$ on $\partial\Omega'$ is meant explicitly that for every $\varepsilon > 0$ there is a neighborhood U of $\partial\Omega'$ such that $u(x) \leq v(x) + \varepsilon$ in $U \cap \Omega'$.]

It is exactly Theorem 1.2 which justifies the terminology "subharmonic".

THEOREM 1.3. – *Let u and v be subharmonic in Ω. If $c = \text{const.} > 0$, then also*

$$cu, \qquad u + v, \qquad \max\{u, v\}$$

are subharmonic in Ω. See Hayman and Kennedy, p. 41, and Helms, Theorem 4.12.

Two somewhat more delicate properties of subharmonic functions, not always emphasized, are given in the following results. For a function $v \in L^1_{loc}(\Omega)$ we use the notation $A(v; x, \delta)$ to denote the average of v over the (open) ball $B(x, \delta)$ centered at x with radius δ, of course with δ so small that $B(x, \delta) \subset \Omega$.

THEOREM 1.4. – *Let u be subharmonic in Ω. Then $u \in L^1_{loc}(\Omega)$ and*

$$A(u; x_0, \delta) \geq u(x_0)$$

at each point $x_0 \in \Omega$ and for $\delta > 0$ sufficiently small. See Helms, Theorem 4.10 and Lemma 4.9.

THEOREM 1.5. – *Let u be subharmonic in Ω. Then each point of Ω is in the Lebesgue set of u.*

[The Lebesgue set of a function $v \in L^1_{loc}(\Omega)$ consists of all points x_0 of Ω such that

$$\lim_{\delta \to 0} A(v; x_0, \delta) = v(x_0).$$

The possibility that both $A(v; x_0, \delta)$ and $v(x_0)$ are $\pm\infty$ is allowed. Lebesgue's great differentiation theorem states that almost every point of Ω is in the Lebesgue set of v.]

PROOF OF THEOREM 1.5. – By upper semi-continuity, for each x_0 in Ω we have $u(y) \leq u(x_0) + \varepsilon(\delta)$ for $y \in B(x_0, \delta)$, where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Thus

$$A(u; x_0, \delta) \leq u(x_0) + \varepsilon(\delta),$$

while on the other hand, for sufficiently small δ,

$$A(u; x_0, \delta) \geq u(x_0))$$

as a direct consequence of the subharmonic condition $L(u; x_0, \delta) \geq u(x_0)$ and a simple integration. It then follows that

$$\lim_{\delta \to 0} A(u; x_0, \delta) = u(x_0),$$

for each $x_0 \in \Omega$, as required.

Theorem 1.5 shows that the "typical" subharmonic function can be considered as continuous at each point where the function is finite, and to satisfy

$$\lim_{y \to x_0} u(y - x_0) = -\infty$$

whenever $u(x_0) = -\infty$, with of course u being integrable in the neighborhood of x_0. The finer structure of u is discussed by Helms (Section 4.6).

A final result of interest is the following Liouville theorem.

THEOREM 1.6. – *Let u be subharmonic and bounded above in $\mathbb{R}^2$. If also $u \in C^2(\Omega)$ then $u \equiv$* constant *in $\mathbb{R}^2$.*

The background of Theorem 1.6 is not known to the author; he first learned it in 1950 from David Gilbarg. A proof appears in the monograph [5] of Protter and Weinberger, and a related proof can be found in [6], page 100.

In the Corollary to Theorem 3.6 below we show that the differentiability condition $u \in C^2(\Omega)$ in Theorem 1.6 can be omitted.

2. – Weakly Subharmonic Functions

A function $u : \Omega \to R_e$ with $u \in L^1_{loc}(\Omega)$ will be called *weakly subharmonic* provided that

$$\int_\Omega u\Delta\eta \geq 0 \tag{2.1}$$

for all non-negative functions η of class $C^2(\Omega)$ with compact support in Ω.

Expressed in the language of distribution theory, this says that u is weakly subharmonic in Ω if $\Delta u \geq 0$ weakly in Ω, this being a natural extension of the fact that C^2 subharmonic functions satisfy the condition $\Delta u \geq 0$ in the usual pointwise sense. It is obvious from a direct integration by parts that every harmonic function in Ω is also weakly subharmonic.

Before turning to the main results of the section, we first introduce the useful concept of mollified functions.

Let $v \in L^1_{loc}(\Omega)$. The standard ρ-mollification $v_\rho(x)$ of v is defined by

$$v_\rho(x) = \int v(y)k_\rho(y-x)dy,$$

where $k_\rho(t)$ is a C^2 standard mollification kernel with $k_\rho(t) = 0$ when $|t| \geq \rho$ (see Evans [1], Appendix C5). Putting

$$\Omega_\rho = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > \rho\},$$

it follows that, e.g. Evans, Appendix C5,

(i) v_ρ is of class C^2 in $\Omega_{2\rho}$, with

$$D^i v_\rho(x) = \int v(y)D^i_{(x)}k_\rho(y-x)dy, \qquad i = 1, 2,$$

(ii) $v_\rho(x) \to v(x)$ for every x in the Lebesgue set of u, and consequently almost everywhere in Ω,

(iii) $c_\rho = c$ for constants c and $h_\rho = h$ for harmonic functions $h = h(x)$,

(iv) $0 \leq k_\rho(t) \leq C/\rho^n$ for some constant C.

LEMMA 2.1. – *Let $v \in L^1_{loc}(\Omega)$. Then for every $x \in \Omega$ and for all sufficiently small values δ, ρ, we have*

$$A(v_\rho; x, \delta) = A(v; x, \delta)_\rho.$$

PROOF. – With $\delta + \rho < \mathrm{dist}\,(x, \partial\Omega)$ we find (with ω_n being the volume of the unit ball in $\mathbb{R}^n$)

$$A(v_\rho; x, \delta) = \frac{1}{\omega_n \delta^n} \int\limits_{|z|\le\delta} v_\rho(y)dz \qquad (y = x + z)$$

$$= \frac{1}{\omega_n \delta^n} \int\limits_{|z|\le\delta} \int v(\hat{y}) k_\rho(\hat{y} - y)\, dz\, d\hat{y}$$

$$= \frac{1}{\omega_n \delta^n} \int\limits_{|z|\le\delta} \int v(w + y)\, k_\rho(w) dw\, dz$$

using the substitution $w = \hat{y} - y$. Then by Fubini's theorem there follows

$$= \frac{1}{\omega_n \delta^n} \int k_\rho(w) \left(\int\limits_{|z|\le\delta} v(w + y) dz \right) dw$$

$$= \int k_\rho(w) A(v; x + w, \delta) dw$$

$$= \int A(v; \hat{y}, \delta) k_\rho(\hat{y} - x)\, d\hat{y} = \{A(v; x, \delta)\}_\rho,$$

as required

The first main result of the section is the following

THEOREM 2.1. – *If u is subharmonic in Ω, then u is weakly subharmonic in Ω.*

PROOF. – Let u be subharmonic. It follows from Theorems 1.4 and 1.5 that

$$u(x) \le A(u; x, \delta) \tag{2.2}$$

Then, for an appropriate constant C, for fixed $x \in \Omega$, and for ρ sufficiently small,

$$u_\rho(x) - u(x) = \int\limits_{|y-x|\le\rho} (u(y) - u(x))\, k_\rho(y - x) dy \le CA(u - u(x); x, \rho) \ge 0$$

by (2.2), that is

$$u_\rho(x) \ge u(x).$$

We claim, moreover, that u_ρ is subharmonic in $\Omega_{2\rho}$. Indeed, by Lemma 2.1 and the fact that u is subharmonic,

$$L(u_\rho; x, \delta) = L(u; x, \delta)_\rho \ge u_\rho(x),$$

as claimed. Then since $u_\rho \in C^2$ it follows also that $\Delta u_\rho \ge 0$ by Theorem 1.1.

Let η be a non-negative C^2 function with compact support $\Omega' \subset \Omega$; clearly $\Omega' \subset \Omega_{2\rho_0}$ for sufficiently small ρ_0. Then obviously

$$\int_\Omega \eta \Delta u_\rho \geq 0 \tag{2.3}$$

when $\rho < \rho_0$. Hence integrating by parts twice in (2.3) gives

$$\int_\Omega u_\rho \Delta \eta \geq 0, \quad \rho < \rho_0. \tag{2.4}$$

Since u is upper semi-continuous it is bounded above on Ω_{ρ_0}, We can now let $\rho \to 0$ in (2.4) and apply the dominated convergence theorem. Indeed $u_\rho \to u$ almost everywhere according to property (ii) of mollified functions. Moreover $u_\rho \leq m$ in Ω' for sufficiently small ρ, while equally $u_\rho \geq u$ in Ω' for sufficiently small ρ. That is, using Theorem 1.4, $|u_\rho|$ is dominated by an integrable function on Ω'. Hence by the dominated convergence theorem,

$$\int_{\Omega'} u\Delta\eta = \lim \int_\Omega u_\rho \Delta\eta \geq 0$$

by (2.4). That is, u satisfies (2.1) and is weakly subharmonic.

The converse result, that a weakly subharmonic function is subharmonic clearly cannot be valid unless one assumes also that u is upper semi-continuous, nor by Theorem 1.5 can it be true unless every point of Ω is a Lebesgue point of u. With this in mind, the following converse holds.

THEOREM 2.2. – *Let u be weakly subharmonic, and assume also that u is upper semi-continuous with every point of Ω a Lebesgue point of u. Then u is subharmonic in Ω.*

In Section 4 we shall give an improved version of Theorem 2.2, see Theorem 4.3.

PROOF. – Let u_ρ be a standard ρ-mollification of u. By property (i) of mollification,

$$\begin{aligned} \Delta u_\rho(x) &= \int u(y) \Delta_{(x)} k_\rho(y-x)dy \\ &= \int u(y) \Delta_{(y)} k_\rho(y-x)dy \geq 0, \end{aligned}$$

by the definition of weakly subharmonic functions. Thus by Theorem 1.1 the ρ-mollification u_ρ of u is a C^2 subharmonic function in $\Omega_{2\rho}$.

Let x be fixed in Ω, and set $\delta_0 = \delta_0(x) = 1/4\ \mathrm{dist}(x, \partial\Omega)$. Then for $\rho < \delta_0$ the mollification u_ρ is defined in $\Omega_{2\delta_0} \supset B(x, 2\delta_0)$. Then by the definition of subharmonic function we have

$$L(u_\rho; x, \delta) \geq u_\rho(x), \qquad \rho,\ \delta < \delta_0. \tag{2.5}$$

Since u is upper semi-continuous it is bounded above on the ball $B(x, 3\delta_0)$, say $u \leq m$ on $B(x, 3\delta)$. Defining $f_\rho = m - u_\rho$, then $f_\rho \geq 0$ on $B(x, 2\delta_0)$ for $\rho \leq \delta_0$. Obviously $u_\rho \to u$ everywhere by the Lebesgue set hypothesis (use property (ii) of mollification), therefore $f_\rho \to m - u \geq 0$ on the sphere $S(x, \delta) \subset B(x, 2\delta_0)$. Thus by Fatou's lemma, as $\rho \to 0$,

$$\begin{aligned}\overline{\lim}\, L(u_\rho; x, \delta) = \overline{\lim}\, L(m - f_\rho; x, \delta) &= m - \underline{\lim}\, L(f_\rho; x, \delta)\\ &\leq m - L(m - u; x, \delta) = L(u; x, \delta).\end{aligned}$$

Hence using (2.5) we obtain

$$L(u; x, \delta) \geq \overline{\lim}\, u_\rho(x) = u(x),$$

and u is subharmonic, as required.

Of course, as noted at the beginning of the paper, $L(u; x, \delta)$ can take the value $-\infty$, in which case also $v(x) = -\infty$ at the given point x.

Theorem 2.2 can be considerably simplified when the class of subharmonic functions is restricted to those which are continuous and satisfy the condition (ii). In this case we have the simplified

THEOREM 2.3. – *Let u be weakly subharmonic and continuous in Ω. Then u is subharmonic (and continuous) in Ω.*

The proof involves noting only that the Lebesgue set of a continuous function in Ω is all of Ω.

3. – Properties of Weakly Subharmonic Functions

With the close connection of weakly subharmonic functions and subharmonic function thus established, one may ask whether weakly subharmonic functions share any of the properties of subharmonic functions. The following results settle this question for the principle properties considered in Section 1.

THEOREM 3.1. – *If u is weakly subharmonic in Ω and of class $C^2(\Omega)$, then $\Delta u \geq 0$.*

PROOF. – Since $u \in C^2$ we can integrate by parts in the definition (2.1) to obtain

$$\int_{\Omega} \eta \Delta u \geq 0,$$

and the result follows at once since the non-negative function η can be chosen arbitrarily.

THEOREM 3.2. – *Let u be weakly subharmonic in Ω, and define $\bar{u}$ to equal u on the set of Lebesgue points of u, and to be $-\infty$ otherwise. Then $\bar{u} \leq A(u; x, \delta)$, and $\bar{u}$ is locally bounded above in Ω.*

PROOF. – Let Ω' be a compact subset of Ω, and fix δ so small that $\Omega' \subset \Omega_{2\delta}$. As in Theorem 2.2, u_ρ is a C^2 subharmonic function in Ω' when $\rho < \delta$. Consequently by Theorem 1.4

$$u_\rho \leq A(u_\rho; x, \delta), \qquad x \in \Omega', \quad \rho < \delta,$$

whence by Lemma 2.1

$$u_\rho \leq A(u; x, \delta)_\rho, \qquad x \in \Omega', \quad \rho < \delta.$$

Now let $\rho \to 0$. Then, for $x \in \Omega'$, $u_\rho \to u = \bar{u}$ on the Lebesgue set of u, so that (note $A(u; x, \delta)$ is continuous in the variable x by Helms, Theorem 1.14),

$$\bar{u} \leq A(u; x, \delta) = \frac{1}{\omega_n \delta^n} \int_{B(x,\delta)} u \leq \frac{1}{\omega_n \delta^n} \int_{\Omega_\delta} u = \frac{\bar{m}}{\delta},$$

where $\bar{m}$ is finite because $u \in L^1_{loc}(\Omega)$. A local upper bound for $A(u; x, \delta)$ also arises independently from the fact that A is continuous.

THEOREM 3.3. – *If u is weakly subharmonic in Ω and the representative $\bar{u}$ reaches a maximum value M in Ω, then $\bar{u} \equiv M$ in Ω.*

PROOF. – We can assume that $M = 0$ without loss of generality. Then if $\bar{u}(x_0) = M = 0$ we have

$$0 \geq A(\bar{u}, x_0, \delta) \geq \bar{u}(x_0) = 0$$

by Theorem 3.2, that is

$$A(\bar{u}, x_0, \delta) = 0$$

for suitably small δ. Since $\bar{u} \leq 0$ this implies that $\bar{u} \equiv 0$ in $B(x_0, \delta)$. By a chaining argument it follows that $\bar{u} \equiv 0$ in Ω, and the proof is complete.

The next lemma is crucial for the proof of Theorems 3.4 and 3.5.

LEMMA 3.1. – *Suppose that* $v \in L^1_{loc}(\Omega)$ *is such that*

$$v(x) \le A(v; x, \delta) \tag{3.1}$$

for any fixed $x \in \Omega$ *and for* δ *sufficiently small. Then*

$$v_\rho(x) \ge v(x)$$

for ρ *sufficiently small.*

If moreover $v \in C^2(\Omega)$ *then* $\Delta v \ge 0$.

PROOF. – *First part.* As in the first step of the proof of Theorem 2.1 we have for an appropriate constant C

$$v_\rho(x) - v(x) = \int\limits_{|y-x|\le\rho} (v(y) - v(x))\, k_\rho(y-x)\, dy \le CA(v - v(x); x, \rho) \ge 0$$

by (3.1), as required.

Second part. – If $\Delta v < 0$ at a point $x_0 \in \Omega$, then also $\Delta v < 0$ in the ball $B = B(x_0, R)$ for R small. Suppose $r < R$ and let h be the harmonic function in $B(x_0, r)$ such that $h = v$ on $S(x_0, r)$. Then $v > h$ in $B(x_0, r)$ by the well known maximum principle for second order elliptic equations; (more specifically, we have $\Delta v < 0$ and $\Delta h = 0$, and the conclusion is immediate, see e.g. [5] or [1], page 327).

Hence in particular

$$v(x_0) > h(x_0) = L(h; x_0; x, r) = L(v; x_0; x, r).$$

We rewrite this as

$$n\omega_n r^{n-1} v(x_0) > \int\limits_{|y|=r} v(x_0 + y) d\sigma(y).$$

where $d\sigma(y)$ is the element of surface area on the sphere $|y| = r$. Integration with respect to r from 0 to δ yields (spherical polar coordinates)

$$\omega_n \delta^n v(x_0) > \int\limits_{|y|\le\delta} v(x_0 + y) dy,$$

that is, $v(x_0) > A(v; x_0, \delta)$, a contradiction with (3.1).

THEOREM 3.4. – *Let* u *be weakly subharmonic in* Ω. *Given any domain* Ω' *with compact support in* Ω, *and a function* $h(x)$ *harmonic in* Ω' *and such that*

$$u(x) \le h(x) \tag{3.2}$$

on $\partial\Omega'$, *then also* $\bar{u}(x) \le h(x)$ *in* Ω'.

PROOF. – We interpret (3.1) to mean that $u \le h(x) + \varepsilon$ almost everywhere in some neighborhood $U \cap \Omega'$. Let u_ρ be the mollification of u. Then

$$u_\rho \le h_\rho + \varepsilon$$

in any compact subset of $U \cup \Omega'$, say on $\partial\Omega_{2\delta}$ for δ small enough and $\rho < \delta$. Then since u_ρ is subharmonic (so $\Delta u_\rho \ge 0$) and $h_\rho + \varepsilon = h + \varepsilon$ is harmonic ($\Delta h = 0$) we get from the maximum principle

$$u_\rho \le h + \varepsilon \quad \text{in } \Omega'.$$

Using Theorem 3.2, and Lemma 3.1 with $v = \bar{u}$, gives $\bar{u} \le u_\rho$ whence

$$\bar{u} \le h + \varepsilon$$

in Ω', and the result follows since ε is arbitrary.

REMARK. – The above proof gives an alternate method for obtaining Theorem 1.2.

THEOREM 3.5. – *Let u and v be weakly subharmonic in Ω. Then if $c > 0$ also*

$$cu, \qquad u + v, \qquad \max\{u, v\}$$

are weakly subharmonic in Ω.

PROOF. – That cu and $u + v$ are weakly subharmonic is immediate. To prove the third case, we first observe by Theorem 3.2 that

$$\bar{u}(x) \le A(u; x.\delta) \le A(\max(u, v); x, \delta)$$

$$\bar{v}(x) \le A(v; x.\delta) \le A(\max(u, v); x, \delta)$$

and therefore

$$\max(\bar{u}(x), \bar{v}(x)) \le A(\max(u, v); x, \delta).$$

For simplicity we define

$$w = w(x) = \max(\bar{u}(x), \bar{v}(x)) \qquad x \in \Omega,$$

so the previous line becomes

$$w(x) \le A(w; x, \delta). \tag{3.3}$$

By mollification of (3.3)

$$w_\rho \le \{A(w; x, \delta)\}_\rho = A(w_\rho; x, \delta) \tag{3.4}$$

by Lemma 2.1. Since mollification produces C^2 functions, it now follows from (3.4) and the second part of Lemma 3.1, with $v = w_\rho$, that $\Delta w_\rho \ge 0$ In turn, for a non-negative test function $\eta(x)$ with compact support in Ω, and for sufficiently small ρ we have

$$\int_\Omega \eta \Delta w_\rho \geq 0.$$

Integrating by parts twice then gives

$$\int_\Omega w_\rho \Delta \eta \geq 0. \tag{3.5}$$

Our purpose is to let $\rho \to 0$ in (3.5), and to apply (once more) the dominated convergence theorem. This requires some preparation. First, by (3.3) and Lemma 3.1, with $v = w$, we find that

$$w_\rho(x) \geq w(x) = \max(\bar{u}(x), \bar{v}(x)) \geq \bar{u}(x).$$

Since $\bar{u} = u$ almost everywhere it follows that w_ρ is bounded below by a locally integrable function. Moreover

$$w_\rho \leq (\bar{u} + \bar{v})_\rho,$$

so by Theorem 3.2 w_ρ is bounded above by a locally integrable function. That is, $|w_\rho|$ is dominated on $\Omega_{2\rho}$ by an integrable function, for sufficiently small ρ. But $w_\rho \to w$ almost everywhere, whence by the dominated convergence theorem applied to (3.5) there follows

$$\int_\Omega w \Delta \eta \geq 0;$$

But $w = \max(\bar{u}, \bar{v}) = \max(u, v)$ almost everywhere, whence $\max(u, v)$ is weakly subharmonic. (2)

THEOREM 3.6. – *Let u be weakly subharmonic and bounded above in $\mathbb{R}^2$. Then* $u \equiv$ constant *in* $\mathbb{R}^2$.

PROOF. – u_ρ is subharmonic in $\mathbb{R}^2$ and bounded above by hypothesis. Then by Theorem 1.6 we have $u_\rho \equiv$ constant(ρ). Letting $\rho \to 0$ yields $u =$ constant almost everywhere. By common understanding this means $u \equiv$ constant.

(2) A surprisingly difficult proof for what at first glance seems an obvious result (as it is for subharmonic functions, Theorem 1.3.)

COROLLARY. – *Let u be subharmonic and bounded above in $\mathbb{R}^2$. Then* $u \equiv$ constant.

PROOF. – Since u is subharmonic it is weakly subharmonic, so Theorem 3.6 shows that $u =$ constant almost everywhere. But by Theorem 1.5 every point of Ω is a Lebesgue point of u and in turn $u \equiv$ constant.

REMARKS. – In view of the conclusions of Theorems 3.1–3.6 one can envision revising the theory of subharmonic functions by replacing the definition given in Section 1 by the definition in Section 2, that is by defining u to be subharmonic if $u \in L^1_{loc}(\Omega)$ and

$$\int u\Delta\eta \geq 0$$

for all non-negative test functions η in Ω. The theory developed above in Section 3 is in fact somewhat easier than the standard theory of subharmonic functions, at least given an acquaintance with basic Lebesgue theory. At the same time it has two advantages, first the simplicity and transparency of the new definition, and second the relatively larger class of functions allowed, in particular upper semi-continuity is no longer required.

The relaxation of the requirement that u be upper semi-continuous is partially paid for by the additional condition that u be locally integrable. Of course, the latter condition is already a consequence of subharmonicity so in fact nothing is lost. That is, the relative simplicity of the definition of weakly subharmonic functions is attained by building into the definition one of the more delicate aspects of the theory of subharmonic functions.

The author naturally is not willing to go beyond simply raising the question of whether the above alternate definition of subharmonic functions is worth adopting as the principal meaning of a subharmonic function. At the same time, this point of view raises further questions of interest, with main matters being the Perron method in the new context, and whether the Riesz representation of subharmonic functions applies in an appropriate sense to weakly subharmonic functions.

4. – The Lebesgue Set of a Weakly Subharmonic Function

The conclusion of Theorem 1.5 clearly cannot hold for a weakly subharmonic function since u is only defined almost everywhere. Nevertheless we shall show that for an appropriate representative of u the conclusion does hold. We need a further lemma.

LEMMA 4.1. – *Let $v \in C(\Omega)$. Then, for fixed x and δ sufficiently small, the average function $L(v;x,\delta)$ is continuous in the variable x and*

$$A(v;x,\delta) = \frac{n}{\delta^n}\int_0^\delta r^{n-1}L(v;x,r)\,dr. \tag{4.1}$$

Moreover, for $\delta < \delta_0$ and δ_0 sufficiently small, we have $A(v;x,\delta) \in C^1(0,\delta_0)$ and

$$\frac{dA(v;x,\delta)}{d\delta} = \frac{n}{\delta}\{L(v;x,\delta) - A(v;x,\delta)\}. \tag{4.2}$$

PROOF. – We have

$$\begin{aligned} A(v_\rho;x,\delta) &= \frac{1}{\omega_n\delta^n}\int_{|z|\le\delta} v(y)\,dz \qquad (y = x+w)\\ &= \frac{1}{\omega_n\delta^n}\int_0^\delta\int_{|w|=r} v(y)\,r^{n-1}d\sigma(w)\,dr \quad \text{(spherical polar coordinates)}\\ &= \frac{n}{\delta^n}\int_0^\delta r^{n-1}L(v;x,r)\;dr \qquad (\text{recall } \omega_n = \omega_{n-1}/n), \end{aligned}$$

proving (4.1).

That $L(v;x,\delta)$ is continuous when $v \in C(\Omega)$ follows at once from the representation

$$L(v;x,\delta) = \frac{1}{n\omega_n\delta^{n-1}}\int_{|z|=1} v(x+z)d\sigma(z).$$

Then from (4.1) we find that

$$\frac{dA(v;x,\delta)}{d\delta} = \frac{n}{\delta}L(v;x,\delta) - \frac{n^2}{\delta^{n+1}}\int_0^\delta r^{n-1}L(v;x,r)\,dr,$$

and (4.2) now follows from (4.1).

THEOREM 4.1. – *Let u be weakly subharmonic. Then*

$$A(u;x,\delta)$$

is an increasing function of δ.

NOTE. – Theorem 4.1 is essentially Helms, Lemma 4.18. For completeness we give a somewhat simpler proof.

PROOF. – Fix x and let δ_0 be so small that $B(x,\delta_0) \subset \Omega$. For $\delta < \delta_0$ and sufficiently small ρ, define $h(x)$ to be the solution of the Dirichlet problem $\Delta h = 0$ in $B(x,\delta)$ with $h = u_\rho$ on $\partial B(x,\delta)$. As in the proof of Lemma 3.1, by the Poisson integral representation it is clear that h exists. Moreover, using the fact that u_ρ is subharmonic, we find from Theorem 1.2 that $h \geq u_\rho$ in $B(x,\delta)$. We have next (!)

$$L(u_\rho; x, \delta_1) \leq L(h; x, \delta_1) = h(x) = L(h; x, \delta) = L(u_\rho; x, \delta),$$

that is, $L(u_\rho; x, \delta)$ is an increasing function of δ. Thus from (4.1)

$$\begin{aligned} A(u_\rho; x, \delta) &= \frac{n}{\delta^n} \int_0^\delta r^{n-1} L(u_\rho; x, r)\, dr \\ &\leq \frac{n}{\delta^n} \left(\int_0^\delta r^{n-1} dr \right) L(u_\rho; x, \delta) = L(u_\rho; x, \delta). \end{aligned}$$

In turn, from (4.2) then follows $dA(u_\rho; x, \delta)/d\delta \geq 0$. Thus $A(u_\rho; x, \delta)$ is an increasing function of δ. Finally by Lemma 2.1

$$A(u_\rho; x, \delta) = \{A(u; x, \delta)\}_\rho \to A(u; x, \delta),$$

since $A(u; x, \delta)$ is continuous. That is, $A(u; x, \delta)$ is an increasing function of δ since it is the limit of increasing functions.

We can now define the *canonical representative* $\hat{u}$ of u by

$$\hat{u}(x) = \lim_{\delta \to 0} A(u; x, \delta), \quad x \in \Omega.$$

Clearly $\hat{u} = u$ almost everywhere by the Lebesgue set theorem. In the sequel, where u is treated as a pointwise extended real valued function $u(x)$ on Ω, we make the following *canonical convention*:

$$u(x) = \hat{u}(x) = \lim_{\delta \to 0} A(u; x, \delta).$$

We can now state the first main result of the section.

THEOREM 4.2. – *Let u be weakly subharmonic in Ω. Then each point of Ω is in the Lebesgue set of u.*

The proof is obvious once one notes that $A(u; x, \delta) = A(\hat{u}; x, \delta)$.

As an immediate consequence of Theorems 2.2 and 4.2 we obtain the following definitive improvement of Theorem 2.2.

THEOREM 4.3. – *Let u be weakly subharmonic. Then u is subharmonic in Ω if and only if u is upper semi-continuous.*

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Received November 24, 2010 and in revised form April 13, 2011