Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.2, p. 187–212.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_2_187_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 187-212

Survey on Probabilistic Methods for the Study of Kac-like Equations

FEDERICO BASSETTI - ESTER GABETTA

Dedicated to the memory of Carlo Cercignani

Sunto. – *This mainly explanatory paper shows how direct application of probabilistic methods, pertaining to central limit general theory, can enlighten us about the relaxation to equilibrium of the solutions of one–dimensional Boltzmann type equations. In particular, conditions under which the solutions of these equations converge to suitable scale mixture of stable distributions are reviewed. In addition, some recent results about the rate of convergence to steady states, with respect to various metrics, are summarized. Finally, by resorting to the above mentioned probabilistic methods, some new results related to a linear kinetic model are proven.*

1. – Introduction

Dynamical processes in many body systems are often modeled by kinetic equations, that describe the dynamics of a single particle distribution. In these equations the so called *collisional term* takes into account the complicated interaction between particles. A classical problem related to kinetic equations is the study of the asymptotic behavior of the solution as the time goes to infinity, that is the study of the *relaxation to equilibrium* [21, 22, 23, 25, 26]. This problem is very challenging for physically realistic (multidimensional) kinetic models. For this reason, simplified one-dimensional and spatially homogeneous equations, such as the celebrated *Kac equation* [44], has been introduced and studied. Since the simplified models usually preserve many essential features of the more realistic ones (for example the non-linearity), their analysis provides a pattern for subsequent possible generalizations. To study the above mentioned problem, many specific tools has been fruitfully employed in the analytic setting, such as: closed moment equations, Fourier transforms, contracting distances, etc. An alternative approach to the study of the relaxation to equilibrium has been developed starting from the idea of H.P. McKean Jr. of relating the solution of the Kac equation to a stochastic process. See [45, 46, 56, 57, 55]. In point of fact, in the last decade, the transition from the analytic to the probabilistic approach produced interesting new results. The aim of the present review is to show how probabilistic techniques, essentially linked to *the central limit theorem* (CLT) of probability theory, permitted to solve some open problems and to refine some already known results concerning the Kac equation and some of its generalizations.

The paper is organized as follows. In Section 2 we introduce the Kac model and its generalizations. In Section 3 we provide the probabilistic representation of the solution of the generalized Kac equation. In Section 4 we summarize the relevant material on stable distributions and central limit theorems. Section 5 deals with the stationary solutions, showing that they are suitable scale mixtures of stable laws. In Section 6 we review the main results concerning the relaxation to equilibrium of the solution of the Kac equation and of the kinetic equations presented in Section 2. In Section 7 we give a brief exposition of some recent results about the rate of convergence to equilibrium with respect to various distances. Some new results related to a linear kinetic model are proven in the last section.

2. – The Kac equation and some of its generalizations

An important example of kinetic equation is the classical homogeneous Boltzamnn equation with Maxwell-type interactions, where scattering probability rates of the two particles at time of the interaction are independent of their relative velocity [21]. In order to better understand the connection between the n-particles system and the corresponding kinetic equation, M. Kac [44] introduced and studied a one-dimensional caricature of a Maxwellian gas. He considered a system of n interacting particles on the real axis and, under suitable conditions, he got, for $n \to +\infty$, the following analogous of the Boltzmann equation:

$$\begin{cases} \dfrac{\partial}{\partial t} f(v,t) + f(v,t) = Q^+(f(\cdot,t), f(\cdot,t))(v) \\ f(v,0) := f_0(v) \qquad (t > 0, v \in \mathbb{R}) \end{cases} \tag{2.1}$$

where $f(\cdot,t)$ stands for the probability density function of the velocity of a molecule at time t and the bilinear collisional term is

$$Q^+(f(\cdot,t), f(\cdot,t))(v) := \int_{\mathbb{R}\times[0,2\pi)} f(vc(\theta) - ws(\theta), t) f(vs(\theta) + wc(\theta), t) \frac{dw d\theta}{2\pi} \tag{2.2}$$

with $c(\theta) := \cos\theta$ and $s(\theta) := \sin\theta$. It is easy to check that the Fourier transform $\phi(\xi,t) := \int e^{i\xi v} f(v,t) dv$ of $f(\cdot,t)$ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(\xi,t) + \phi(\xi,t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \phi(\xi s(\theta), t)\phi(\xi c(\theta), t) d\theta \\ \phi(\xi, 0) := \phi_0(\xi) \qquad (t > 0, \xi \in \mathbb{R}), \end{cases} \tag{2.3}$$

where ϕ_0 stands for the Fourier transform of f_0. See [9].

Relatively recently, models of granular gases were introduced in the mathematical physics framework [10, 24]. In this case one considers pseudo-Maxwellian particles approximating dissipative hard spheres. The undergoing interparticle inelastic collisions are described by a Boltzamnn-like collision term that does not preserve the kinetic energy. To model one-dimensional caricature of granular gases suitable modifications of the Kac equation (2.1) have been introduced. See [3, 8, 52]. For instance, [52] used equation (2.3) with $c(\theta) = \cos(\theta)|\cos(\theta)|^p$ and $s(\theta) = \sin(\theta)|\sin(\theta)|^p$, where the positive constant p measures the degree of inelasticity during collisions. From now on we shall call this equation *inelastic Kac equation*.

The previous one-dimensional models can be seen as special cases of a more general model, introduced in [4], governed by a Kac like non-linear kinetic equation of the form

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t,\xi) + \phi(t,\xi) = \hat{Q}^+(\phi(t,\cdot),\phi(t,\cdot))(\xi) \qquad (t \geq 0, \xi \in \mathbb{R}) \\ \phi(0,\xi) = \phi_0(\xi) \end{cases} \tag{2.4}$$

where the collisional gain term $\hat{Q}^+$ is a *generalized Wild convolution*,

$$\hat{Q}^+(\phi(t,\cdot),\phi(t,\cdot))(\xi) := \mathbb{E}\left[\phi(t;L\xi)\phi(t;R\xi)\right] \qquad (\xi \in \mathbb{R}). \tag{2.5}$$

Above, (L,R) is a random vector defined on a probability space $(\Omega,\mathcal{F},P)$ and $\mathbb{E}$ denotes the expectation with respect to the probability measure P. The initial condition ϕ_0 is a prefixed Fourier-Stieltjes transform of a probability measure μ_0, i.e. $\phi_0(\xi) = \int e^{i\xi v}\mu_0(dv)$. Hence, $\phi(t;\cdot)$ can be viewed as a Fourier-Stieltjes transform of a time dependent probability measure μ_t, i.e. $\phi(t;\xi) = \hat{\mu}_t(\xi) = \int_{\mathbb{R}} e^{i\xi v}\mu_t(dv)$. In the following, we will say that μ_t is a solution of the *generalized Kac equation* (2.4), with initial condition μ_0, provided that its Fourier-Stieltjes transform $\phi(t,\cdot)$ is a solution of (2.4) with initial condition ϕ_0. Finally, we shall denote by F_0 the probability distribution function of μ_0, that is $F_0(x) := \int_{(-\infty,x]} \mu_0(dv)$.

A fundamental assumption on (L,R) in this kind of equation is that there exists an a in $(0,2]$ such that

$$\mathbb{E}\left[|L|^a + |R|^a\right] = 1. \tag{2.6}$$

It is worth noticing that (2.6) expresses a conservation law. For instance if $a = 1$ or 2 condition (2.6) entails the conservation of momentum and energy, respectively.

We recall that the generalized Kac equation (2.4)–(2.5) is strictly related to a class of Maxwell-type equations introduced and studied, by analytic techniques, in [11].

The *Kac equation* (2.3) is obtained as a special case of (2.4) for the particular choice $L = \cos\Theta$ and $R = \sin\Theta$, where Θ is uniformly distributed on $[0, 2\pi)$. In this case, since $L^2 + R^2 = \sin^2\Theta + \cos^2\Theta = 1$ almost surely (a.s.), (2.6) holds with $a = 2$. The above–mentioned inelastic Kac equation fits into the framework of (2.4)–(2.5) letting $L = \cos\Theta|\cos\Theta|^p$ and $R = \sin\Theta|\sin\Theta|^p$. Indeed, for $a = 2/(p+1)$, one has

$$|L|^a + |R|^a = 1, \qquad \text{a.s.} \tag{2.7}$$

and thus also (2.6) holds.

Finally, we mention that equation (2.4)-(2.5) with $a = 1$ has been used also to model the temporal distribution of wealth, represented by μ_t, in a simplified economy. See [49] and references therein. In these models the relaxed condition (2.6) is very important, as it takes into account stochastic gains and losses due to the trade with risky investments. In particular, wealth distributions with heavy tails are consistent with certain models satisfying (2.6), but are excluded under the stricter condition (2.7) of deterministic trading. See [49, 4].

It is worth pointing out that a semi-explicit expression for the Fourier-Stieltjes transform of the solution of (2.4)-(2.5) is given by the *Wild sum*

$$\phi(t;\xi) := \sum_{n=0}^{\infty} e^{-t}(1-e^{-t})^n \hat{q}_n(\xi) \qquad (t \geq 0, \xi \in \mathbb{R}) \tag{2.8}$$

where $\hat{q}_n$ is recursively defined by

$$\begin{cases} \hat{q}_0(\xi) := \phi_0(\xi) \\ \hat{q}_n(\xi) := \dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1} \mathbb{E}[\hat{q}_j(L\xi)\hat{q}_{n-1-j}(R\xi)] \qquad (n = 1, 2, \dots). \end{cases} \tag{2.9}$$

Originally, the series (2.8) has been derived in [62] for the solution of the Kac equation. It is easy to prove, following the same line of reasoning of the original paper of Wild [62], that (2.8) is also the unique solution of equation (2.4) in the class of the Fourier-Stieltjes transforms. See, e.g., Proposition 1 in [4].

We conclude this section by recalling that the aim of the present paper is to survey some results concerning the relaxation to equilibrium of the solution of (2.4)-(2.5), obtained via the application of probabilistic methods.

In particular, we will investigate the conditions under which $\phi(t, \cdot)$ converges, as $t \to +\infty$, to a *stationary profile* ϕ_∞, solution of the integral equation

$$\phi_\infty(\xi) = \hat{Q}^+(\phi_\infty, \phi_\infty)(\xi) \qquad (\xi \in \mathbb{R}). \tag{2.10}$$

3. – Probabilistic representation of the solutions

As already recalled in the introduction, the starting point of the probabilistic approach is a suitable representation of the solution of the generalized Kac equation, derived from the ideas of McKean [45, 46]. In point of fact, in [36] it has been proved that the solution ϕ of the Kac equation is the characteristic function of a sum of a random number of randomly weighted random variables. Later, in [4] the validity of a slightly different representation has been provided for the solution of the generalized Kac equation (2.4)-(2.5).

Now, a description of all the elements, which are necessary for the understanding of this probabilistic representation, is given.

Let the probability space $(\Omega, \mathcal{F}, P)$, mentioned apropos of (2.5), be large enough to support the following random elements:

- a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and identically distributed random variables with probability distribution function F_0, i.e. $P\{X_i \le x\} = F_0(x)$;
- a sequence $\big((L_n, R_n)\big)_{n\in\mathbb{N}}$ of independent and identically distributed random vectors, distributed as (L, R);
- a sequence $(I_n)_{n\in\mathbb{N}}$ of independent integer random variables, each I_n being uniformly distributed on the indices $\{1, 2, \dots, n\}$;
- a stochastic process $(\nu_t)_{t\ge 0}$, with $\nu_t \in \mathbb{N}$ and $P\{\nu_t = n\} = e^{-t}(1-e^{-t})^{n-1}$, $n \ge 1$.

Moreover, let $(I_n)_{n\in\mathbb{N}}$, $(L_n, R_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$ and $(\nu_t)_{t\ge 0}$ be stochastically independent.

Next, define a random array of weights $\bar{\beta} := [\beta_{j,n} : j = 1, \dots, n]_{n\ge 1}$ recursively:

$$\beta_{1,1} := 1, \qquad (\beta_{1,2}, \beta_{2,2}) := (L_1, R_1)$$

and, for any $n \ge 3$,

$$\begin{aligned}(3.1)\qquad &(\beta_{1,n+1}, \dots, \beta_{n+1,n+1})\\ &\quad := (\beta_{1,n}, \dots, \beta_{I_n-1,n}, L_n\beta_{I_n,n}, R_n\beta_{I_n,n}, \beta_{I_n+1,n}, \dots, \beta_{n,n}).\end{aligned}$$

Finally, set

$$(3.2)\qquad W_n := \sum_{j=1}^{n} \beta_{j,n} X_j \quad \text{and} \quad V_t := W_{\nu_t} = \sum_{j=1}^{\nu_t} \beta_{j,\nu_t} X_j.$$

We are now in a position to formulate the announced representation.

PROPOSITION 3.1 (Probabilistic representation of $\phi(t)$, [36]-[4]). – *The characteristic function $\phi(t, \cdot)$ of V_t, is given by*

$$\phi(t, \xi) := \mathbb{E}[e^{i\xi V_t}] = \sum_{n=0}^{\infty} e^{-t}(1-e^{-t})^n \mathbb{E}[e^{i\xi W_{n+1}}] \qquad (t > 0,\ \xi \in \mathbb{R}).$$

It represents the unique and global solution to equation (2.4)-(2.5), *with initial condition* $\phi_0(0,\xi) = \phi_0(\xi)$.

The probabilistic representation contained in the previous proposition is based on the idea, due to McKean, of re-writing the Wild series in terms of a random walk on a class of binary trees, the so-called *McKean trees*, [45, 46]. In point of fact, each finite sequence $\mathcal{I}_n = (I_1, I_2, \dots, I_{n-1})$ corresponds to a McKean tree with n leaves. The tree associated with $\mathcal{I}_{n+1}$ is obtained from the tree associated to $\mathcal{I}_n$ upon replacing the I_n-th leaf (counting from the left) by a binary branching with two new leaves. The left of the new branches is labeled with L_n, and the right branch with R_n. The weights $\beta_{j,n}$ are associated with the leaves of the $\mathcal{I}_n$-tree: $\beta_{j,n}$ is the product of the labels assigned to the branches along the ascending path connecting the j-th leaf to the root. See Figure 1 for an illustration. For more information on the connection between the McKean trees and the Wild series, see [15, 6, 36]. It is worth recalling that, in the probabilistic literature, McKean trees are usually referred to as *random binary search trees*. See, e.g., [31].

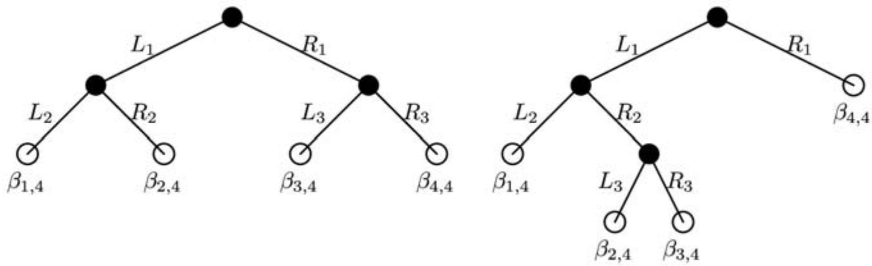

Fig. 1. Two 4-leafed McKean trees, with relative weights $\beta_{j,4}$: the left tree is generated by $\mathcal{I}_4 = (1,1,3)$ and its weights are $\beta_{1,4} = L_1L_2$, $\beta_{2,4} = L_1R_2$, $\beta_{3,4} = R_1L_3$, $\beta_{4,4} = R_1R_3$; the right tree is generated by $\mathcal{I}_4 = (1,1,2)$ and its weights are $\beta_{1,4} = L_1L_2$, $\beta_{2,4} = L_1R_2L_3$, $\beta_{3,4} = L_1R_2R_3$, $\beta_{4,4} = R_1$.

By Proposition 3.1 it is clear that the behavior of $\phi(t)$ as $t \to +\infty$ is determined by the behavior of the law of W_n, as $n \to +\infty$. Apropos of this, we anticipate that direct application of the classical central limit theorem is not allowed to investigate the weak limit of W_n, since the weights in (3.2) are not mutually independent. However, as we shall see in Section 6, by resorting to suitable forms of conditioning for W_n, one can take advantage of classical propositions pertaining to the central limit problems. The idea of using the probabilistic representation of the solution $\phi(t,\cdot)$, in combination with a suitable conditioning argument, to study the relaxation to equilibrium of Kac like equations, has been used for the first time in [36], and it is in part inspired to a technique developed in [33] to prove central limit theorems for triangular arrays of exchangeable random variables.

From now on we make the following assumption:

(H_1) *L and R are positive random variables.*

Since the original Kac model assumes uniform distribution of (L, R) on the circle, the assumption (H_1) could seem very severe. However, the study of both the classical Kac model and of the inelastic Kac model can be reduced to the study of the positive case. Indeed, it is well-known that their solution can be written as

$$\phi(t, \xi) = e^{-t} Im(\phi_0(\xi)) + \phi^*(t, \xi),$$

where ϕ^* is the solution to problem (2.4)-(2.5), with $Re(\phi_0)$ in the place of ϕ_0, $L = |\sin(\Theta)|^{1+p}$ and $R = |\cos(\Theta)|^{1+p}$ ($p \geq 0$). Hence, note that if μ_0 is a symmetric probability distribution, then $\phi(t, \cdot)$ itself is a solution of (2.4)-(2.5), with $L = |\sin(\Theta)|^{1+p}$ and $R = |\cos(\Theta)|^{1+p}$.

Finally, in addition to hypothesis (H_1), we will assume that

(H_2) *μ_0 is symmetric.*

This assumption is unnecessarily restrictive for the validity of the results we will present, but it simplifies the statements of the results and certain types of computations. The reader will be referred to the suitable references for the results concerning the non–symmetric initial datum.

In spite of the above remark about the direct applicability of the classical central limit theorem, such a distinguished theorem of the probability theory provides a fundamental direction toward the results we want to review. Then, in the next section, some basic facts concerning stable distributions and central limit theorems will be recalled.

4. – Stable laws and central limit theorem

We recall that a *symmetric a-stable distribution* (for $a \in (0, 2]$) – in symbol Γ_{a,k_0} – is a probability measure with Fourier-Stieltjes transform

$$\hat{\gamma}_{a,k_0}(\xi) = e^{-k_0|\xi|^a} \qquad (\xi \in \mathbb{R}). \tag{4.1}$$

Here $k_0 > 0$ is a positive parameter, and $k_0^{1/a}$ is the so called *scale parameter*. Note that if $a = 2$ then $\hat{\gamma}_{a,k_0}$ is the Fourier-Stieltjes transform of a Gaussian distribution with variance $\sigma^2 = 2k_0$ and zero mean. It is well–known that any stable distribution is absolutely continuous and its density will be denoted by

γ_{a,k_0}, in particular

$$e^{-k_0|\xi|^a} = \int_{\mathbb{R}} e^{i\xi v}\gamma_{a,k_0}(v)dv.$$

Stable laws are strictly related to the limit in distribution of normalized sums of independent and identically distributed random variables, i.e. to a particular form of the *central limit problem* of probability theory.

Before proceeding, it is worth recalling that a sequence of probability measures $(\mu_n)_{n\geq 1}$ converges weakly to a probability measure μ if, for every bounded and continuous real valued function g,

$$\lim_{n\to+\infty} \int g(y)\mu_n(dy) = \int g(y)\mu(dy).$$

Analogously, a sequence of random variables $(Y_n)_{n\geq 1}$, with probability distributions $(\mu_n)_{n\geq 1}$, is said to converge in distribution to a random variable Y, with law μ, if $(\mu_n)_{n\geq 1}$ converges weakly to μ, or, what is the same, if

$$\lim_{n\to+\infty} \mathbb{E}[g(Y_n)] = \mathbb{E}[g(Y)],$$

for every bounded continuous function g. Recall also that the Lévy continuity theorem, see e.g. Theorem 8.28 in [14], states that $(Y_n)_n$ converges in distribution to Y if and only if the characteristic function of Y_n converges pointwise to the characteristic function of Y, i.e.

$$\lim_{n\to+\infty} \mathbb{E}[e^{i\xi Y_n}] = \mathbb{E}[e^{i\xi Y}] \qquad (\xi \in \mathbb{R}).$$

Now we are in a position to formulate the characterization of a (symmetric) stable distribution in terms of limit of a normed sum of random variables. According to the hypothesis that the initial datum is symmetric, we confine ourselves to considering only the case of symmetric summands. For the general statement of this well-known result, as well as for the general definition of stable law, see, for instance, [14].

Let $X_1, X_2, \ldots$ be independent and identically distributed real–valued random variables, with symmetric probability distribution, and set

$$S_n := \frac{1}{n^{1/a}} \sum_{i=1}^{n} X_i. \tag{4.2}$$

A random variable X_∞ is the limit in distribution of (4.2) *if and only if X_∞ has characteristic function* (4.1) *for some $k_0 \geq 0$.*

A symmetric distribution function F_0 is said to be an element of the *normal domain of attraction* of a symmetric stable law of exponent *a if, for any sequence of independent and identically distributed real-valued random variables*

$(X_n)_{n\geq 1}$, *with common distribution function* F_0, S_n *converges in distribution to a random variable* X_∞ *with symmetric stable law of exponent* a.

In other words, the normal domain of attraction of a stable law is the class of all the probability distributions of X_i for which a central limit theorem holds for the normed sum (4.2).

We summarize here some relevant theorems on the characterization of the normal domain of attraction of a stable law - for the proofs see, for instance, [38] or [14].

If $a = 2$, F_0 *belongs to the normal domain of attraction of a Gaussian law if and only if it has finite variance*

$$\sigma_0^2 := \int x^2 dF_0(x) < +\infty.$$

In contrast, *if* $a \neq 2$, *then a symmetric distribution function* F_0 *belongs to the normal domain of attraction of a symmetric a-stable law if and only if it satisfies*

$$\lim_{x\to+\infty} x^a(1 - F_0(x)) = \lim_{x\to-\infty} |x|^a F_0(x) = c^+ < +\infty. \tag{4.3}$$

The parameter k_0 in (4.1) can be expressed as a function of c^+ by

$$k_0 = \frac{2\pi c^+}{2\Gamma(a)\sin(\pi a/2)}. \tag{4.4}$$

Among the mathematicians who have given important contributions to the theory of stable laws as limiting distributions and their domain of attraction one has to recall P. Lévy, A. Ya. Khintchine, W. Feller, B. V. Gnedenko and W. Doeblin. See, e.g., [41]. A deep investigation into the area is given in the fundamental book of Gnedenko and Kolmogorov [38].

In order to characterize the normal domain of attraction of a stable distribution in terms of the Fourier-Stieltjes transform $\phi_0(\xi) := \int_{\mathbb{R}} e^{i\xi v} dF_0(v)$, it is useful to mention the following result:

A symmetric probability distribution function F_0 *belongs to the normal domain of attraction of an* $a-$*stable law if and only if*

$$1 - \phi_0(\xi) = (k_0 + v_0(\xi))|\xi|^a \qquad (\xi \in \mathbb{R}), \tag{4.5}$$

where v_0 *is bounded and* $|v_0(\xi)| = o(1)$ *as* $|\xi| \to 0$. See Theorem 2.6.5 in [43].

We conclude this section by stating a central limit theorem for *weighted sums* of random variables, which will be useful in the next section. Let $(b_{j,n})_{j,n}$ be an array of positive weights. Given any sequence of identically distributed random variables $(X_j)_{j\geq 1}$ with a symmetric distribution function F_0, set

$$\tilde{S}_n = \sum_{j=1}^n b_{j,n} X_j.$$

PROPOSITION 4.1. – *Assume that, for some a in* $(0, 2]$,

$$\lim_{n\to+\infty} \sum_{j=1}^{n} b_{j,n}^{a} = m_{\infty} < +\infty, \qquad \text{and} \qquad \lim_{n\to+\infty} \max_{j=1,\dots,n} b_{j,n} = 0. \tag{4.6}$$

If the symmetric probability distribution function F_0 belongs to the normal domain of attraction of a symmetric a-stable law, then

$$\lim_{n\to+\infty} \mathbb{E}[e^{i\xi\tilde{S}_n}] = \hat{\gamma}_{a,k_0 m_{\infty}}(\xi) = e^{-k_0 m_{\infty}|\xi|^a} \tag{4.7}$$

for every $\xi \in \mathbb{R}$, with k_0 defined in (4.4) *for $a \neq 2$ and $k_0 = \sigma_0^2/2$ for $a = 2$.*

PROOF. – The proof can be obtained as a consequence of the central limit theorem for triangular arrays. We give here a simple direct proof. First note that, by (4.6),

$$\psi_n(\xi) := \prod_{j=1}^{n} \hat{\gamma}_{a,k_0}(\xi b_{j,n}) = e^{-k_0|\xi|^a \sum_{j=1}^{n} |b_{j,n}|^a} \to e^{-k_0|\xi|^a m_{\infty}}.$$

Hence it suffices to prove that $|\phi_n(\xi) - \psi_n(\xi)| \to 0$ where

$$\phi_n(\xi) := \mathbb{E}[e^{i\xi\tilde{S}_n}] = \mathbb{E}\left[e^{i\xi \sum_{j=1}^{n} b_{j,n} X_j}\right] = \prod_{j=1}^{n} \mathbb{E}[e^{i\xi b_{j,n} X_j}] = \prod_{j=1}^{n} \phi_0(b_{j,n}\xi).$$

Since an a–stable law trivially belongs to its normal domain of attraction, recalling that F_0 belongs to the normal domain of attraction of an a-stable distribution, by (4.5), one gets

$$\phi_0(\xi) - 1 + k_0|\xi|^a = -v_0(\xi)|\xi|^a \qquad \hat{\gamma}_{a,k_0}(\xi) - 1 + k_0|\xi|^a = -v_1(\xi)|\xi|^a \tag{4.8}$$

with v_i bounded and $|v_i(\xi)| = o(1)$ as $\xi \to 0$ $(i = 0, 1)$. Since for every complex numbers $z_1, \dots, z_n$ and $z'_1, \dots, z'_n$ of modulus ≤ 1

$$\left| \prod_{j=1}^{n} z_j - \prod_{j=1}^{n} z'_j \right| \leq \sum_{j=1}^{n} |z_j - z'_j|, \tag{4.9}$$

one has

$$\begin{aligned} |\phi_n(\xi) - \psi_n(\xi)| &\leq \sum_{j=1}^{n} \left|\phi_0(b_{j,n}\xi) - \gamma_{a,k_0}(b_{j,n}\xi)\right| \\ &\leq \sum_{j=1}^{n} \left|\phi_0(b_{j,n}\xi) - 1 + k_0|b_{j,n}\xi|^a\right| + \sum_{j=1}^{n} \left|\hat{\gamma}_{a,k_0}(b_{j,n}\xi) - 1 + k_0|b_{j,n}\xi|^a\right| \end{aligned}$$

and by (4.8) one obtains

$$|\phi_n(\xi) - \psi_n(\xi)| \le \sum_{j=1}^{n} \left|v_0(b_{j,n}\xi)|b_{j,n}\xi|^a\right| + \sum_{j=1}^{n} \left|v_1(b_{j,n}\xi)|b_{j,n}\xi|^a\right|.$$

After setting $b_{(n)} := \max_{j=1,\dots,n} b_{j,n}$, one gets

$$\sum_{j=1}^{n} \left|v_0(b_{j,n}\xi)|b_{j,n}\xi|^a\right| \le |\xi|^a \sup_{|u|\le|b_{j,n}\xi|} |v_0(u)| \sum_{j=1}^{n} |b_{j,n}|^a \to 0$$

since $\sum_{j=1}^{n} |b_{j,n}|^a \to m_\infty$ and $\sup_{|u|\le|\xi b_{(n)}|} |v_0(u)| \to 0$ by (4.6). Analogously one proves that

$$\sum_{j=1}^{n} \left|v_1(b_{j,n}\xi)|b_{j,n}\xi|^a\right| \to 0.$$

□

5. – Steady states and fixed point equations for distributions

An application of the probabilistic representation given in Proposition 3.1 leads to state that the possible limits ϕ_∞ of $\phi(t,\cdot)$, as $t \to +\infty$, are solutions of (2.10), i.e.

$$\phi_\infty(\xi) = \mathbb{E}[\phi_\infty(L\xi)\phi_\infty(R\xi)] \qquad (\xi \in \mathbb{R}). \tag{5.1}$$

A class of solutions of equation (5.1) is described in the remaining part of the present section.

Let us start by noticing that, if $L^a + R^a = 1$ a.s. for $a \in (0,2]$, as in the (elastic and inelastic) Kac model, it is immediate to check that $\hat{\gamma}_{a,k_0}$, defined in (4.1), is a solution of (5.1). Indeed, in this case,

$$\mathbb{E}[e^{-k_0|L\xi|^a} e^{-k_0|R\xi|^a}] = \mathbb{E}[e^{-k_0|\xi|^a(L^a+R^a)}] = \mathbb{E}[e^{-k_0|\xi|^a}] = e^{-k_0|\xi|^a}.$$

More generally, since W_n is a randomly weighted sum of independent and identically distributed random variables, in the light of the statements of the previous section (see Proposition 4.1), it is natural to search possible solutions of (5.1) as mixtures of stable laws.

Let us start by recalling that a probability measure is said to be a *scale mixture* of (symmetric) a-stable distributions if its Fourier-Stieltjes transform is given by

$$\hat{\rho}(\xi) = \int_{\mathbb{R}^+} e^{-m|\xi|^a} \nu(dm) \qquad (\xi \in \mathbb{R}) \tag{5.2}$$

for a probability measure ν on $\mathbb{R}^+ := [0, +\infty)$. Note that, $\hat{\rho}(\xi)$ is the Fourier-Stieltjes transform of the probability measure

$$R(\,\cdot\,) = \int_{\mathbb{R}^+} \Gamma_{a,m}(\,\cdot\,)\nu(dm),$$

which turns out to be absolutely continuous, with density

$$\rho(v) = \int_{\mathbb{R}^+} \gamma_{a,m}(v)\nu(dm),$$

if $\nu\{0\} = 0$.

At this stage observe that if ϕ_∞ is defined to be (5.2), then (5.1) reduces to

$$\begin{aligned}\int_{\mathbb{R}^+} e^{-m|\xi|^a}\nu(dm) \\ = \mathbb{E}\Big[\int_{\mathbb{R}^+} e^{-m_1|L|^a|\xi|^a}\nu(dm_1)\int_{\mathbb{R}^+} e^{-m_2|R|^a|\xi|^a}\nu(dm_2)\Big] \qquad (\xi \in \mathbb{R}),\end{aligned} \tag{5.3}$$

i.e.

$$\psi(u) = \mathbb{E}[\psi(L^a u)\psi(R^a u)] \qquad (u \geq 0), \tag{5.4}$$

where ψ stands for the Laplace transform of ν

$$\psi(u) := \int_{\mathbb{R}^+} e^{-um}\nu(dm) \qquad (u \geq 0).$$

In other words:

(5.5) *If ν is a probability measure on $\mathbb{R}^+$, whose Laplace transform ψ solves* (5.4) *for some a in* $(0, 2]$, *then $\hat{\rho}$ defined in* (5.2) *is a solution of* (5.1)

It should be emphasized that equations of the kind of (5.1) and (5.4) are known, in probability theory, as *fixed point equations for distributions* or *fixed point equations for smoothing transformations*. In point of fact, in the probabilistic literature, several results concerning the solutions of equations like (5.4) have been proven. See, e.g., [32, 47, 48].

We shall summarize some of these results in the next proposition, in which $\mathcal{S} : [0, \infty) \to [-1, \infty]$ is the convex function defined by

$$\mathcal{S}(s) := \mathbb{E}[L^s + R^s] - 1, \tag{5.6}$$

with the convention that $0^0 = 0$.

PROPOSITION 5.1 ([32],[47],[48]). – *Let* (H_1) *be in force. Assume that condition* (2.6) *holds true, that is* $\mathcal{S}(a) = 0$.

(i) *If* $L^a + R^a = 1$ *almost surely, then there is a unique probability distribution* ν *with* $\int_{\mathbb{R}^+} \nu(dv) = \int_{\mathbb{R}^+} v\nu(dv) = 1$, *and Laplace transform satisfying equation* (5.4). *In this case* ν *corresponds to the unit mass at* 1 $(\delta_1(\,\cdot\,))$.

(ii) *If* $P\{L^a + R^a = 1\} < 1$ *and if* $\mathcal{S}(s) < 0$ *for some* $s > a$, *then there is a unique probability distribution* ν *with* $\int_{\mathbb{R}^+} \nu(dv) = \int_{\mathbb{R}^+} v\nu(dv) = 1$ *and Laplace transform* ψ *satisfying* (5.4). *Moreover* ν *is non-degenerate and, for any* $p > a$, $\int_{\mathbb{R}^+} v^{\frac{p}{a}}\nu(dv) < +\infty$ *if and only if* $\mathcal{S}(p) < 0$.

Combining (5.5) with Proposition 5.1 we obtain the characterization of possible steady states as specific mixture of stable laws, as stated in the next proposition.

PROPOSITION 5.2. – *Assume that condition* (2.6) *holds true along with* $\mathcal{S}(s) < 0$ *for some* $s > a$. *Then*

$$\hat{\rho}(\xi) = \int_{[0,+\infty)} e^{-mk_0|\xi|^a}\nu(dm),$$

is a solution of (2.10) *for every* $k_0 > 0$, *when* ν *is the probability distribution characterized in* Proposition 5.1.

Note that when $L^a + R^a = 1$ almost surely, then the Fourier-Stieltjes transform $\hat{\rho}$, defined in the previous proposition, is simply the Fourier-Stieltjes transform of a stable distribution, while, if $P\{L^a + R^a = 1\} < 1$, then $\hat{\rho}$ is a non-trivial mixture of Fourier-Stieltjes transforms of stable distributions.

6. – Central limit theorems for generalized Kac equations

In this section we shall determine conditions on the initial datum assuring that the solution $\phi(t, \cdot)$ of (2.4) converges pointwise to a stationary solution, which is a mixture of stables distributions as described in the previous section. Recall that we are assuming (H_1) and (H_2), that is that L and R in (2.5) are positive random variables and that μ_0 is a symmetric probability measure.

Although, as already recalled, W_n is not a sum of independent random variables, one can apply the central limit theorem, as stated in Proposition 4.1, to study the conditional law of W_n, given the array of weights $\bar{\beta} = [\beta_{j,n} : j = 1, \ldots, n; n \geq 1]$. To this end, according to Proposition 4.1, it suffices to prove that

$$M_n^{(a)} := \sum_{j=1}^{n} \beta_{j,n}^{a} \tag{6.1}$$

converges a.s. to a limit $M_\infty^{(a)}$, as $n \to +\infty$, and that $\max_{j=1,\dots,n} \beta_{j,n}$ converges to zero in probability. This allows to apply Proposition 4.1 to the conditional law of W_n given $\bar{\beta}$ and to prove that the latter converges weakly to an a–stable law, rescaled by $\left(M_\infty^{(a)}\right)^{1/a}$. Hence, one obtains that the limit law of W_n is a scale mixture of a-stable laws.

Here, convergence of $M_n^{(a)}$ to $M_\infty^{(a)}$ is derived from the fact that, when $\mathcal{S}(a) = 0$, the sequence is a martingale, i.e., more precisely,

$$\mathbb{E}[M_{n+1}^{(a)} | \mathcal{G}_n] = M_n^{(a)} \qquad (n \geq 1),$$

$\mathcal{G}_n$ being, for every n, the σ-field generated by $(I_i, L_i, R_i)_{i=1,\dots,n-1}$. For a proof of this fact, see [4]. Then, since any positive martingale converges almost surely, see e.g. Theorem 5.14 in [14], one immediately gets the existence of $M_\infty^{(a)}$. Actually, much more can be said about this limit, as stated in the next proposition.

PROPOSITION 6.1 (The mixing measure, [4]). – *If* $\mathcal{S}(a) = 0$ *and* $\mathcal{S}(\gamma) < 0$ *for some* $\gamma > 0$, *then* $M_n^{(a)}$ *converges almost surely to a non-negative random variable* $M_\infty^{(a)}$ *and* $\max\limits_{j=1,\dots,n} \beta_{j,n}$ *converges to zero in probability. In particular, if* $\mathcal{S}(\gamma) < 0$ *for some* $0 < \gamma < a$, *then* $M_\infty^{(a)} = 0$ *a.s., while, if* $\mathcal{S}(\gamma) < 0$ *for some* $a < \gamma$, *then* $M_\infty^{(a)}$ *is distributed according to the law* ν *described in* Proposition 5.1.

A fundamental ingredient in the proof of the previous proposition is the identity

$$\mathbb{E}\Big[\sum_{j=1}^{n} \beta_{j,n}^{s}\Big] = \frac{\Gamma(n + \mathcal{S}(s))}{\Gamma(\mathcal{S}(s) + 1)\Gamma(n)}, \tag{6.2}$$

valid for every $s > 0$ such that $\mathbb{E}[L^s + R^s] < +\infty$, where $\Gamma(z) := \int_0^{+\infty} x^{z-1} e^{-x} dx$ is the Euler gamma function. In point of fact, formula (6.2) was derived for the first time in [35], for the special case $L = |\sin(\Theta)|$ and $R = |\cos(\Theta)|$, and fruitfully used in the study of the speed of convergence of the Kac equation in [36, 29, 37].

Proposition 4.1, combined with Proposition 6.1 and the conditioning argument sketched above, yields the main convergence theorem, i.e. that, under suitable assumptions, the solution of the generalized Kac equation converges weakly to a mixture of stable laws.

THEOREM 6.2 (CLT for the generalized Kac equation, [4]). – *Let* ϕ *be the solution of the generalized Kac equation* (2.4). *Assume that* $\mathcal{S}(a) = 0$ *for some*

$a \in (0,2]$ and that $\mathcal{S}(\gamma) < 0$ for some $\gamma > 0$. Assume also that the symmetric distribution function F_0 belongs to the normal domain of attraction of a symmetric a-stable distribution. If $\gamma < a$, then $\lim_{t\to+\infty} \phi(t,\xi) = 1$, while, if $\gamma > a$, then

$$\phi_\infty(\xi) := \lim_{t\to+\infty} \phi(t,\xi) = \int_{[0,+\infty)} e^{-k_0 m|\xi|^a} \nu(dm) \qquad (\xi \in \mathbb{R}), \tag{6.3}$$

where ν is the same as in Proposition 5.1 *and the parameter k_0 is defined in* (4.4) *for $a < 2$ and $k_0 = \sigma_0^2/2$ for $a = 2$.*

In spite of differences of mathematical language, the previous theorem can be referred to results contained in [11], which, on the one hand, are generally based on hypotheses slightly stronger than those assumed in Theorem 6.2 and, on the other hand, are valid for more general models. A detailed comparative analysis does not seem important here. With a view to the aim of the present paper, it appears more interesting to mention how the probabilistic approach leads to weaken the hypotheses relating both to the initial data and to the random collision coefficients (L,R). First, the martingale convergence theorem plays a fundamental role to prove the convergence of $M_n^{(a)}$. Second, starting from this convergence, one can apply well-known results, concerning fixed point equations for distributions, to extract useful information on the mixing measure ν. Third, the characterization of the normal domain of attraction of stable laws indicates how to refine conditions on ϕ_0.

It is worth noticing that expression (6.3) permits to obtain information on the tail behavior of the limit law by using well-known properties of stable distributions in combination with Proposition 5.1. The interested reader is referred to [4], where many variants of Theorem 6.2 are also proven. In particular the non-symmetric case and the positive case have been treated in full detail. It should be emphasized that, when $\nu \neq \delta_1$, in general it is not easy to give an analytic expression of ϕ_∞. In [7] some explicit examples, related to the economic applications, are provided.

As already noted, when $L = |\sin(\Theta)|$ and $R = |\cos(\Theta)|$, equation (2.5) reduces to the original Kac equation (2.1). In this case it is clear that $M_n^{(2)} = 1$ a.s. since $L^2 + R^2 = 1$ a.s. Hence, whenever $\int x^2 dF_0(x) = \sigma_0^2 < +\infty$, the solution μ_t converges weakly to the Gaussian law of variance σ_0^2. The fact that the finiteness of the variance, in the case of the Kac model, is a sufficient condition for the convergence to a Gaussian steady state is well-known, see for instance [20]. As far as the necessity of this condition is concerned, while it cannot be doubted from a physical intuitive standpoint, the first mathematical proof of this fact is contained in [36]. The next theorem states this result.

THEOREM 6.3 (CLT for the Kac equation, [36]). – *The solution* μ_t *of the Kac equation* (2.3) *converges weakly, as* $t \to +\infty$, *if and only if* μ_0 *has finite variance* σ_0^2.

We point out that the determination of a necessary and sufficient condition for the generalized Kac equation (2.4)-(2.5) is still an open problem, even in the case of the inelastic Kac equation. In this last case, a necessary condition for the convergence to a steady state is

$$\liminf_{x \to +\infty} x^{2/(1+p)}(1 - F_0(x)) < +\infty,$$

that is weaker than the sufficient condition – c.f. Theorem 6.2 –

$$\lim_{x \to +\infty} x^{2/(1+p)}(1 - F_0(x)) = c_0^+.$$

See [5].

We conclude this section by recalling that Theorem 6.3 has been complemented in [19] by providing a detailed analysis of what does actually happen to the solution of (2.3) when the initial energy is infinite. More precisely it has been proven that, as $t \to +\infty$, the total mass of the limiting distribution splits into two equal masses (of value $1/2$ each) adherent to $-\infty$ and $+\infty$, respectively. In the same paper an estimate of the quantitative rate at which such a phenomenon takes place is provided. In other words, when the initial energy is infinite, all of the mass "explodes to infinity" at a rate governed by the tails behavior of μ_0. In [18] this result has been extended to the multidimensional case of the Bolztmann equation for pseudo-Maxwellian molecules and an interesting connection with the so called *eternal solutions* of Bobylev and Cercignani [12, 13] has been established.

7. – Rate of convergence to equilibrium

Probabilistic methods have been successfully used also to obtain explicit rates of convergence to equilibrium (in suitable metrics) for the solution of both the original Kac equation and of its generalizations.

Before proceeding let us recall the definition of the metrics that we will use in the rest of the paper.

Let μ_1 and μ_2 be two probability measures on $\mathcal{B}(\mathbb{R})$, the Borel sets of $\mathbb{R}$. Let $F_i(x) = \mu_i\{(-\infty, x]\}$ $(i = 1, 2)$ be their probability distribution functions and $\phi_i(\xi) = \int e^{i\xi v}\mu_i(dv)$ $(i = 1, 2)$ their Fourier-Stieltjes transforms.

(D1) The *Kolmogorov uniform distance* between μ_1 and μ_2 is defined by

$$K(\mu_1, \mu_2) := \sup_{x \in \mathbb{R}} |F_1(x) - F_2(x)|.$$

(D2) The *minimal* L_p*-metric* (1) *(*$p>0$*) between* μ_1 *and* μ_2 *is defined by*

$$l_p(\mu_1,\mu_2) := \inf_{h\in\mathcal{H}(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^2} |x-y|^p h(dxdy) \right)^{\min(1,1/p)},$$

where $\mathcal{H}(\mu_1,\mu_2)$ is the class of probability measures on $\mathcal{B}(\mathbb{R}^2)$ with marginals μ_1 and μ_2, that is the probability measures h such that $h(\,\cdot\times\mathbb{R}) = \mu_1(\,\cdot\,)$ and $h(\mathbb{R}\times\cdot) = \mu_2(\,\cdot\,)$.

(D3) The *weighted* χ*-metric* of order $p>0$ between μ_1 and μ_2 is defined by

$$\chi_p(\mu_1,\mu_2) := \sup_{\xi\neq 0} \frac{|\phi_1(\xi)-\phi_2(\xi)|}{|\xi|^p}.$$

(D4) The *total variation distance* between μ_1 and μ_2 is defined by

$$TV(\mu_1,\mu_2) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mu_1(A)-\mu_2(A)|.$$

Recall that if μ_1 and μ_2 are absolutely continuous with respect to the Lebesgue measure, with density functions f_1 and f_2, respectively, then

$$TV(\mu_1,\mu_2) := \frac{1}{2}\int_{\mathbb{R}} |f_1(v)-f_2(v)|dv = \frac{1}{2}\|f_1-f_2\|_1.$$

We list now a few important properties of these distances. Consider a sequence of probability measures $(\mu_n)_{n\geq 1}$ and a probability measures μ.

(P1) If $K(\mu_n,\mu)\to 0$, as $n\to+\infty$, then μ_n converges weakly to μ. Conversely, by a classical result due to Pólya, if μ_n converges weakly to μ and the probability distribution function of μ is continuous, then $K(\mu_n,\mu)\to 0$. See, e.g., Theorem 1.11 in [51].

(P2) If $l_p(\mu_n,\mu)\to 0$, as $n\to+\infty$, then μ_n converges weakly to μ. If $\int|x|^p\mu_n(dx) < +\infty$ and $\int|x|^p\mu(dx) < +\infty$, then $l_p(\mu_n,\mu)\to 0$ if and only if μ_n converges weakly to μ and $\int|x|^p\mu_n(dx)\to\int|x|^p\mu(dx)$. See, e.g., Remark 7.1.11 in [2].

(P3) If $\chi_p(\mu_n,\mu)\to 0$, then μ_n converges weakly to μ.

For more information on these distances see, for instance, [54].

(1) The minimal L_p-metric was invented historically several times from different perspectives. Maybe historically the name Gini-Dall'Aglio-Kantorovich-Wasserstein-Mallows metric would be correct for this class of metrics. For simplicity reasons it seems preferable to use the name minimal "L_p-metric", and write it as l_p. See, e.g., [2, 54, 60, 61].

7.1 – *Rate of convergence for the Kac equation*

In [36] a first bound for the Kolmogorov's uniform metric between the solution at time t of the Kac equation and the Gaussian distribution has been proven.

In [37], in addition to Kolmogorov's uniform metric, the behavior of the l_1 and l_2 metrics (the latter also known as Tanaka functional) and weighted χ-metrics of order $p \geq 2$ are analyzed. Explicit rates, new or improvements on already well-known ones, are deduced both under the necessary assumption that initial data have finite energy, without assuming existence of moments of order greater than 2, and under the condition that the $(2+\delta)$-moment of the initial distribution is finite for some $\delta > 0$. It is worth noticing that (P1)-(P2)-(P3) combined with Theorem 6.3, imply that the metrics defined in (D1)-(D2)-(D3) turn out to be equivalent with respect to the convergence to equilibrium.

In [29] the study of the rate of convergence, with respect to the (more significant physically speaking) total variation distance, is dealt with to prove that the exponential rate of convergence $-1/4$, conjectured as optimal by McKean [45], is actually reached. We emphasize that the constant $1/4$ is the *spectral gap* of the linearized collisional operator of the Kac equation. Thanks to suitable developments of the probabilistic standpoint, in [29] the optimal rate is achieved under hypotheses which are definitely weaker than those considered in previous papers. See [17, 16, 34]. Instead of the finiteness of all absolute moments of the initial datum [16], one assumes the finiteness of the first four moments only; the finiteness assumption of the Linnik functional at f_0 is substituted with a weaker one on the behavior at infinity of the Fourier transform of the initial datum. The result is embodied in the next theorem, where

$$\gamma_\sigma(v) := \gamma_{2,2\sigma}(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2}{2\sigma^2}}$$

is the Gaussian density of variance σ^2.

THEOREM 7.1 (Optimal rate in total variation for the Kac equation: upper bound, [29]). – *Let f be the solution to the Kac equation* (2.1) *with initial condition f_0. Assume that $\int v^4 f_0(v)dv < +\infty$ and suppose that*

$$\phi_0(\xi) = \int e^{i\xi v} f_0(v)dv = o(|\xi|^{-p}) \qquad |\xi| \to +\infty$$

for some strictly positive p. Then there is a constant C^+, depending only on f_0, for which

$$\|f(\cdot,t) - \gamma_{\sigma_0}\|_1 \leq C^+ e^{-t/4}$$

where $\sigma_0^2 = \int v^2 f_0(v)dv$.

In other words this proves that the L^1–distance between the solution $f(\cdot, t)$ of the Kac's equation and the Gaussian density has an upper bound which goes to zero, as $t \to +\infty$, with an exponential rate equal to $-1/4$. For a detailed review of the results concerning the rates of relaxation to equilibrium in the Kac model, see [53].

We conclude this subsection by recalling that in [30] a lower bound which decreases exponentially to zero with rate $-1/4$ has been proved, provided that f_0 has nonzero fourth cumulant.

Let us recall that the "fourth cumulant" of a symmetric density function f_0 is defined by

$$k_4(f_0) := \int v^4 f_0(v)dv - 3\Big(\int v^2 f_0(v)dv\Big)^2.$$

The precise statement of the fact that the rate $-1/4$ may be the best possible one is contained in the following theorem.

THEOREM 7.2 (Optimal rate in total variation for the Kac equation: lower bound, [30]). – *Let f be the solution to the Kac equation* (2.1) *with initial condition f_0. If $\int v^4 f_0(v)dv < +\infty$, $\sigma_0^2 = \int v^2 f_0(v)dv$ and $k_4(f_0) \neq 0$, then there exists a constant C^-, depending only on f_0, for which*

$$\|f(\cdot, t) - \gamma_{\sigma_0}\|_1 \geq C^- e^{-t/4}$$

for every $t > 0$.

In [30], together with the above stated theorem, it is also proved that $\|f(\cdot, t) - \gamma_{\sigma_0}\|_1 \leq Ce^{-t/4}\rho_\delta(t)$, when $\int v^{4+\delta} f_0(v)dv < +\infty$ for some $0 \leq \delta < 2$ and $k_4(f_0) = 0$, with ρ_δ vanishing at infinity. In addition, generalizations of these statements are presented, together with some remarks about non-Gaussian initial conditions which yield the insuperable barrier of -1 for the rate of convergence.

7.2 – *Rate of convergence for the generalized Kac equation*

In [5] rates of convergence to equilibrium for the inelastic Kac model with respect to Kolmogorov's uniform metric and χ-weighted metrics have been derived. Explicit bounds are obtained under the sole assumption that the initial datum belongs to the normal domain of attraction of a stable law of exponent $a = 2/(1 + p)$. Sharper bounds, of an exponential type, are exhibited in the presence of additional assumptions concerning either the behavior, close to the origin, of the initial characteristic function, or the behavior, at infinity, of the initial probability distribution function. Exponential bounds on the χ-weighted distances were previously obtained (by analytic techniques) in [52], under slightly stronger assumptions. We state here a result concerning the case in which an exponential bound is reached. For all the other results the reader is referred to [5] .

THEOREM 7.3 (Rates for the inelastic Kac models w.r.t. Kolmogorov distance, [5]). – *Let* $\phi(t,\xi) = \int e^{i\xi v}\mu_t(dv)$ *be the solution to the inelastic Kac equation with initial condition* ϕ_0. *Assume that*

$$1 - \phi_0(\xi) = k_0|\xi|^a + O(|\xi|^{a+\delta}) \qquad \xi \to 0$$

for $a = 2/(1+p)$ *and some* $\delta > 0$. *Let* $\Gamma_{a,k_0}(dv) = \gamma_{a,k_0}(v)dv$ *be a symmetric a-stable distribution of scale parameter* $(k_0)^{1/a}$. *Then, there are (explicitly computable) positive constants* b_0 *and* b_1 *such that*

$$K(\mu_t, \Gamma_{a,k_0}) \leq b_0 e^{-b_1 t}.$$

The methods used to obtain these bounds have been inspired by the works of H. Cramér [27, 28] and by its developments in [42], concerning bounds for the Kolmogorov's metric for the sum of independent and identically distributed random variables belonging to the normal domain of attraction of a stable law.

Exponential bounds for the speed of convergence to equilibrium, with respect to l_p-distances, for the solution of the general model (2.4)-(2.5) are proven in [4], under additional hypotheses on the tail behavior of the initial datum. In the same paper some sufficient conditions for the convergence with respect to the L^1 distance are presented, even if no explicit bounds are given. In [50] the probabilistic representation of Proposition 3.1 has been used to derive explicit rates of convergence for the L^1 distance between the solution of the general problem (2.4)-(2.5) and its steady states when $a = 1, 2$.

The optimality of all the above mentioned rates is still an open problem.

8. – A linear Kac like equation.

As already recalled in Section 6, usually the nonlinearity of the models (2.4)-(2.5) does not permit to obtain explicit expressions for the steady states when $P\{L^a + R^a = 1\} < 1$. For this reason, in [58] a linear kinetic equation has been introduced to model the wealth redistribution in presence of taxation. In the same work the wealth distribution is assumed to be driven by collisions undergone with a fixed background. The aim of this section is to show how the probabilistic methods can be adapted to deal with this kind of equations.

To be specific, let us consider the following model

$$\begin{cases} \partial_t\phi(t;\xi) + \phi(t;\xi) = \widehat{Q}_l[\phi(t;\cdot)](\xi) & (t>0, \xi\in\mathbb{R}) \\ \phi(0;\xi) = \phi_0(\xi) \end{cases} \tag{8.1}$$

with

$$\widehat{Q}_l[\phi(t;\cdot)](\xi) := \mathbb{E}[\phi(t;A\xi)e^{i\xi B}] \tag{8.2}$$

where (A,B) is a random vector, $\phi_0(\xi) := \int e^{i\xi v}dF_0(v)$ for an initial probability distribution function F_0 and, as usual, ϕ is a Fourier-Stieltjes transform. Here B models the interactions with the background.

In particular, choosing $A = 1-\varepsilon$ and $B = \varepsilon Z$, where $\varepsilon\in(0,1)$ is a constant and Z is a random variable with characteristic function $\hat{M}$ such that $\mathbb{E}[Z] = 1$, (8.1) reduces to a special case of the above mentioned redistribution models studied in [58]. For this special choice of (A,B), by using contracting distances techniques, in [58] it is shown that, whenever $\int v^2 dF_0(dv) < +\infty$, the limit of $\phi(t,\xi)$, as $t\to+\infty$, is

$$\phi_\infty(\xi) = \prod_{k\geq 1}\hat{M}(\xi\varepsilon(1-\varepsilon)^{k-1}).$$

Here we want to show how one can extend these results to dynamics driven by a general random vector (A,B). First of all we start with the following elementary proposition.

The unique solution of (8.1) *is*

$$\phi(t;\xi) = \sum_{n\geq 0}\frac{t^n e^{-t}}{n!}\widehat{Q}_l^{(n)}[\phi_0(\,\cdot\,)](\xi) \tag{8.3}$$

where $\widehat{Q}_l^{(0)}[\psi](\xi) = \psi(\xi)$ *and*

$$\widehat{Q}_l^{(n)}[\psi] = \widehat{Q}_l[\widehat{Q}_l^{n-1}\psi].$$

Now we recast (8.3) in a probabilistic way. Let $V = (A_n, B_n)_{n\geq 1}$ be a sequence of independent and identically distributed random vectors with the same law of (A,B), let $\nu = (\nu_t, t\geq 0)$ be a Poisson process of parameter 1 and let X_0 be a random variable with distribution function F_0. Assume that V, ν and X_0 are independent. Set $Z_0 := X_0$ and, for any $n\geq 1$,

$$Z_n := \prod_{i=1}^n A_i X_0 + B_1 + \sum_{j=2}^n B_j\left(\prod_{i=1}^{j-1}A_i\right).$$

At this stage, we are in a position to give the probabilistic representation of the solution ϕ.

PROPOSITION 8.1. – *Under the previous hypotheses, for every* $n \geq 1$

$$\mathbb{E}[e^{i\xi Z_n}] = \widehat{Q_l}^{(n)}[\phi_0(\,\cdot\,)](\xi) \tag{8.4}$$

and hence

$$\phi(t;\xi) = \mathbb{E}[e^{i\xi Z_{\nu_t}}].$$

PROOF. – For every $k \geq 1$, Z_k has the same law of the random variable

$$Z_k^* = \prod_{i=1}^{k} A_i X_0 + B_k + \sum_{j=1}^{k-1} B_j \left(\prod_{i=j+1}^{k} A_k \right),$$

and, by induction, it follows easily that

$$\mathbb{E}[e^{i\xi Z_k^*}] = \widehat{Q}_l^{(k)}[\phi_0(\,\cdot\,)](\xi).$$

□

Given this representation, we turn our attention to the asymptotic behavior of the solution. Under suitable hypotheses, we will prove the convergence of the solution to a steady profile with respect to the l_1 at an exponential rate.

We need some more notation. Set

$$S_\infty := B_1 + \sum_{j\geq 2} B_j \left(\prod_{i=1}^{j-1} A_i \right),$$

$$\phi_\infty(\xi) := \mathbb{E}[e^{i\xi S_\infty}],$$

and finally denote by μ_∞ the probability distribution of S_∞ and by μ_t the law of Z_{ν_t}, i.e. $\phi(t,\xi) = \int e^{i\xi v} \mu_t(dv)$.

PROPOSITION 8.2. – *Assume that* A, B *and* X_0 *are such that* $\mathbb{E}[|X_0|] < +\infty$, $\mathbb{E}[|A|] < 1$ *and* $\mathbb{E}[|B|] < +\infty$. *Then* $P\{|S_\infty| < +\infty\} = 1$ *and* $\mathbb{E}[S_\infty] = \mathbb{E}[B]/(1-\mathbb{E}[A])$. *Moreover* $\phi(t)$ *converges pointwise to* ϕ_∞, *which is solution of the following integral equation*

$$\phi_\infty(\xi) = \widehat{Q_l}(\phi_\infty)(\xi) \qquad (\xi \in \mathbb{R}). \tag{8.5}$$

In addition

$$l_1(\mu_t, \mu_\infty) \leq \left(\mathbb{E}[|X_0|] + \frac{\mathbb{E}[|B|]\mathbb{E}[|A|]}{1 + \mathbb{E}[|A|]} \right) e^{-t(1-\mathbb{E}[|A|])}.$$

Once again, equation (8.5) is well–known in probability theory. In particular, the random variable S_∞, whose characteristic function is a solution of equation (8.5), is called *perpetuity*. See, e.g., [59, 39, 1] and references therein.

The connection between steady states of equation (8.1) and perpetuities allows to use some already known results to describe the tail behavior of the stationary solution. Apropos of this, we mention here two important results.

(i) $\mathbb{E}|S_\infty|^p < +\infty$ *if and only if* $\mathbb{E}|A|^p < 1$ *and* $\mathbb{E}|B|^p < +\infty$. See [1].

(ii) *If* $P\{|A| \le 1\} = 1$ *and* $\mathbb{E}[e^{\varepsilon|B|}] < +\infty$ *for some* $\varepsilon > 0$, *then* $\mathbb{E}[e^{\rho|S_\infty|}] < +\infty$ *for* $0 \le \rho < \sup\{\theta : \mathbb{E}[e^{\theta|B|}|A|] < 1\}$. See [1, 39, 40].

PROOF OF PROPOSITION 8.2. Let us show that

$$S_\infty = B_1 + \sum_{j\ge 2} B_j \left(\prod_{i=1}^{j-1} A_i\right)$$

is an a.s. absolutely convergent series. To do this let us show that

$$\limsup_{n\to+\infty} \mathbb{E}[S_n^+] < +\infty$$

where $S_n^+ = |B_1| + \sum_{j=2}^{n} |B_j| \left(\prod_{i=1}^{j-1} |A_i|\right)$. We have $\mathbb{E}[S_n^+] = \mathbb{E}[|B|] \sum_{k=0}^{n-1} \mathbb{E}[|A|]^k$ and hence $\limsup_{n\to+\infty} \mathbb{E}[S_n^+] < +\infty$ which gives $P\{|S_\infty| < +\infty\} = 1$. Now note that $Z_n = S_n + R_n$ with $S_n = B_1 + \sum_{j=2}^{n} B_j \left(\prod_{i=1}^{j-1} A_i\right)$ and $R_n = \prod_{i=1}^{n} A_i X_0$. Since $\mathbb{E}[|R_n|] = E[|A|]^n \mathbb{E}[|X_0|]$, then R_n converges in L^1 to zero and hence Z_n converges in distribution to S_∞. This means that $\phi_n(\xi) := \mathbb{E}[e^{i\xi Z_n}] \to \phi_\infty(\xi)$ for every ξ as $n \to +\infty$. Since

$$\phi_n(\xi) = \widehat{Q}_l(\phi_{n-1}(\,\cdot\,))(\xi),$$

the monotone convergence theorem shows that ϕ_∞ is a fixed point of $\widehat{Q}_l$. To conclude the proof, observe that

$$\begin{aligned}
l_1(\mu_t, \mu_\infty) &\le \mathbb{E}[|Z_{\nu_t} - S_\infty|] \le \sum_{n\ge 0} \frac{t^n e^{-t}}{n!} \mathbb{E}[|Z_n - S_\infty|] \\
&\le \sum_{n\ge 0} \frac{t^n e^{-t}}{n!} \left(\mathbb{E}[|R_n|] + \mathbb{E}[| \sum_{j\ge n+1} B_j (\prod_{i=1}^{j-1} A_i)|] \right) \\
&= \sum_{n\ge 0} \frac{t^n e^{-t}}{n!} \left(\mathbb{E}[|A|]^n \mathbb{E}[|X_0|] + \frac{\mathbb{E}[|B|]\mathbb{E}[|A|]^{n+1}}{1 + \mathbb{E}[|A|]} \right) \\
&= (\mathbb{E}[|X_0|] + \frac{\mathbb{E}[|B|]\mathbb{E}[|A|]}{1 + \mathbb{E}[|A|]}) \sum_{n\ge 0} \frac{t^n e^{-t}}{n!} \mathbb{E}[|A|]^n
\end{aligned}$$

□

Acknoledgments. The authors are grateful to Eugenio Regazzini for helping to improve an earlier version of the paper. They also thank an anonymous referee for his comments and suggestions.

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Received November 30, 2010 and in revised form January 10, 2011