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The mathematical work of Aldo Andreotti ranges over an impressive varieties of subjects. His interests and achievements covered a large spectrum of arguments, approached by various techniques and different perspectives. Once, as far as I remember the argument came about because I had to indicate a discipline for a job application, he lectured me about the difference between being a geometer, an algebraist, an analyst, or simply a mathematician. The last one was the highest prize to strive for. This view, in which aesthetic quality prevails on technique, fully reflects his consideration of beautiful mathematics as an Art. He was familiar with Art. In fact, his father Libero was a sculptor and he was a relative of and shared the first name with the painter Aldo Carpi. He was a Florentine. His education and attitude, as I could assess by listening, talking to him, seeing him working and working with him, made me think of the Renaissance. Speaking of himself, he modestly pretended being some craftsman, but we know that during the Renaissance there was no clear-cut distinction between artisans and artists.

His generation bore the heroic effort to restore and reconnect with the world Italian science and culture, after the debris of Fascism and war. His work on that was priceless. He gave a major contribution to build the strong mathematical department of Pisa of the sixties.

I had the chance of being his collaborator in the last part of his work, which was mostly devoted to the general theory of systems of partial differential operators. Some of Andreotti's papers on this subject were collected in a special volume [3] published in 1999 by the Scuola Normale di Pisa.

When we first met, the Mathematics Department in Pisa was stepping from its golden to some less noble metal age. I heard of that mythical time of joint seminars, where everybody was aware and contributed to the work of everybody else, and there was cooperation and no jealousy. Sound evidence of this past were the notes of the E.E. Levi seminar. The name commemorated an Italian mathematician who succeeded, in his too short life, to significantly contribute to most areas of Mathematics. Andreotti was the organizer, and all senior students and colleagues attended. It broke up in the seventies, when the climate in Pisa had changed and Aldo moved to the States. He held positions in Oregon, California and later in France, before reentering, as a professor of the Scuola Normale Superiore at the end of the decade.

Aldo was a curious and eager listener of any sort of interesting mathematics. Attending seminars, he took always careful notes that he later arranged and stored in his well ordered series of notebooks, always adding his own comments and remarks. He also prepared meticulous notes for his lectures. All statements and proofs were written in the greatest detail and perfectly organized, in preparation for his inspiring expositions in which the unessential technical steps were eventually dropped to let the ideas and the beauty stand out.

In Pisa he lived in an old large apartment house with a nice garden. Usually we met for mathematics at his place. A huge ceramicist and carpenter desk, much more impressive than the small writing table pulled to a central pillar, occupied part of his studio. Artistic handicraft was one of his favorite ways to relax. His pieces of ceramics were restricted to the wall of this room, but the whole house was a gallery, with paintings and some original and many plaster proofs of his father's works. Our collaboration was routinely organized according to a fixed schedule, providing for weekly meetings where we checked the progresses and fixed the goals for the next. To this we added several occasions of more leisurely and informal discussions, ranging from various humanities to politics, religion, sciences, travels, experiences, usually while taking long walks. Our opposite views on most subjects lead us often to converging attitudes on issues. Aldo joined willpower and balance, that he considered necessary for any serious activity, to the transgressive spirit of a true anarchist. He was often against power and authority, also when it concerned mathematics and himself. At times he was making a point that, at the very moment one achieves the highest results and recognition in a field, it is the right time to quit and silently move into a different direction, to ease the pressure and quietly follow his own track.

As I said at the beginning, a striking point of his mathematical career is the broad scope of his research. At the end of the forties he begun as the most brilliant young Italian algebraic geometer to find himself, in the seventies, investigating general complexes of partial differential operators by methods of hard analysis.

In the middle, his interest in Complex Analysis and Geometry, stemming from an effort to strengthen the foundations of Algebraic Geometry, grew to make Aldo, because of his results, one of the outstanding leading scientists in the field. After the seminal paper [7], he started to use techniques from the theory of partial differential equations to investigate questions relating to the cohomology of holomorphic objects, see, e.g. [24, 25, 26, 27]. The success of these methods in Complex Analysis fostered a broader interest in the general theory of p.d.o.'s.

His first paper in this field, [11], is connected to [8, 9, 10]. In these articles it was explained, and extended to the cohomology classes of the tangential Cauchy-Riemann complex, the famous example of Hans Lewy of a linear partial differential operator L, with polynomial coefficients in $\mathbb{R}^3$, for which the equation $Lu = f$ does not admit not even local distribution solution for most right hand sides in $\mathcal{C}^\infty(\mathbb{R}^3)$. The aim of [11] was to extend these ideas to general complexes of linear partial differential operators. Already the title: "Complexes of differential operators", with the subtitle "The Mayer-Vietoris sequence", testimonies his aim to rebuild, from a geometrical perspective, the theory of p.d.o.'s. Consideration of systems with arbitrary numbers of data and unknown made apparent the need of introducing homological and cohomological methods and formulations. The Mayer-Vietoris sequence, restated in this new context, was the key for investigating all sorts of boundary value problems; in a similar way, as the solution to the Riemann-Hilbert problem in one complex variable can be used to study the Dirichelet and the Neumann problem for the Laplacian in the plain.

During the a.a. 1972/73 Andreotti lectured at the Scuola Normale Superiore di Pisa on the subject, carefully discussing examples taken from the classical equations of mathematical physics, to illustrate how his new theory adjusted to known results.

Andreotti's project connected to various contemporary trends. D.C. Spencer and his school were developing general constructions for the study of differential systems, that related to pseudogroups and were inspired by the deformation theory of complex structures. Also, in these years, the classical method of the parametrix evolved into the theory of pseudodifferential and Fourier integral operators. Furthermore, in parallel, from Sato's hyperfunctions originated the notion of microfunction and the Japanese school of microlocal analysis.

Andreotti's approach was purposely naive. The new general geometric theory of p.d.o.'s, that would include overdeterminded systems, had to retrace, as much as possible, the historical development of the scalar case. The very first step was to understand linear constant coefficients, starting from the work of Ehrenpreis, Palamodov, Malgrange on division of functions and distributions. The next class to be considered consisted of complexes of constant strength, which very much behave as those obtained by disregarding lower order terms and freezing at a point the coefficients of their principal parts. Truly variable coefficients come into the picture when completely new phenomena arise, like local non solvability.

The progress of this ambitious program, which would eventually also deal with nonlinearity, would measure against the capability of understanding and including whatever new and special appears while moving from one to the next more general context. In this long journey, new methods and ideas had to be constantly confronted with problems from mathematical physics, differential geometry, complex geometry.

Andreotti's approach was distant from the others. Spencer's constructions were very general, but so complicated that it seemed very hard to discuss any application. Pseudodifferential operators were too closely related to linearity and determined problems. Microlocal analysis restricted to the analytic category.

Also the brand of the mathematical analysis in Pisa deeply influenced Aldo's design. At the time, the foremost methods where a priori estimates and functional analysis, also more suitable for generalizing to nonlinear equations and systems.

Let me briefly describe the actual contents of Andreotti's contribution to the theory of partial differential operators. The papers [2, 13, 14] are mostly devoted to constant coefficients. The last one uses the Hilbert resolution to suitably inscribe a given operator into a complex, and computes resolutions in the generic cases. This work was conceived as a first chapter of a comprehensive theory of p.d.o.'s, which would include also [11, 12]. A draft of the state of the art at that time is given in his J.K. Whittemore lectures [1].

As I already explained, the exigency of finding substantial applications to specific problems was a constant issue and motivation. In the early seventies, Cattabriga and De Giorgi proved the global solvability of analytic p.d.e.'s with constant coefficients in two variables, and next De Giorgi and Piccinini gave a counterexample to global analytic solvability for the heath equation in three variables. A similar counterexample, using the Bochner-Martinelli kernel for several complex variables was obtained in [13], while analytic convexity was later investigated in full generality in the series of papers [16, 17, 19] and in the book [20], where the Phragmén-Lindelöf type estimates of Hörmander for scalar operators were suitably extended to the overdetermined case.

In [15] a condition is given, for systems of p.d.o.'s with smooth coefficients, under which local solvablity implies exactness at the formal power series level for the complex in which the system is inserted. The rationale is that the Poincaré lemma means flatness of the ring of formal power series over rings involving smooth categories. This was pursued in [22], where it was shown how Hilbert's syzygies theorem yields good resolutions also in the real analytic category, yielding complexes for which the analytic and formal Poincaré lemmas are valid outside proper analytic sets.

In [15, 18] the theory of holomorphic convexity is extended to general elliptic complexes, and results are obtained which generalize E.E.Levi convexity and the Cartan-Tullen theory.

In [21] the subject is again the Mayer-Vietoris sequence for complexes of p.d.o.'s. Here, compared with [9, 10, 11, 12], there is a deeper concern for the invariance of the notion of non characteristic and formally non characteristic hypersurface. Moreover, the general theory is tested on the classical question of characterizing the traces of the pluriharmonic functions.

The last researches of Andreotti were centered on local solvability (see [22, 23, 5, 6]). Besides the afore mentioned [22], I want to call the attention on [6], where the generalization of the H. Lewy example of [10] is extended to *CR* manifolds of arbitrary *CR* codimension. He had already considered general real analytic *CR* manifolds in [4]. The tangential Cauchy-Riemann complex was the simplest natural example of a non elliptic complex of p.d.o.'s with smooth coefficients. The proof of the non validity of the Poincaré lemma, using a completely different technique from [10], was intended to test a method that could be applied also to more general complexes.

Andreotti died in his office at the Scuola Normale Superiore di Pisa, just before starting a meeting that he had organized to explain the results already obtained and to involve a number of young mathematicians into his research project in p.d.o.'s.

Parts of Andreotti's program were continued after his death. Certainly his project lost its momentum and a large portion of it still remains as a legacy for the future.

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