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On a Problem Posed by Luigi Campedelli

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To my two Teachers D. Gallarati and C. P. Ramanujam

Abstract. – *We construct an irreducible degree* 10 *plane curve* C_{10}*, giving the explicit equation with real coefficients, having six [*$3,3$*] points that are not on a conic. The fairly simple equation of* C_{10} *enables the shape of a curve, whose equation has been researched without success since* 1932*, to be visualized in the real plane.*

Introduction

In 1932 L. Campedelli [C_1, C_2] constructed his classical double plane branched along a degree 10 plane curve C_{10} with six $[3,3]$ points that are not on a conic. A $[3,3]$ point is a triple point with an infinitely near ordinary triple point in the first neighbourhood. A degree 10 plane curve with six $[3,3]$ points that are not on a conic is called a *Campedelli branch curve*. The desingularization of the Campedelli double plane is a regular surface of general type having the geometric genus $p_g = 0$ and the bigenus $P_2 = 3$. For a study of these surfaces cf. [D].

Campedelli otained the degree 10 curve split into four irreducible components: three conics that are bitangent two by two, and a quartic curve that is tangent to the conics at their six contact points.

In his article [C_2], p. 542, in an old-fashioned Italian, Campedelli wondered: – *Tra tali* C_{10} *ne saranno di irriducibili? Non sembra facile poterlo dimostrare, ...* –. [*Among such* C_{10} *are there any irreducible ones? It does not seem easy to demonstrate as much, ...*].

M. Reid in [R] gives a proof of the existence of an irreducible Campedelli branch curve, without providing any equations of the curve. We do not know if the curve in the present paper is a special case of Reid's examples.

The purpose of the present paper is to produce a direct construction, "à la Campedelli", of an irreducible C_{10}, i.e. writing its equation that has six $[3,3]$ points that are not on a conic (cf. section 4). To solve an algebraic system of three equations, we need the ground field k to contain $\sqrt{5}$. So, it was possible to find a polynomial defining the curve with coefficients either in the ring $\mathbb{Z}[\sqrt{5}]$, where $\mathbb{Z}$ is the ring of integer numbers, or in the field $\mathbb{F}_q[\sqrt{5}]$, where $\mathbb{F}_q$ is a finite field of q elements, such that it does not produce trivial cases for C_{10}, such as a coincidence of the $[3,3]$ points, multiple components on C_{10}.

In particular, if we assume that k is the complex field, then the equation of the curve has real coefficients and the twelve triple points have real coordinates.

The curve C_{10} is rational and, in the projective real plane, it has one component connected in the Euclidean topology, according to Harnack's theorem.

The equation is quite simple, so the curve can be plotted with any mathematical computer program, showing its shape (cf. section 6). At the six $[3,3]$ points, only one of the three local branches generating the $[3,3]$ point is real. We do not know whether an irreducible Campedelli branch curve can be found that has at least one $[3,3]$ point where the three branches generating the $[3,3]$ point are all real, as in some reducible cases.

Let us illustrate the method that led to the equation of the irreducible Campedelli branch curve.

In the construction, the first fact to consider is that we cannot impose the singularities of a Campedelli branch curve in a general position because imposing such fixed singularities produces 72 linear conditions on the 66 homogeneous coefficients of a generic degree 10 curve. So, to construct this branch curve, be it irreducible or not, we have to impose the actual or infinitely near singularities in a particular position. Before trying to construct the curve, it is therefore important to decide the position of the singular points.

A good strategy for finding the right position of the singularities is to consider curves with special symmetries. When we construct these curves, we thus leave it to the curves themselves to choose the position of the singularities, which have the same symmetries as the curve.

This method enabled us to construct new Campedelli branch curves, but it was not strong enough to produce an irreducible one (cf. $[S_1]$ and $[W]$, where Campedelli branch curves with three irreducible components are obtained and cf. $[S_2]$, where a branch curve C_{10} with two irreducible components is obtained $C_{10} = C_1 + C_9$: a line + a degree 9 curve).

In the last construction of the degree 9 curve C_9, which is symmetrical to a line, we had 30 homogeneous coefficients and 32 linear conditions on them. So, after considering all the imposed conditions, we had three algebraic relations in three variables, i.e. an algebraic system of three equations given by three polynomials in three variables. The three variables arose from the coordinates of the singularities. These three algebraic relations were very important because their solutions, the symmetry of C_9 and the symmetry of the singularities, contained the conditions for the existence of the irreducible C_9.

Bearing all these facts in mind, the best approach to constructing the irreducible Campedelli branch curve involved:

1) considering a very particular position of the singularities that had not previously been considered in $[S_1]$ and $[S_2]$, i.e. putting three $[3,3]$ points on the vertices of a triangle and the other three on the sides of the same triangle; the

latter three $[3,3]$ points are not collinear; and clearly the six $[3,3]$ points are not on a conic;

2) considering curves that are invariant with respect to a rotation of the three homogeneous coordinates of the projective plane;

3) allowing the coordinates of the twelve triple points generating the six $[3,3]$ points to depend on three variables.

Construction of the irreducible Campedelli branch curve

1. – Explanation of the statements in 1), 2) and 3)

The following is an explanation of the statements in 1), 2) and 3) at the end of the Introduction.

1′) In the projective plane $\mathbb{P}^2$, over the ground field k, of homogeneous coordinates (x_1, x_2, x_3), we fix the fundamental triangle $x_1x_2x_3 = 0$.

2′) We consider the rotation of the coordinates given by $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$. Considering a degree 10 curve f_{10}, which is invariant with respect to this rotation and has a triple point at the vertices of the fundamental triangle, its equation is given by

$$f_{10} : a_1(x_1^7x_2^3 + x_2^7x_3^3 + x_1^3x_3^7) + a_2(x_1^7x_2^2x_3 + x_1x_2^7x_3^2 + x_1^2x_2x_3^7) +$$
$$a_3(x_1^7x_2x_3^2 + x_1^2x_2^7x_3 + x_1x_2^2x_3^7) + a_4(x_1^7x_3^3 + x_2^7x_1^3 + x_2^3x_3^7) +$$
$$a_5(x_1^6x_2^4 + x_2^6x_3^4 + x_1^4x_3^6) + a_6(x_1^6x_2^3x_3 + x_1x_2^6x_3^3 + x_1^3x_2x_3^6) +$$
$$a_7(x_1^6x_2^2x_3^2 + x_1^2x_2^6x_3^2 + x_1^2x_2^2x_3^6) + a_8(x_1^6x_2x_3^3 + x_1^3x_2^6x_3 + x_1x_2^3x_3^6) +$$
$$a_9(x_1^6x_3^4 + x_2^6x_1^4 + x_2^4x_3^6) + a_{10}(x_1^5x_2^5 + x_2^5x_3^5 + x_1^5x_3^5) +$$
$$a_{11}(x_1^5x_2^4x_3 + x_1x_2^5x_3^4 + x_1^4x_2x_3^5) + a_{12}(x_1^5x_2^3x_3^2 + x_1^2x_2^5x_3^3 + x_1^3x_2^2x_3^5) +$$
$$a_{13}(x_1^5x_2^2x_3^3 + x_1^3x_2^5x_3^2 + x_1^2x_2^3x_3^5) + a_{14}(x_1^5x_2x_3^4 + x_1^4x_2^5x_3 + x_1x_2^4x_3^5) +$$
$$a_{15}(x_1^4x_2^4x_3^2 + x_1^2x_2^4x_3^4 + x_1^4x_2^2x_3^4) + a_{16}(x_1^4x_2^3x_3^3 + x_1^3x_2^4x_3^3 + x_1^3x_2^3x_3^4) = 0,$$

where $a_i \in k$.

3′) We put the remaining three $[3,3]$ points on the sides of the fundamental triangle at $(-b_1, 0, b_2)$, $(b_2, -b_1, 0)$ and $(0, b_2, -b_1)$, $b_j \in k$.

Moreover, let us consider the vertex $(0,0,1)$ where the above f_{10} has a triple point. Infinitely near this triple point, we impose another triple point in the direction of the line $x_2 + Ax_1 = 0$, $A \in k$, i.e. we want the singular tangent line at the $[3,3]$ point $(0,0,1)$ to be $x_2 + Ax_1 = 0$.

The singular tangent line at $(0,1,0)$ and at $(1,0,0)$ remains fixed by rotation.

Finally, we set the singular tangent line at the $[3,3]$ point $(-b_1, 0, b_2)$ as $x_2 + B(b_2x_1 + b_1x_3) = 0,\ B \in k$. The singular tangent line at the $[3,3]$ points $(b_2, -b_1, 0)$ and $(0, b_2, -b_1)$ remains fixed by rotation.

So the six $[3,3]$ points are not fixed a priori, they depend on three variables as stated above, in 3); i.e. the three variables are given by $A, B, d = \dfrac{b_2}{b_1}$.

2. – Preparing the linear system of conditions

Since we must put a triple point at $(-b_1, 0, b_2)$, it is convenient to intersect f_{10} with $x_2 = 0$

$$\begin{cases} f_{10} = 0 \\ x_2 = 0 \end{cases} = \begin{cases} x_1^3x_3^3(a_1x_3^4 + a_5x_1x_3^3 + a_{10}x_1^2x_3^2 + a_9x_1^3x_3 + a_4x_1^4) = 0 \\ x_2 = 0 \end{cases}$$

and impose the equality

$$a_1x_3^4 + a_5x_1x_3^3 + a_{10}x_1^2x_3^2 + a_9x_1^3x_3 + a_4x_1^4 = (b_2x_1 + b_1x_3)^3(c_2x_1 + c_1x_3).$$

This is a simplification for the linear system of the conditions, i.e. we immediately have the five values of a_i: $a_1 = b_1^3c_1,\ a_5 = \cdots$. In particular, we have explained all the six $[3,3]$ points that the curve f_{10} has on the fundamental triangle $x_1x_2x_3 = 0$.

3. – Simplifications for solving the algebraic system in the three variables $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{d} = \dfrac{\boldsymbol{b_2}}{\boldsymbol{b_1}}$

Having solved the linear system, we can go on to solve an algebraic system of three equations in the three variables $A, B, d = \dfrac{b_2}{b_1}$.

The variable B appears in the three polynomials with the lower powers, so we eliminate B first. In the elimination of B, there are several coefficients to simplify, as we saw in $[S_2]$. This is very important because otherwise there will be problems with the computer's memory space. Having eliminated B, we obtain two polynomials in A and d. The resultant of the two polynomials in relation to A is factorized by:

$$(d^2 - d - 1)(d^2 + d - 1)(d^4 - 3d^3 + 3d^2 - 2d + 2)(2d^4 - 2d^3 + 3d^2 - 3d + 1)$$
$$(4d^5 + 3d^4 + 5d^3 + 2d^2 + d + 1)(d^5 + d^4 + 2d^3 + 5d^2 + 3d + 4)$$
$$(74d^{10} - 162d^9 + 617d^8 - 463d^7 + 713d^6 - 1126d^5 + 713d^4 - 463d^3 + 617d^2 - 162d + 74)$$
$$(2d^2 - 1)^2(d^2 - 2)^2(d^2 - d + 1)^4(d^2 + 1)^4(d^2 + d + 1)^{47}(d + 1)^{56}(d - 1)^{125}d^{397}$$
$$(\textit{a very long polynomial of degree } 128)^2.$$

Multiple roots of this polynomial usually produce trivial solutions, e.g. the curve f_{10} split into a cubic f_3 with multiplicity 3, in symbols: $f_{10} = f_3^3(\cdots)$. For this reason, and because the polynomial is one of the shortest, we choose the polynomial $d^2 - d - 1$ and its root $\bar{d} = \frac{1}{2} + \frac{\sqrt{5}}{2}$.

It should be noted that we have not considered the roots of the other polynomials.

Considering the chosen root, we find that the two values of A and B that solve our algebraic system are given by $\bar{A} = 1 - \frac{2}{5}\sqrt{5}$ and $\bar{B} = \frac{5}{2} + \frac{\sqrt{5}}{2}$, respectively.

4. – The equation

Solving the linear system of conditions in line with the above values $\bar{d}, \bar{A}$ and $\bar{B}$, the equation of our irreducible Campedelli branch curve C_{10} is given by

$$
\begin{aligned}
&C_{10} : 5(-275 + 123\sqrt{5})(x_1^7x_2^3 + x_2^7x_3^3 + x_1^3x_3^7)+\\
&75(-29 + 13\sqrt{5})(x_1^7x_2^2x_3 + x_1x_2^7x_3 + x_1^2x_2x_3^7)+\\
&75(-15 + 7\sqrt{5})(x_1^7x_2x_3^2 + x_1^2x_2^7x_3 + x_1x_2^2x_3^7)+\\
&125(-1 + \sqrt{5})(x_1^7x_3^3 + x_1^3x_2^7 + x_2^3x_3^7)+\\
&5(685 - 303\sqrt{5})(x_1^6x_2^4 + x_2^6x_3^4 + x_1^4x_3^6)+\\
&5(1081 - 417\sqrt{5})(x_1^6x_2^3x_3 + x_1x_2^6x_3^3 + x_1^3x_2x_3^6)+\\
&39(235 - 59\sqrt{5})(x_1^6x_2^2x_3^2 + x_1^2x_2^6x_3^2 + x_1^2x_2^2x_3^6)+\\
&5(1391 - 335\sqrt{5})(x_1^6x_2x_3^3 + x_1^3x_2^6x_3 + x_1x_2^3x_3^6)+\\
&10(145 - 52\sqrt{5})(x_1^6x_3^4 + x_1^4x_2^6 + x_2^4x_3^6)+\\
&15(-205 + 97\sqrt{5})(x_1^5x_2^5 + x_2^5x_3^5 + x_1^5x_3^5)+\\
&15(155 + 37\sqrt{5})(x_1^5x_2^4x_3 + x_1x_2^5x_3^4 + x_1^4x_2x_3^5)+\\
&3(3575 + 1437\sqrt{5})(x_1^5x_2^3x_3^2 + x_1^2x_2^5x_3^3 + x_1^3x_2^2x_3^5)+\\
&15(643 + 465\sqrt{5})(x_1^5x_2^2x_3^3 + x_1^3x_2^5x_3^2 + x_1^2x_2^3x_3^5)+\\
&15(-299 + 307\sqrt{5})(x_1^5x_2x_3^4 + x_1^4x_2^5x_3 + x_1x_2^4x_3^5)+\\
&12(905 + 963\sqrt{5})(x_1^4x_2^4x_3^2 + x_1^4x_2^2 + x_1^2x_2^4x_3^4)+\\
&3(9407 + 5741\sqrt{5})(x_1^4x_2^3x_3^3 + x_1^3x_2^4x_3^3 + x_1^3x_2^3x_3^4) = 0.
\end{aligned}
$$

The equation is quite simple and all the properties of the curve can be deduced directly from its equation. The irreducibility of C_{10} can also be deduced from its parametric equations (cf. next section).

5. – The parametric equations

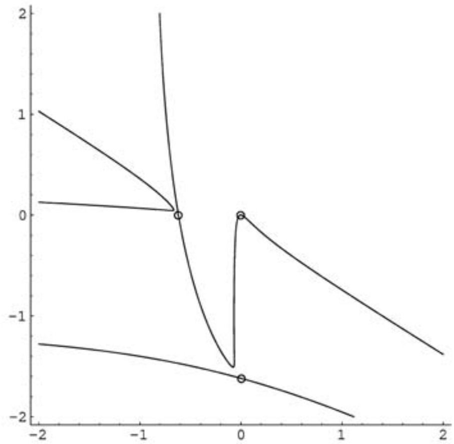
The curve C_{10} is rational. It can be parametrized by the linear pencil of octic curves C_8, having the same $[3, 3]$ point at $(0, 0, 1)$ as C_{10}, with five tacnodes at the remaining five $[3, 3]$ points of C_{10}, and with tacnodal tangent lines that coincide with the tangent singular lines at the $[3, 3]$ points on C_{10}, and finally the octic curves pass through the three simple points on C_{10} lying on the three sides of the fundamental triangle $x_1x_2x_3 = 0$.

The parametric equations of C_{10}, that are obtained using the above pencil of octic curves C_8 are rather long. If we consider the affine coordinates $x = \dfrac{x_1}{x_3}$, $y = \dfrac{x_2}{x_3}$, then the affine parametric equations of C_{10}, with parameter t, are given by

$$\begin{cases} x = \dfrac{f(t)}{g(t)} \\ y = \dfrac{h(t)}{k(t)} \end{cases},$$

where $f(t), g(t), h(t), k(t)$ are polynomials of degree seven with coefficients given by numbers with 46-54 digits - making them too long to be reproduced here.

6. – Plotting with "Mathematica"



In the picture there are three $[3, 3]$ points, their location is drawn with a small circle "∘". At each $[3, 3]$ point, there are two of the three local branches, generating such singularity, that are complex branches. So, in the picture only one real local branch appears and the singular points appear as simple points. We do not know whether an irreducible Campedelli branch curve exists having one or more $[3,3]$ points with three real local branches, as it happens in the case of reducible Campedelli branch curves. The shape of such a curve would be very interesting, because it is rational and it has only one connected component in the Euclidean topology of the projective plane.

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