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On a Problem Posed by Luigi Campedelli

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To my two Teachers D. Gallarati and C. P. Ramanujam

Abstract. – We construct an irreducible degree 10 plane curve $C_{10}$, giving the explicit equation with real coefficients, having six $[3, 3]$ points that are not on a conic. The fairly simple equation of $C_{10}$ enables the shape of a curve, whose equation has been researched without success since 1932, to be visualized in the real plane.

Introduction

In 1932 L. Campedelli [$C_1, C_2$] constructed his classical double plane branched along a degree 10 plane curve $C_{10}$ with six $[3, 3]$ points that are not on a conic. A $[3, 3]$ point is a triple point with an infinitely near ordinary triple point in the first neighbourhood. A degree 10 plane curve with six $[3, 3]$ points that are not on a conic is called a Campedelli branch curve. The desingularization of the Campedelli double plane is a regular surface of general type having the geometric genus $p_g = 0$ and the bigenus $P_2 = 3$. For a study of these surfaces cf. [D].

Campedelli obtained the degree 10 curve split into four irreducible components: three conics that are bitangent two by two, and a quartic curve that is tangent to the conics at their six contact points.

In his article [C$_2$, p. 542, in an old-fashioned Italian, Campedelli wondered: – Tra tali $C_{10}$ ne saranno di irreducibili? Non sembra facile poterlo dimostrare, ... –. [Among such $C_{10}$ are there any irreducible ones? It does not seem easy to demonstrate as much, ...].

M. Reid in [R] gives a proof of the existence of an irreducible Campedelli branch curve, without providing any equations of the curve. We do not know if the curve in the present paper is a special case of Reid’s examples.

The purpose of the present paper is to produce a direct construction, “à la Campedelli”, of an irreducible $C_{10}$, i.e. writing its equation that has six $[3, 3]$ points that are not on a conic (cf. section 4). To solve an algebraic system of three equations, we need the ground field $k$ to contain $\sqrt{5}$. So, it was possible to find a polynomial defining the curve with coefficients either in the ring $\mathbb{Z}[\sqrt{5}]$, where $\mathbb{Z}$ is the ring of integer numbers, or in the field $\mathbb{F}_q[\sqrt{5}]$, where $\mathbb{F}_q$ is a finite field of $q$ elements, such that it does not produce trivial cases for $C_{10}$, such as a coincidence of the $[3, 3]$ points, multiple components on $C_{10}$. 
In particular, if we assume that $k$ is the complex field, then the equation of the curve has real coefficients and the twelve triple points have real coordinates.

The curve $C_{10}$ is rational and, in the projective real plane, it has one component connected in the Euclidean topology, according to Harnack’s theorem.

The equation is quite simple, so the curve can be plotted with any mathematical computer program, showing its shape (cf. section 6). At the six $[3, 3]$ points, only one of the three local branches generating the $[3, 3]$ point is real. We do not know whether an irreducible Campedelli branch curve can be found that has at least one $[3, 3]$ point where the three branches generating the $[3, 3]$ point are all real, as in some reducible cases.

Let us illustrate the method that led to the equation of the irreducible Campedelli branch curve.

In the construction, the first fact to consider is that we cannot impose the singularities of a Campedelli branch curve in a general position because imposing such fixed singularities produces 72 linear conditions on the 66 homogeneous coefficients of a generic degree 10 curve. So, to construct this branch curve, be it irreducible or not, we have to impose the actual or infinitely near singularities in a particular position. Before trying to construct the curve, it is therefore important to decide the position of the singular points.

A good strategy for finding the right position of the singularities is to consider curves with special symmetries. When we construct these curves, we thus leave it to the curves themselves to choose the position of the singularities, which have the same symmetries as the curve.

This method enabled us to construct new Campedelli branch curves, but it was not strong enough to produce an irreducible one (cf. $[S_1]$ and $[W]$, where Campedelli branch curves with three irreducible components are obtained and cf. $[S_2]$, where a branch curve $C_{10}$ with two irreducible components is obtained $C_{10} = C_1 + C_9$: a line + a degree 9 curve).

In the last construction of the degree 9 curve $C_9$, which is symmetrical to a line, we had 30 homogeneous coefficients and 32 linear conditions on them. So, after considering all the imposed conditions, we had three algebraic relations in three variables, i.e. an algebraic system of three equations given by three polynomials in three variables. The three variables arose from the coordinates of the singularities. These three algebraic relations were very important because their solutions, the symmetry of $C_9$ and the symmetry of the singularities, contained the conditions for the existence of the irreducible $C_9$.

Bearing all these facts in mind, the best approach to constructing the irreducible Campedelli branch curve involved:

1) considering a very particular position of the singularities that had not previously been considered in $[S_1]$ and $[S_2]$, i.e. putting three $[3, 3]$ points on the vertices of a triangle and the other three on the sides of the same triangle; the
latter three \([3, 3]\) points are not collinear; and clearly the six \([3, 3]\) points are not on a conic;

2) considering curves that are invariant with respect to a rotation of the three homogeneous coordinates of the projective plane;

3) allowing the coordinates of the twelve triple points generating the six \([3, 3]\) points to depend on three variables.

**Construction of the irreducible Campedelli branch curve**

1. – **Explanation of the statements in 1), 2) and 3)**

The following is an explanation of the statements in 1), 2) and 3) at the end of the Introduction.

1’) In the projective plane \(\mathbb{P}^2\), over the ground field \(k\), of homogeneous coordinates \((x_1, x_2, x_3)\), we fix the fundamental triangle \(x_1 x_2 x_3 = 0\).

2’) We consider the rotation of the coordinates given by \(x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1\). Considering a degree 10 curve \(f_{10}\), which is invariant with respect to this rotation and has a triple point at the vertices of the fundamental triangle, its equation is given by

\[
\begin{align*}
f_{10} : & \quad a_1(x_1^7 x_2^2 + x_2^7 x_3^2 + x_3^7 x_1^2) + a_2(x_1^7 x_2^3 x_3 + x_1 x_2^7 x_3^2 + x_2^7 x_3 x_1^2) + \\
& \quad a_3(x_1^5 x_2 x_3^2 + x_1^5 x_2^2 x_3 + x_1 x_2 x_3^2) + a_4(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) + \\
& \quad a_5(x_1^6 x_2^2 + x_1^6 x_3^2 + x_1^2 x_2^6) + a_6(x_1^6 x_2^3 x_3 + x_1 x_2^6 x_3^2 + x_2^6 x_3 x_1^2) + \\
& \quad a_7(x_1^6 x_2^2 x_3^2 + x_1^2 x_2^6 x_3^2 + x_2^6 x_3 x_1^2) + a_8(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) + \\
& \quad a_9(x_1^6 x_2^2 + x_1^6 x_3^2 + x_1^2 x_2^6) + a_{10}(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) + \\
& \quad a_{11}(x_1^5 x_2^2 x_3 + x_1 x_2^5 x_3^2 + x_1 x_2^2 x_3^5) + a_{12}(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) + \\
& \quad a_{13}(x_1^5 x_2^2 x_3 + x_1 x_2^5 x_3^2 + x_1 x_2^2 x_3^5) + a_{14}(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) + \\
& \quad a_{15}(x_1^5 x_2^2 x_3 + x_1 x_2^5 x_3^2 + x_1 x_2^2 x_3^5) + a_{16}(x_1^5 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_2^3 x_3 x_1^2) = 0,
\end{align*}
\]

where \(a_i \in k\).

3’) We put the remaining three \([3, 3]\) points on the sides of the fundamental triangle at \((-b_1, 0, b_2), (b_2, -b_1, 0)\) and \((0, b_2, -b_1), b_j \in k\).

Moreover, let us consider the vertex \((0, 0, 1)\) where the above \(f_{10}\) has a triple point. Infinitely near this triple point, we impose another triple point in the direction of the line \(x_2 + Ax_1 = 0, A \in k\), i.e. we want the singular tangent line at the \([3, 3]\) point \((0, 0, 1)\) to be \(x_2 + Ax_1 = 0\).
The singular tangent line at \((0, 1, 0)\) and at \((1, 0, 0)\) remains fixed by rotation.

Finally, we set the singular tangent line at the \([3, 3]\) point \((- b_1, 0, b_2)\) as 
\[x_2 + B(b_2 x_1 + b_1 x_3) = 0,\quad B \in k.\]
The singular tangent line at the \([3, 3]\) points 
\((b_2, -b_1, 0)\) and \((0, b_2, -b_1)\) remains fixed by rotation.

So the six \([3, 3]\) points are not fixed a priori, they depend on three variables as 
stated above, in \(3\); i.e. the three variables are given by \(A, B, d = \frac{b_2}{b_1}\).

2. – Preparing the linear system of conditions

Since we must put a triple point at \((- b_1, 0, b_2)\), it is convenient to intersect 
\(f_{10}\) with \(x_2 = 0\)
\[
\begin{align*}
  f_{10} &= 0 \\
  x_2 &= 0
\end{align*}
\]
and impose the equality
\[
a_1 x_3^4 + a_5 x_1 x_3^3 + a_{10} x_1^2 x_3^2 + a_9 x_1^3 x_3 + a_4 x_1^4 = (b_2 x_1 + b_1 x_3)^3 (c_2 x_1 + c_1 x_3).
\]
This is a simplification for the linear system of the conditions, i.e. we immediately 
have the five values of \(a_i; a_1 = b_1^3 c_1, a_5 = \ldots\) In particular, we have explained all 
the six \([3, 3]\) points that the curve \(f_{10}\) has on the fundamental triangle 
\(x_1 x_2 x_3 = 0\).

3. – Simplifications for solving the algebraic system in the three variables

\(A, B, d = \frac{b_2}{b_1}\)

Having solved the linear system, we can go on to solve an algebraic system of 
three equations in the three variables \(A, B, d = \frac{b_2}{b_1}\).

The variable \(B\) appears in the three polynomials with the lower powers, so we 
eliminate \(B\) first. In the elimination of \(B\), there are several coefficients to 
simplify, as we saw in \([S_2]\). This is very important because otherwise there will be 
problems with the computer’s memory space. Having eliminated \(B\), we obtain 
two polynomials in \(A\) and \(d\). The resultant of the two polynomials in relation to \(A\) 
is factorized by:
\[
\begin{align*}
  (d^2 - d - 1)(d^2 + d - 1)(d^4 - 3d^3 + 3d^2 - 2d + 2)(2d^4 - 2d^3 + 3d^2 - 3d + 1) \\
  (4d^5 + 3d^4 + 5d^3 + 2d^2 + d + 1)(d^5 + d^4 + 2d^3 + 5d^2 + 3d + 4) \\
  (74d^{10} - 162d^9 + 617d^8 - 463d^7 + 713d^6 - 1126d^5 + 713d^4 - 463d^3 + 617d^2 - 162d + 74) \\
  (2d^2 - 1)^2(d^2 - 2)^2(d^2 - d + 1)^4(d^2 + 1)^4(d^2 + d + 1)^47(d + 1)^{56}(d - 1)^{125}d^{397} \\
  (a \text{ very long polynomial of degree } 128)^2.
\]
Multiple roots of this polynomial usually produce trivial solutions, e.g. the
curve $f_{10}$ split into a cubic $f_3$ with multiplicity 3, in symbols: $f_{10} = f_3^3(\cdots)$. For this
reason, and because the polynomial is one of the shortest, we choose the poly-
nomial $d^2 - d - 1$ and its root $\bar{d} = \frac{1}{2} + \frac{\sqrt{5}}{2}$.

It should be noted that we have not considered the roots of the other poly-
nomials.

Considering the chosen root, we find that the two values of $A$ and $B$ that solve
our algebraic system are given by $A = 1 - \frac{2}{5}\sqrt{5}$ and $B = \frac{5}{2} + \frac{\sqrt{5}}{2}$, respectively.

4. – The equation

Solving the linear system of conditions in line with the above values $\bar{d}, \bar{A}$ and
$\bar{B}$, the equation of our irreducible Campedelli branch curve $C_{10}$ is given by

\[
C_{10} : 5(-275 + 123\sqrt{5})(x_1^3x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) + \\
75(-29 + 13\sqrt{5})(x_1^3x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2) + \\
75(-15 + 7\sqrt{5})(x_1^3x_2x_3^2 + x_1^2x_2^2x_3 + x_1x_2x_3^3) + \\
125(-1 + \sqrt{5})(x_1^3x_3^2 + x_1^2x_2^2 + x_1x_3x_2^2) + \\
5(685 - 303\sqrt{5})(x_1^3x_2^3 + x_1^2x_2x_3^2 + x_1x_3x_2^3) + \\
5(1081 - 417\sqrt{5})(x_1^6x_2x_3 + x_1x_2^3x_3^3 + x_1^2x_2x_3^5) + \\
39(235 - 59\sqrt{5})(x_1^6x_2^2x_3^2 + x_1^4x_2x_3^4 + x_1x_2^3x_3^6) + \\
5(1391 - 335\sqrt{5})(x_1^6x_2x_3^3 + x_1x_2^5x_3^3 + x_1^2x_2x_3^5) + \\
10(145 - 52\sqrt{5})(x_1^4x_3^2 + x_1^3x_2^2 + x_1^2x_3x_2^2) + \\
15(-205 + 97\sqrt{5})(x_1^5x_2^2 + x_1^5x_3^3 + x_1^5x_2^3) + \\
15(155 + 37\sqrt{5})(x_1^4x_2x_3 + x_1^3x_2x_3^3 + x_1^2x_3x_2^3) + \\
3(3575 + 1437\sqrt{5})(x_1^5x_2^3 + x_1^4x_2x_3^3 + x_1^3x_3x_2^3) + \\
15(643 + 465\sqrt{5})(x_1^5x_2^3x_3^2 + x_1^4x_2^2x_3^4 + x_1^3x_2x_3^2x_2^3) + \\
15(-299 + 307\sqrt{5})(x_1^5x_2x_3^3 + x_1^4x_2^2x_3^3 + x_1x_3^3x_2^2) + \\
12(905 + 963\sqrt{5})(x_1^4x_2^2 + x_1^3x_2^2 + x_1^2x_2x_3^4) + \\
3(9407 + 5741\sqrt{5})(x_1^4x_2x_3^3 + x_1^3x_2^2x_3^3 + x_1^2x_2x_3^2x_3) = 0.
\]

The equation is quite simple and all the properties of the curve can be de-
duced directly from its equation. The irreducibility of $C_{10}$ can also be deduced
from its parametric equations (cf. next section).
5. – The parametric equations

The curve $C_{10}$ is rational. It can be parametrized by the linear pencil of octic curves $C_8$, having the same $[3,3]$ point at $(0,0,1)$ as $C_{10}$, with five tacnodes at the remaining five $[3,3]$ points of $C_{10}$, and with tacnodal tangent lines that coincide with the tangent singular lines at the $[3,3]$ points on $C_{10}$, and finally the octic curves pass through the three simple points on $C_{10}$ lying on the three sides of the fundamental triangle $x_1 x_2 x_3 = 0$.

The parametric equations of $C_{10}$, that are obtained using the above pencil of octic curves $C_8$ are rather long. If we consider the affine coordinates $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$, then the affine parametric equations of $C_{10}$, with parameter $t$, are given by

\[
\begin{align*}
    x &= \frac{f(t)}{g(t)}, \\
    y &= \frac{h(t)}{k(t)}
\end{align*}
\]

where $f(t), g(t), h(t), k(t)$ are polynomials of degree seven with coefficients given by numbers with 46-54 digits - making them too long to be reproduced here.

6. – Plotting with “Mathematica”

In the picture there are three $[3,3]$ points, their location is drawn with a small circle “o”. At each $[3,3]$ point, there are two of the three local branches, generating such singularity, that are complex branches. So, in the picture only one
real local branch appears and the singular points appear as simple points. We do not know whether an irreducible Campedelli branch curve exists having one or more \([3,3]\) points with three real local branches, as it happens in the case of reducible Campedelli branch curves. The shape of such a curve would be very interesting, because it is rational and it has only one connected component in the Euclidean topology of the projective plane.

REFERENCES

[S2] E. Stagnaro, What decides the existence of a branch curve? (A new Campedelli branch curve \(C_1 + C_0\)), to appear on “Atti Acc. Ligure di Sc. e Lett.”.