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From the Boltzmann Equation to Hydrodynamic Equations in thin Layers

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In memory of Carlo Cercignani (1939-2010)

Abstract. – *The present paper discusses an asymptotic theory for the Boltzmann equation leading to either the Prandtl incompressible boundary layer equations, or the incompressible hydrostatic equations. These results are formal, and based on the same moment method used in [C. Bardos, F. Golse, D. Levermore, J. Stat. Phys 63 (1991), pp. 323–344] to derive the incompressible Euler and Navier-Stokes equations from the Boltzmann equation.*

1. – The Boltzmann equation and hydrodynamic models for thin layers of fluid.

The Boltzmann equation governs the evolution of a monatomic gas, following the principles of kinetic theory, founded by J. Clerk Maxwell and L. Boltzmann in the second half of 19th century. In this theory, the state of a monatomic gas is defined by its distribution function, which is the single-particle phase space number density of gas molecules. In other words, the distribution function $F \equiv F(t, x, v)$ is the number density with respect to the phase space volume element $dxdv$ of gas molecules to be found at time t at the position x with velocity v. If the influence of external force fields (such as gravity) on the dynamics of the gas molecules can be somehow neglected, the distribution function F satisfies

$$(\partial_t + v \cdot \nabla_x)F = \mathcal{C}(F) \tag{1}$$

where $\mathcal{C}(F)$ is the collision integral.

For each continuous function $f \equiv f(v)$ on $\boldsymbol{R}^3$ decaying rapidly enough as $|v| \to +\infty$, the collision integral $\mathcal{C}(f)$ is defined by the formula

$$\mathcal{C}(f)(v) := \iint_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} (f(v')f(v'_*) - f(v)f(v_*))b(v - v_*, \omega)dv_*d\omega\,, \tag{2}$$

where $v' \equiv v'(v, v_*, \omega) \in \boldsymbol{R}^3$ and $v'_* \equiv v'_*(v, v_*, \omega) \in \boldsymbol{R}^3$ are given in terms of

$v, v_* \in \boldsymbol{R}^3$ and $\omega \in \boldsymbol{S}^2$ by the formulas

$$\begin{cases} v' = v - (v - v_*) \cdot \omega\omega \,, \\ v'_* = v_* + (v - v_*) \cdot \omega\omega \,, \end{cases} \tag{3}$$

and where $b(v - v_*, \omega)$ is the collision kernel. Specifically, $b(v - v_*, \omega)$ is an a.e. positive function that depends on the interaction between gas molecules. In any case, it satisfies the symmetries

$$b(v - v_*, \omega) = b(v_* - v, \omega) = b(v' - v'_*, \omega) \tag{4}$$

a.e. in $(v, v_*, \omega) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{S}^2$. For instance, in the (somewhat academic) case where gas molecules behave like perfectly elastic hard spheres, the collision kernel is of the form

$$b(v - v_*, \omega) = \frac{1}{2} d_m^2 |(v - v_*) \cdot \omega| \,, \tag{5}$$

where d_m is the diameter of gas molecules. Henceforth, the notation $\mathcal{C}(F)(t, x, v)$ stands for $\mathcal{C}(F(t, x, \cdot))(v)$, meaning that the time and space variables are parameters in the collision integral, which acts on the velocity variable v only.

How the kinetic theory of gases is related to earlier descriptions in terms of fluid mechanics is an obviously important question, investigated in Maxwell's 1866 paper [33]. It is explicitly mentioned in Hilbert's 6th problem as a motivation for a "mathematical treatment of the axioms of physics". From then on, the problem of hydrodynamic limits of the Boltzmann equation has been an object of considerable interest among mathematicians and specialists of rarefied gas dynamics. We refer to [37] for a very detailed presentation of the most recent progress on these questions in the context of formal asymptotic analysis. Specifically, the hydrodynamic models derived in this book are based on asymptotic expansions in powers of a small dimensionless parameter, the Knudsen number that is the ratio of the mean free path of gas molecules to some macroscopic length scale of the flow considered. Such asymptotic expansions have been proposed by Hilbert, Chapman and Enskog in the 1910's.

Complete mathematical results in that direction have also been obtained in the last 35 years. For instance the Euler system of gas dynamics has been rigorously derived from the Boltzmann equation in [34] and [13]. Incompressible fluid models have also been established rigorously as scaling limits of the Boltzmann equation: see [32, 23] for the case of the Stokes equations, [10, 32, 35] for the case of the incompressible Euler equations, and [19, 8, 24, 25, 30] for the case of the incompressible Navier-Stokes equations. The basis for these derivations is a program laid out in [3, 4, 5]. These mathematical results should not be mistaken as an attempt to derive the incompressible Euler or Navier-Stokes equations from first principles, since the Boltzmann equation is not a first principle equation itself. Besides, the incompressible Euler or Navier-Stokes systems are well-established models in continuum mechanics that apply to (Newtonian) fluids in general – for instance to liquids – and not only to gases. As a matter of fact, the so-called incompressible hydrodynamic limits of the Boltzmann equation lead to correct motion equations in the special case of incompressible fluids with constant density; the energy equations obtained in this limit are however slightly different from the ones that would hold in a genuinely incompressible fluid. The interested reader is referred to footnotes 6 on p. 93 in [37] and 43 on p. 107 in [38], together with section 3.7.2 in [38] for a detailed account of this subtle point. Hydrodynamic limits of the Boltzmann equation should rather be viewed as qualitative information on the behavior of solutions of the Boltzmann equation in some asymptotic regimes – whose compatibility with the physical hypothesis on which the kinetic theory is based should always be checked on principle.

As stated in the title of this contribution, we are concerned with hydrodynamic limits of the Boltzmann equation leading to a class of well-known models used for thin layers of incompressible fluids. Specifically, the target hydrodynamic equations of interest in this paper is either the Prandtl system of equations used in the theory of viscous boundary layers

$$\left\{\begin{aligned} &\partial_t u_\| + u\cdot\nabla_x u_\| + \nabla_\| p = \nu\partial_\perp^2 u_\|\,,\\ &\mathrm{div}_x u = 0\,,\qquad \partial_\perp p = 0\,,\end{aligned}\right. \tag{6}$$

or the hydrostatic Euler equations

$$\left\{\begin{aligned} &\partial_t u_\| + u\cdot\nabla_x u_\| + \nabla_\| p = 0\,,\\ &\mathrm{div}_x u = 0\,,\qquad \partial_\perp p = 0\,.\end{aligned}\right. \tag{7}$$

Both these systems can be derived, at the formal level at least, from either the incompressible Navier-Stokes equations (in the case of the Prandtl boundary layer equations) or the incompressible Euler equations (in the case of the hydrostatic equations), assuming that the fluid flow takes place in a very thin layer. Therefore, the space variable $x\in\boldsymbol{R}^3$ is split as $x=(x_\|,x_\perp)$, where $x_\|$ is the two-dimensional space variable parallel to the layer's direction, and $x_\perp$ the one-dimensional variable orthogonal to the layer's direction. The notation $u_\|$, $\nabla_\|$ and $\partial_\perp$ refers to the component of the fluid velocity field u or the ∇ vector parallel to the direction of the layer, and to the partial derivative with respect to the perpendicular variable $x_\perp$.

After reviewing briefly the basic properties of the Boltzmann equation (section 2), and discussing the relevant scaling assumptions for gas flows confined in thin layers (section 3), we explain in section 4 how these systems of fluid dynamic equations can be formally derived from the Boltzmann equation following the moment method described in [4]. This is the main result in the present paper, stated below as Theorem 4.3. Some very brief indications concerning the boundary conditions are to be found in section 5. However, there seems to be serious difficulties in obtaining complete mathematical derivations of these models along the lines of the program presented in [5]. We shall comment on this in the final section of this paper.

Although the present paper deals with gas flows in thin layers, it leaves asides the important question of Knudsen layers. Knudsen layers are thin regions, usually located near boundaries of the gas container or of some body immersed in the gas, where the distribution function is not well approximated by a local Maxwellian form, for instance because of the nature of the gas surface interaction. In other words, the evolution of the distribution function in Knudsen layers is not governed by any fluid dynamic equation, but requires solving half-space problems for the steady Boltzmann equation: see for instance [6] for a survey on this topic as of 2006, with a list of references. Knudsen layers are much thinner (typically, of the order of a few mean free paths) than the layers considered in the present paper, where some fluid dynamic model is supposed to drive the distribution function. (For instance, the aspect ratio of the Prandtl viscous boundary layer is typically of the order of the square-root of that of the Knudsen layer.)

Carlo Cercignani will be remembered as a great scientific leader in the theory of rarefied gases, a topic of considerable importance for the past 60 years in view of its applications to space flight, microfluidics and other modern technologies. It is of course impossible in a few lines to do justice to his own work in this fundamental scientific field, which bear on mathematical as well as physical issues. His (about) 300 articles and research monographs speak for themselves. Yet no list of publications, however impressive, can give an exact idea Carlo Cercignani's influence on the kinetic theory of gases. Through the exceptional clarity of his series of books on the analysis of the Boltzmann equation, with his indefatigable enthusiasm and generosity in sharing his scientific insight, he has inspired many of his younger colleagues, and indeed several major results in the mathematical theory of kinetic models originate from conjectures proposed by Carlo Cercignani. The present work is dedicated to his memory.

2. – Basic structure of the Boltzmann equation.

In this section, we present the fundamental properties of the Boltzmann collision integral, which are of crucial importance in the derivation of hydrodynamic models from the kinetic theory of gases.

We begin with the symmetries of the collision relations (3): if, for some $\omega \in \boldsymbol{S}^2$, the vectors $v, v_*, v', v'_* \in \boldsymbol{R}^3$ are related by (3), then

$$\left\{ \begin{aligned} v' + v'_* &= v + v_* \,, \\ |v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2 \,. \end{aligned} \right. \tag{8}$$

In particular, for each $\omega \in \boldsymbol{S}^2$, the mapping $\mathcal{J}_\omega : (v, v_*) \mapsto (v', v'_*)$ is a linear isometry of $\boldsymbol{R}^6$, so that $dvdv_* = dv'dv'_*$. In view of (4), one has

$$\tag{9} \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \Phi(v, v_*)b(v-v_*, \omega)dvdv_* = \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \Phi(v', v'_*)b(v-v_*, \omega)dvdv_*$$
$$= \iint_{\boldsymbol{R}^3\times\boldsymbol{R}^3} \frac{1}{4}(\Phi(v, v_*)+\Phi(v_*, v)-\Phi(v', v'_*)+\Phi(v'_*, v'))b(v - v_*, \omega)dvdv_*$$

for each $\Phi \in C_c(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$.

Applying this identity to $\Phi(v, v_*) = (f(v')f(v'_*) - f(v)f(v_*))\phi(v)$ for each $\omega \in \boldsymbol{S}^2$, where $f \in C_c(\boldsymbol{R}^3)$ and $\phi \in C(\boldsymbol{R}^3)$ (for simplicity), one finds that

$$\tag{10} \int_{\boldsymbol{R}^3} \mathcal{C}(f)(v)\phi(v)dv = \frac{1}{4} \iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f(v')f(v'_*) - f(v)f(v_*))$$
$$\times (\phi(v)+\phi(v_*)-\phi(v')-\phi(v'_*))b(v-v_*, \omega)dvdv_*d\omega.$$

In particular, if $\phi(v) = 1$, $\phi(v) = v$ or $\phi(v) = |v|^2$, on account of (8), one finds that, for each $f \in C_c(\boldsymbol{R}^3)$ – more generally for each measurable f decaying fast enough at infinity,

$$\tag{11} \begin{aligned} \int_{\boldsymbol{R}^3} \mathcal{C}(f)dv &= 0\,, \\ \int_{\boldsymbol{R}^3} \mathcal{C}(f)vdv &= 0\,, \\ \int_{\boldsymbol{R}^3} \mathcal{C}(f)\frac{1}{2}|v|^2dv &= 0\,. \end{aligned}$$

The first identity is the conservation of mass (or, equivalently, of the number of particles), the second is the conservation of momentum, while the third is the conservation of kinetic energy by the collision process.

Indeed, if F is a solution of the Boltzmann equation decaying rapidly enough as $|v| \to +\infty$, one deduces from (11) the local conservation laws of mass, momentum and energy

$$\tag{12} \begin{aligned} \partial_t\int_{\boldsymbol{R}^3} Fdv + \mathrm{div}_x\int_{\boldsymbol{R}^3} vFdv &= 0\,, \\ \partial_t\int_{\boldsymbol{R}^3} vFdv + \mathrm{div}_x\int_{\boldsymbol{R}^3} v^{\otimes 2}Fdv &= 0\,, \\ \partial_t\int_{\boldsymbol{R}^3} \frac{1}{2}|v|^2Fdv + \mathrm{div}_x\int_{\boldsymbol{R}^3} v\frac{1}{2}|v|^2Fdv &= 0\,. \end{aligned}$$

Next we apply (9) to $\Phi(v,v_*)=(f(v')f(v'_*)-f(v)f(v_*))\ln f(v)$, where $f\in C(\boldsymbol{R}^3)$ is positive and rapidly decaying at infinity while $\ln f$ has at most polynomial growth at infinity, to find

$$\text{(13)}\quad \int_{\boldsymbol{R}^3} \mathcal{C}(f)\ln f dv = -\frac{1}{4}\iiint_{\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{S}^2} (f(v')f(v'_*)-f(v)f(v_*)) \times \ln\frac{f(v')f(v'_*)}{f(v)f(v_*)} b(v-v_*,\omega)dvdv_*d\omega \le 0\,,$$

with equality if and only if f is a Maxwellian, i.e. is of the form

$$\text{(14)}\qquad f(v)=\mathcal{M}_{(\rho,u,\theta)}(v):=\frac{\rho}{(2\pi\theta)^{3/2}}\exp\left(-\frac{|v-u|^2}{2\theta}\right).$$

In particular, $\mathcal{C}(f)=0$ if and only if f is a Maxwellian, i.e. if and only if there exists $\rho,\theta>0$ and $u\in\boldsymbol{R}^3$ such that $f=\mathcal{M}_{(\rho,u,\theta)}$.

One deduces from (13) the local version of Boltzmann's H Theorem: if F is a solution of the Boltzmann equation rapidly decaying while $\ln F$ has at most polynomial growth as $|v|\to+\infty$,

$$\text{(15)}\qquad \partial_t\int_{\boldsymbol{R}^3} F\ln F dv + \operatorname{div}_x\int_{\boldsymbol{R}^3} vF\ln F dv = -D(F)\le 0\,,$$

where $D(F)$ is the entropy production

$$D(F) = -\int_{\boldsymbol{R}^3}\mathcal{C}(f)\ln f dv\,.$$

Since Maxwellians are the only equilibrium distributions for the collision integral, studying the linearized collision integral about Maxwellians is a natural question. Henceforth, we denote for simplicity

$$M=\mathcal{M}_{(1,0,1)}\,.$$

Let $\mathcal{L}_M$ be the linear operator defined as follows

$$\text{(16)}\qquad \mathcal{L}_M f = -M^{-1}D\mathcal{C}(M)\cdot(Mf)$$

whose explicit expression is

$$\text{(17)}\qquad \mathcal{L}_M f(v)=\iint_{\boldsymbol{R}^3\times\boldsymbol{S}^2}(f(v)+f(v_*)-f(v')-f(v'_*))b(v-v_*,\omega)M(v_*)dv_*d\omega\,.$$

If the collision kernel b satisfies certain growth properties, corresponding to molecular interactions of the type known as "hard cutoff potentials", the operator

$\mathcal{L}_M$ is an unbounded, self-adjoint nonnegative Fredholm operator on the Hilbert space $L^2(\boldsymbol{R}^3; Mdv)$ with domain

$$\mathrm{Dom}(\mathcal{L}_M) = \{f \in L^2(\boldsymbol{R}^3; Mdv) \,|\, (\overline{b} \star M) f \in L^2(\boldsymbol{R}^3; Mdv)\}\,,$$

where $\star$ denotes the convolution in $\boldsymbol{R}^3$ while

$$\overline{b}(z) = \int_{\boldsymbol{S}^2} b(z, \omega) d\omega\,.$$

This remarkable result is due to H. Grad [26]. In the hard sphere case, the collision kernel b given by (5) satisfies precisely these growth conditions. If the molecular interaction is given by a radial, inverse power potential, the contribution of grazing collisions to the kernel b must be artificially truncated. Otherwise, $b(z, \omega) \to +\infty$ as $z \cdot \omega \to 0$ with $|z|$ bounded away from 0, and $\mathcal{C}(F)$ is a distribution of positive order.

An important property of the linearized collision integral is the characterization of its nullspace:

$$\mathrm{Ker}\,\mathcal{L}_M = \mathrm{span}\{1, v_1, v_2, v_3, |v|^2\}\,. \tag{18}$$

In particular there exists a pseudo-inverse

$$\mathcal{L}_M^{-1} : (\mathrm{Ker}\,\mathcal{L}_M)^\perp \to (\mathrm{Ker}\,\mathcal{L}_M)^\perp \cap \mathrm{Dom}\,\mathcal{L}_M$$

such that

$$\mathcal{L}_M(\mathcal{L}_M^{-1}\phi) = \phi \quad \text{for each } \phi \in (\mathrm{Ker}\,\mathcal{L}_M)^\perp\,.$$

In the sequel, we shall often encounter the tensor field

$$A(v) := v^{\otimes 2} - \frac{1}{3}|v|^2 I\,. \tag{19}$$

Notice that $A_{ij}(v) \perp \mathrm{Ker}\,\mathcal{L}_M$ for all $i,j = 1,2,3$, so that $\mathcal{L}_M^{-1}A \in (\mathrm{Ker}\,\mathcal{L}_M)^\perp \cap \mathrm{Dom}\,\mathcal{L}_M$ is well-defined.

A lucid account of all these basic properties of the Boltzmann equation can be found in [17].

3. – Hydrodynamic scalings for thin layers.

We start from the general dimensionless formulation of the Boltzmann equation (see [37])

$$\mathrm{Sh}\,\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}}\mathcal{C}(F)\,, \tag{20}$$

where Kn is the Knudsen number and Sh the kinetic Strouhal number.

Since we are concerned with incompressible flows, the distribution function F is sought as a perturbation of some uniform equilibrium state, which we can always choose to be the reduced, centered Maxwellian distribution M without loss of generality. The choice of this Maxwellian state sets the scale of the speed of sound $c_M = \sqrt{5/3}$. The macroscopic velocity field

$$u_F := \frac{\displaystyle\int_{\mathbf{R}^3} vF dv}{\displaystyle\int_{\mathbf{R}^3} F dv} \tag{21}$$

is compared to c_M, thereby defining a local Mach number $\mathrm{Ma} = u_F/c_M$.

The asymptotic limit of interest in the present discussion assumes $\mathrm{Kn} \ll 1$ corresponding with a hydrodynamic regime, $\mathrm{Ma} \ll 1$ corresponding with an incompressible flow, and $\mathrm{Sh} \sim \mathrm{Ma}$ corresponding with an unsteady, nonlinear hydrodynamic model. For more information on these prescriptions, the reader is referred to section 4.9 of [37] (and especially to p. 111 therein.)

At this point we introduce a further scaling assumption of particular relevance in the case of flows in thin layers of fluid. The position variable x is split as $x = (x_\parallel, x_\perp)$, where $x_\parallel$ and $x_\perp$ are respectively the position variables parallel to and orthogonal to the direction of the layer. Similarly, the velocity variable is split as $v = (v_\parallel, v_\perp)$.

Define $\varepsilon > 0$ to be the thickness of the fluid layer divided by the typical length scale of the flow in the direction parallel to the layer. The space variable x is rescaled as

$$\hat{x}_\parallel = x_\parallel\,, \qquad \hat{x}_\perp = \frac{x_\perp}{\varepsilon}\,.$$

Likewise, we expect that the ratio

$$\frac{u_{F,\perp}}{|u_{F,\parallel}|} = \frac{\displaystyle\int_{\mathbf{R}^3} v_\perp F dv}{\left|\displaystyle\int_{\mathbf{R}^3} v_\parallel F dv\right|} = O(\varepsilon)\,. \tag{22}$$

Next we explain how the dimensionless parameters Sh, Ma and Kn are related to ε. If the target hydrodynamic model is the Prandtl boundary layer system of equations, a first constraint is that the Reynolds number Re satisfies $\varepsilon \simeq \mathrm{Re}^{-1/2}$ — indeed, as is well known, the Prandtl boundary layer has thickness $\mathrm{Re}^{-1/2}$. In view of the von Karman relation $\mathrm{Kn} = \mathrm{Ma}/\mathrm{Re}$ (up to multiplication by some universal constant, see formula (3.95) on p. 60 in [37]), this entails $\mathrm{Kn} = \varepsilon^2\mathrm{Ma}$. Since the limiting model of interest involves advection with velocity field u_F, we also prescribe Sh $=$ Ma. Henceforth, we choose Sh $=$ Ma $= \varepsilon^2$, so that Kn $= \varepsilon^4$.

Therefore, the dimensionless form of the Boltzmann equation considered here is

$$\varepsilon^2 \partial_t \hat{F}_\varepsilon + v_\parallel \cdot \nabla_{\hat{x}_\parallel} \hat{F}_\varepsilon + \frac{1}{\varepsilon} v_\perp \partial_{\hat{x}_\perp} \hat{F}_\varepsilon = \frac{1}{\varepsilon^4} \mathcal{C}(\hat{F}_\varepsilon)\,, \tag{23}$$

where

$$F(t,x,v) = \hat{F}_\varepsilon\left(t, x_\parallel, \frac{x_\perp}{\varepsilon}, v\right).$$

On account of the choice Ma $= \varepsilon^2$, we set $\hat{F}_\varepsilon$ to be of the form

$$\hat{F}_\varepsilon(t,\hat{x},v) = M(1 + \varepsilon^2 \hat{g}_\varepsilon(t,\hat{x},v))\,. \tag{24}$$

Indeed, a typical distribution function corresponding with a Mach number Ma $= O(\varepsilon^2)$ is $\mathcal{M}_{(1,\varepsilon^2 u,1)}$, since

$$u_{\mathcal{M}_{(1,\varepsilon^2 u,1)}} = \frac{\displaystyle\int_{\boldsymbol{R}^3} v \mathcal{M}_{(1,\varepsilon^2 u,1)} dv}{\displaystyle\int_{\boldsymbol{R}^3} \mathcal{M}_{(1,\varepsilon^2 u,1)} dv} = \varepsilon^2 u\,,$$

while the speed of sound associated with $\mathcal{M}_{(1,\varepsilon^2 u,1)}$ is $\sqrt{5/3}$. By a Taylor expansion of $\mathcal{M}_{(1,\varepsilon^2 u,1)}$ about $\varepsilon = 0$, one has

$$\mathcal{M}_{(1,\varepsilon^2 u,1)} = M(1 + \varepsilon^2 u \cdot v) + O(\varepsilon^4)\,, \tag{25}$$

which is a particular case of (24).

At this point, we need to explain how the scaling (22), which is natural in the case of hydrodynamic models for thin layers of fluids, is formulated in kinetic theory. In order to gain some intuition on this issue, consider the special case of the distribution function in (25), with $u_\perp = \varepsilon \hat{u}_\perp$, assuming $|u_\parallel|$ to be of order unity. Then

$$\mathcal{M}_{(1,\varepsilon^2 u,1)} = M(1 + \varepsilon^2 u_\parallel \cdot v_\parallel + \varepsilon^3 \hat{u}_\perp v_\perp) + O(\varepsilon^4)\,. \tag{26}$$

This suggests to seek the relative fluctuation of distribution function in the form

$$\hat{g}_\varepsilon = \hat{g}_{\varepsilon,+} + \varepsilon \hat{g}_{\varepsilon,-} \tag{27}$$

where $\hat{g}_{\varepsilon,+}$ is the even component of $\hat{g}_\varepsilon$ in $v_\perp$ and $\hat{g}_{\varepsilon,-}$ its odd component. The idea of using this decomposition was suggested to us by [7], where another symmetry group is used (specifically, the symmetry $v \mapsto -v$ instead of $v \mapsto (v_\parallel, -v_\perp)$ as here.)

As we shall see, the asymptotic limit of the Boltzmann equation in the scaling (23)-(24) with the symmetry (27) leads to the Prandtl equations (6).

We shall also derive the hydrostatic system by a variant of the scaling and symmetry assumptions above, where the viscosity is scaled to 0. This is done by keeping $\mathrm{Sh} = \mathrm{Ma} = \varepsilon^2$, where ε is the thickness of the layer of fluid considered, while setting $\mathrm{Kn} = o(\varepsilon^4)$, for instance $\mathrm{Kn} = \varepsilon^q$ with $q > 4$, so that the scaled Boltzmann equation reads

$$\varepsilon^2 \partial_t \hat{F}_\varepsilon + v_\parallel \cdot \nabla_{\hat{x}_\parallel} \hat{F}_\varepsilon + \frac{1}{\varepsilon} v_\perp \partial_{\hat{x}_\perp} \hat{F}_\varepsilon = \frac{1}{\varepsilon^q} \mathcal{C}(\hat{F}_\varepsilon)\,, \tag{28}$$

The hydrostatic system (7) is derived from the Boltzmann equation with the new scaling (28), while keeping the distribution function of the form (24) with the symmetry (27).

4. – Formal derivation of the Prandtl and hydrostatic equations.

For simplicity, we henceforth drop all hats on the scaled variables and distribution function, and consider the Boltzmann equation

$$\varepsilon^2 \partial_t F_\varepsilon + v_\parallel \cdot \nabla_{x_\parallel} F_\varepsilon + \frac{1}{\varepsilon} v_\perp \partial_{x_\perp} F_\varepsilon = \frac{1}{\varepsilon^q} \mathcal{C}(F_\varepsilon)\,, \tag{29}$$

for which we seek the solution in the form

$$F_\varepsilon(t,x,v) = M(1 + \varepsilon^2 \hat{g}_{\varepsilon,+}(t,x,v) + \varepsilon^3 \hat{g}_{\varepsilon,-}(t,x,v))\,, \tag{30}$$

with $g_{\varepsilon,+}$ even in $v_\perp$ while $g_{\varepsilon,-}$ is odd in $v_\perp$.

In addition to the linearized collision operator $\mathcal{L}_M$ defined in (16), we introduce

$$\mathcal{Q}_M(\phi,\psi) = \frac{1}{2} M^{-1} D^2 \mathcal{C}(M) \cdot (M\phi, M\psi)\,, \tag{31}$$

whose explicit expression is

$$\mathcal{Q}_M(\phi,\psi)(v) = \frac{1}{2} \iint\limits_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} (\phi'\psi'_* + \psi'\phi'_* - \phi\psi_* - \psi\phi_*) b(v - v_*, \omega) M_* dv_* d\omega\,, \tag{32}$$

using the notation

$$\phi := \phi(v)\,, \quad \phi_* := \phi(v_*)\,, \quad \phi' := \phi(v')\,, \quad \text{and } \phi'_* := \phi(v'_*)\,, \tag{33}$$

as is customary in the literature on the Boltzmann equation.

The following observation greatly simplifies some of the computations below.

LEMMA 4.1. – *For each $\phi,\psi \in \operatorname{Ker}\mathcal{L}_M$, one has*

$$\mathcal{Q}_M(\phi,\psi) = \frac{1}{2}\mathcal{L}_M(\phi\psi)\,.$$

This lemma can be found in [15]; see [4] on p. 338 for a quick proof.

We also recall the action of linear isometries on the Boltzmann collision integral.

LEMMA 4.2. – *For each $R \in O_3(\boldsymbol{R})$, and each $\phi,\psi \in C_c(\boldsymbol{R}^3)$, one has*

$$\mathcal{C}(\phi\circ R) = \mathcal{C}(\phi)\circ R\,,$$

and

$$\mathcal{Q}_M(\phi\circ R,\psi\circ R) = \mathcal{Q}_M(\phi,\psi)\circ R\,.$$

Likewise, for each $g \in \operatorname{Dom}(\mathcal{L}_M)$, one has

$$\mathcal{L}_M(g\circ R) = \mathcal{L}_M(g)\circ R\,.$$

Observe that the tensor field $A(Rv) = RA(v)R^T$ for each $R \in O_3(\boldsymbol{R})$, so that

$$\langle A_{ij}A_{lk}\rangle = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right),\quad i,j,k,l = 1,2,3\,. \tag{34}$$

Likewise, applying the lemma above shows that

$$(\mathcal{L}_M^{-1}A)(Rv) = R(\mathcal{L}_M^{-1}A(v))R^T$$

for each $R \in O_3(\boldsymbol{R})$, so that there exists $\nu > 0$ satisfying

$$\langle \mathcal{L}_M^{-1}(A_{ij})A_{kl}\rangle = \nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})\,,\quad i,j,k,l = 1,2,3\,. \tag{35}$$

Applying this lemma with R defined by $Rv := (v_\parallel, -v_\perp)$, and denoting g_+ (resp. g_-) the even (resp. odd) in $v_\perp$ component of $g \equiv g(v)$, we see that, for each $g \in \operatorname{Dom}(\mathcal{L}_M)$

$$\mathcal{L}_M(g)_+ = \mathcal{L}_M(g_+)\,,\qquad \text{and } \mathcal{L}_M(g)_- = \mathcal{L}_M(g_-)\,. \tag{36}$$

Likewise, for each $\phi,\psi \in C_c(\boldsymbol{R}^3)$, one has

$$\begin{cases} \mathcal{Q}_M(\phi,\psi)_+ = \mathcal{Q}_M(\phi_+,\psi_+) + \mathcal{Q}_M(\phi_-,\psi_-)\,,\\ \mathcal{Q}_M(\phi,\psi)_- = 2\mathcal{Q}_M(\phi_+,\psi_-)\,. \end{cases} \tag{37}$$

With this observation in mind, we first recast the scaled Boltzmann equation (29) in terms of the relative number density fluctuation $g_\varepsilon = g_{\varepsilon,+} + \varepsilon g_{\varepsilon,-}$:

$$\varepsilon^2\partial_t g_\varepsilon + v_\parallel\cdot\nabla_{x_\parallel}g_\varepsilon + \frac{1}{\varepsilon}v_\perp\partial_{x_\perp}g_\varepsilon + \frac{1}{\varepsilon^q}\mathcal{L}_M g_\varepsilon = \frac{1}{\varepsilon^{q-2}}\mathcal{Q}_M(g_\varepsilon,g_\varepsilon)\,. \tag{38}$$

Next, we decompose each side of this equation into its odd and even components:

$$\left\{\begin{aligned}&\varepsilon^2\partial_t g_{\varepsilon,+}+v_\parallel\cdot\nabla_{x_\parallel}g_{\varepsilon,+}+v_\perp\partial_{x_\perp}g_{\varepsilon,-}+\frac{1}{\varepsilon^q}\mathcal{L}_M g_{\varepsilon,+}\\&\qquad=\frac{1}{\varepsilon^{q-2}}\mathcal{Q}_M(g_{\varepsilon,+},g_{\varepsilon,+})+\frac{1}{\varepsilon^{q-4}}\mathcal{Q}_M(g_{\varepsilon,-},g_{\varepsilon,-}),\\&\varepsilon^3\partial_t g_{\varepsilon,-}+\varepsilon v_\parallel\cdot\nabla_{x_\parallel}g_{\varepsilon,-}+\frac{1}{\varepsilon}v_\perp\partial_{x_\perp}g_{\varepsilon,+}+\frac{1}{\varepsilon^{q-1}}\mathcal{L}_M g_{\varepsilon,-}\\&\qquad=\frac{2}{\varepsilon^{q-3}}\mathcal{Q}_M(g_{\varepsilon,+},g_{\varepsilon,-}).\end{aligned}\right. \tag{39}$$

This is a coupled system of Boltzmann type equations, which we are going to analyze by the moment method as in [4]. For the sake of notational simplicity, we henceforth denote

$$\langle\phi\rangle:=\int_{\boldsymbol{R}^3}\phi(v)M dv \tag{40}$$

whenever $\phi\in L^1(M dv):=L^1(\boldsymbol{R}^3;M dv)$.

We also use the notation

$$A_\parallel(v):=v_\parallel^{\otimes 2}-\frac{1}{3}|v|^2 I\in M_2(\boldsymbol{R}),\qquad A_\perp(v):=v_\perp^2-\frac{1}{3}|v|^2\in\boldsymbol{R}. \tag{41}$$

THEOREM 4.3. – *Let F_ε be a family of solutions of the Boltzmann equation* (29), *whose relative fluctuations at scale ε^2, i.e.*

$$g_\varepsilon=\frac{F_\varepsilon-M}{\varepsilon^2 M}=g_{\varepsilon,+}+\varepsilon g_{\varepsilon,-},$$

where $g_{\varepsilon,+}$ (resp. $g_{\varepsilon,-}$) even (resp. odd) in $v_\perp$, satisfy

$$g_{\varepsilon,+}\to g_+,\quad \text{and } g_{\varepsilon,-}\to g_-$$

a.e. and in the sense of distributions, and that

$$\langle\phi g_{\varepsilon,\pm}\rangle\to\langle\phi g_\pm\rangle \text{ and } \langle\phi\mathcal{Q}_M(g_{\varepsilon,\pm},g_{\varepsilon,\pm})\rangle\to\langle\phi\mathcal{Q}_M(g_\pm,g_\pm)\rangle$$

in the sense of distributions for each $\phi\in L^2(M dv)$. Then g_+ and g_- are of the form

$$\left\{\begin{aligned}&g_+(t,x,v)=\rho(t,x)+u_\parallel(t,x)\cdot v_\parallel+\theta(t,x)\frac{1}{2}(|v|^2-3),\\&g_-(t,x,v)=u_\perp(t,x)v_\perp.\end{aligned}\right. \tag{42}$$

where $\rho+\theta=$ Const. and $(u_\parallel,u_\perp)$ satisfy

a) the Prandtl system of equations (6) *if $q=4$, and*

b) the hydrostatic system of equations (7) *if $q>4$.*

PROOF. – The argument is split in several steps.

Step 1: the limiting number density fluctuations.

We deduce from (39) that

$$\begin{aligned}\mathcal{L}_M g_{\varepsilon,+} &= \varepsilon^2 \mathcal{Q}_M(g_{\varepsilon,+}, g_{\varepsilon,+}) + \varepsilon^4 \mathcal{Q}_M(g_{\varepsilon,-}, g_{\varepsilon,-}) \\ &\quad - \varepsilon^{q+2}\partial_t g_{\varepsilon,+} - \varepsilon^q v_\| \cdot \nabla_{x_\|} g_{\varepsilon,+} - \varepsilon^q v_\perp \partial_{x_\perp} g_{\varepsilon,-} \to 0\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_M g_{\varepsilon,-} &= 2\varepsilon^2 \mathcal{Q}_M(g_{\varepsilon,+}, g_{\varepsilon,-}) \\ &\quad - \varepsilon^{q+2}\partial_t g_{\varepsilon,-} - \varepsilon^q v_\| \cdot \nabla_{x_\|} g_{\varepsilon,-} - \varepsilon^{q-2} v_\perp \partial_{x_\perp} g_{\varepsilon,+} \to 0\end{aligned}$$

in the sense of distributions since $q \geq 4$, so that

$$\mathcal{L}_M g_+ = \mathcal{L}_M g_- = 0\,.$$

In view of the parity of g_+ and g_- in the variable $v_\perp$, we conclude that g_+ and g_- are of the form (42).

Step 2: the incompressibility condition[1].

By the local conservation of mass – the first identity in (11) – applied to the first equation in (39),

$$\varepsilon^2 \partial_t \langle g_{\varepsilon,+}\rangle + \mathrm{div}_{x_\|}\langle v_\| g_{\varepsilon,+}\rangle + \partial_{x_\perp}\langle v_\perp g_{\varepsilon,-}\rangle = 0\,.$$

Passing to the limit in the sense of distributions in both sides of this equality, one finds

$$\mathrm{div}_{x_\|} u_\| + \partial_{x_\perp} u_\perp = \mathrm{div}_{x_\|}\langle v_\| g_+\rangle + \partial_{x_\perp}\langle v_\perp g_-\rangle = 0\,. \tag{43}$$

Step 3: the limiting fluctuations of density and temperature.

By the local conservation of longitudinal momentum – the second identity in (11) – applied to the first equation in (39)

$$\varepsilon^2 \partial_t \langle v_\| g_{\varepsilon,+}\rangle + \mathrm{div}_{x_\|}\langle v_\|^{\otimes 2} g_{\varepsilon,+}\rangle + \partial_{x_\perp}\langle v_\perp v_\| g_{\varepsilon,-}\rangle = 0\,.$$

([1]) Strictly speaking, in a fluid with density ρ and velocity field u, the incompressibility condition is $\partial_t \rho + u \cdot \nabla_x \rho = 0$ (see for instance equation (2.2.4) on p. 75 in [9].) When combined with the continuity equation $\partial_t \rho + \mathrm{div}_x(\rho u) = 0$, the incompressibility condition implies that $\rho\,\mathrm{div}_x u = 0$. Conversely, if $\mathrm{div}_x u = 0$, the incompressibility condition follows from the continuity equation. However, in the mathematical literature, it is customary to refer to $\mathrm{div}_x u = 0$ as the incompressibility condition, and we have followed this usage here.

Passing to the limit in the sense of distributions in both sides of this equality, one finds

$$\operatorname{div}_{x_\parallel}\langle v_\parallel^{\otimes 2} g_+\rangle = 0\,.$$

(Indeed

$$\langle v_\perp v_\parallel g_-\rangle = 0$$

since g_- is even in $v_\parallel$.) Substituting the explicit formula for g_+ in the left-hand side of the identity above, one finds

$$\operatorname{div}_{x_\parallel}\left\langle v_\parallel^{\otimes 2}\Big(\rho + u_\parallel\cdot v_\parallel + \theta\frac{1}{2}(|v|^2-3)\Big)\right\rangle = \nabla_{x_\parallel}(\rho+\theta) = 0\,,$$

in view of the identity

$$\langle v^{\otimes 2}\rangle = \left\langle v^{\otimes 2}\frac{1}{2}(|v|^2-3)\right\rangle = \left\langle \frac{1}{6}|v|^2(|v|^2-3)\right\rangle I = I\,. \tag{44}$$

By the local conservation of transverse momentum – the second identity in (11) – applied to the second equation in (39)

$$\varepsilon^4\partial_t\langle v_\perp g_{\varepsilon,-}\rangle + \varepsilon^2\operatorname{div}_{x_\parallel}\langle v_\perp v_\parallel g_{\varepsilon,-}\rangle + \partial_{x_\perp}\langle v_\perp^2 g_{\varepsilon,+}\rangle = 0\,.$$

Passing to the limit in both sides of this equality, we arrive at

$$\partial_{x_\perp}\langle v_\perp^2 g_+\rangle = 0\,.$$

Substituting the explicit formula for g_+ in the left-hand side of the identity above leads to

$$\partial_{x_\perp}\left\langle v_\perp^2\Big(\rho + u_\parallel\cdot v_\parallel + \theta\frac{1}{2}(|v|^2-3)\Big)\right\rangle = \partial_{x_\perp}(\rho+\theta) = 0\,,$$

using again the identity (44).

In the end, we conclude that

$$\rho+\theta = \text{Const.} \tag{45}$$

Step 4: the motion equation.

Multiplying the first equation in (39) by $\varepsilon^{-2}v_\parallel M$ and integrating in $v\in\boldsymbol{R}^3$ leads to

$$\partial_t\langle v_\parallel g_{\varepsilon,+}\rangle + \operatorname{div}_{x_\parallel}\frac{1}{\varepsilon^2}\langle A_\parallel g_{\varepsilon,+}\rangle + \partial_{x_\perp}\frac{1}{\varepsilon^2}\langle v_\perp v_\parallel g_{\varepsilon,-}\rangle + \nabla_{x_\parallel}\frac{1}{\varepsilon^2}\left\langle\frac{1}{3}|v|^2 g_{\varepsilon,+}\right\rangle = 0.$$

Since $A_\parallel$ belongs to $(\operatorname{Ker}\mathcal{L}_M)^\perp$ componentwise and $\mathcal{L}_M$ is self-adjoint, one has

$$\frac{1}{\varepsilon^2}\langle A_\parallel g_{\varepsilon,+}\rangle = \left\langle(\mathcal{L}_M^{-1}A_\parallel)\frac{1}{\varepsilon^2}\mathcal{L}_M g_{\varepsilon,+}\right\rangle.$$

By using the first equation in (39) we express

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathcal{L}_M g_{\varepsilon,+} &= \mathcal{Q}_M(g_{\varepsilon,+},g_{\varepsilon,+}) + \varepsilon^2\mathcal{Q}_M(g_{\varepsilon,-},g_{\varepsilon,-}) \\ &\quad - \varepsilon^q\partial_t g_{\varepsilon,+} - \varepsilon^{q-2}v_\parallel\cdot\nabla_{x_\parallel}g_{\varepsilon,+} - \varepsilon^{q-2}v_\perp\partial_{x_\perp}g_{\varepsilon,-}\,,\end{aligned} \tag{46}$$

and since $q\geq 4$, we conclude that

$$\begin{aligned}\frac{1}{\varepsilon^2}\langle A_\parallel g_{\varepsilon,+}\rangle &\to \langle(\mathcal{L}_M^{-1}A_\parallel)\mathcal{Q}_M(g_+,g_+)\rangle \\ &= \langle(\mathcal{L}_M^{-1}A_\parallel)\frac{1}{2}\mathcal{L}_M(g_+^2)\rangle = \frac{1}{2}\langle A_\parallel g_+^2\rangle\end{aligned}$$

in view of Lemma 4.1. By using the explicit formula (42) for g_+, we find

$$\frac{1}{2}\langle A_\parallel g_+^2\rangle = \frac{1}{2}\langle A_\parallel\otimes v_\parallel^{\otimes 2}\rangle : u_\parallel^{\otimes 2} = u_\parallel^{\otimes 2} - \frac{1}{3}|u_\parallel|^2 I \in M_2(\boldsymbol{R})\,.$$

Likewise, since $v_\perp v_\parallel$ belongs to $(\mathrm{Ker}\mathcal{L}_M)^\perp$ componentwise, one has

$$\frac{1}{\varepsilon^2}\langle v_\perp v_\parallel g_{\varepsilon,-}\rangle = \left\langle \mathcal{L}_M^{-1}(v_\perp v_\parallel)\frac{1}{\varepsilon^2}\mathcal{L}_M g_{\varepsilon,-}\right\rangle .$$

By using the second equation in (39) we express

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathcal{L}_M g_{\varepsilon,-} &= 2\mathcal{Q}_M(g_{\varepsilon,+},g_{\varepsilon,-}) - \varepsilon^{q-4}v_\perp\partial_{x_\perp}g_{\varepsilon,+} \\ &\quad -\varepsilon^q\partial_t g_{\varepsilon,-} - \varepsilon^{q-2}v_\parallel\cdot\nabla_{x_\parallel}g_{\varepsilon,-}\,.\end{aligned} \tag{47}$$

At this point we distinguish the inviscid case $q>4$ from the viscous case $q=4$.

In the inviscid case $q>4$

$$\begin{aligned}\frac{1}{\varepsilon^2}\langle v_\perp v_\parallel g_{\varepsilon,-}\rangle \to\ & 2\langle(\mathcal{L}_M^{-1}v_\perp v_\parallel)\mathcal{Q}_M(g_+,g_-)\rangle = \langle(\mathcal{L}_M^{-1}v_\perp v_\parallel)\mathcal{L}_M(g_+g_-)\rangle \\ &= \langle v_\perp v_\parallel g_+g_-\rangle = \langle v_\perp^2 v_\parallel^{\otimes 2}\rangle\cdot u_\parallel u_\perp = u_\perp u_\parallel\,,\end{aligned}$$

by using (34).

On the other hand,

$$\langle\mathcal{L}_M^{-1}(v_\perp v_\parallel)v_\perp\partial_{x_\perp}g_{\varepsilon,+}\rangle \to \langle\mathcal{L}_M^{-1}(v_\perp v_\parallel)v_\perp v_\parallel\rangle\cdot\partial_{x_\perp}u_\parallel = \nu\partial_{x_\perp}u_\parallel$$

by (35) so that, in the viscous case $q=4$, one has

$$\frac{1}{\varepsilon^2}\langle v_\perp v_\parallel g_{\varepsilon,-}\rangle \to u_\perp u_\parallel - \nu\partial_{x_\perp}u_\parallel$$

as $\varepsilon\to 0^+$.

Therefore, in the viscous case $q = 4$

$$\nabla_{x_\parallel}\left(\frac{1}{\varepsilon^2}\left\langle\frac{1}{3}|v|^2 g_{\varepsilon,+}\right\rangle - \frac{1}{3}|u_\parallel|^2\right) \tag{48}$$
$$\to -\partial_t u_\parallel - \operatorname{div}_{x_\parallel}(u_\parallel^{\otimes 2}) - \partial_{x_\perp}(u_\perp u_\parallel) + \nu\partial_{x_\perp}^2 u_\parallel$$

in the sense of distributions as $\varepsilon \to 0^+$, while in the inviscid case

$$\nabla_{x_\parallel}\left(\frac{1}{\varepsilon^2}\left\langle\frac{1}{3}|v|^2 g_{\varepsilon,+}\right\rangle - \frac{1}{3}|u_\parallel|^2\right) \tag{49}$$
$$\to -\partial_t u_\parallel - \operatorname{div}_{x_\parallel} u_\parallel^{\otimes 2} - \partial_{x_\perp}(u_\perp u_\parallel)\,.$$

Multiplying the second equation in (39) by $\varepsilon^{-1}\, v_\perp M$ and integrating in $v \in \boldsymbol{R}^3$ leads to

$$\varepsilon^2\partial_t\langle v_\perp g_{\varepsilon,-}\rangle + \operatorname{div}_{x_\parallel}\langle v_\parallel v_\perp g_{\varepsilon,-}\rangle + \frac{1}{\varepsilon^2}\partial_{x_\perp}\langle v_\perp^2 g_{\varepsilon,+}\rangle =$$
$$\varepsilon^2\partial_t\langle v_\perp g_{\varepsilon,-}\rangle + \operatorname{div}_{x_\parallel}\langle v_\parallel v_\perp g_{\varepsilon,-}\rangle + \partial_{x_\perp}\frac{1}{\varepsilon^2}\langle A_\perp g_{\varepsilon,+}\rangle + \partial_{x_\perp}\frac{1}{\varepsilon^2}\left\langle\frac{1}{3}|v|^2 g_{\varepsilon,+}\right\rangle = 0\,.$$

On the other hand, using (46) as above, one finds that

$$\frac{1}{\varepsilon^2}\langle A_\perp g_{\varepsilon,+}\rangle = \left\langle(\mathcal{L}_M^{-1}A_\perp)\frac{1}{\varepsilon^2}\mathcal{L}_M g_{\varepsilon,+}\right\rangle \to \langle(\mathcal{L}_M^{-1}A_\perp)\mathcal{Q}_M(g_+,g_+)\rangle$$
$$= \frac{1}{2}\langle A_\perp g_+^2\rangle = \frac{1}{2}\langle A_\perp A_\parallel\rangle : u_\parallel^{\otimes 2} = -\frac{1}{3}|u_\parallel|^2 I$$

by (34), so that

$$\partial_{x_\perp}\left(\frac{1}{\varepsilon^2}\left\langle\frac{1}{3}|v|^2 g_{\varepsilon,+}\right\rangle - \frac{1}{3}|u_\parallel|^2\right) \to -\operatorname{div}_{x_\parallel}\langle v_\parallel v_\perp g_-\rangle = 0 \tag{50}$$

in the sense of distributions as $\varepsilon \to 0^+$.

Step 5: the weak formulation.

Let $\phi \equiv \xi(x) \in \boldsymbol{R}^3$ be a divergence-free, compactly supported vector field. The convergence in (48) or (49) and in (50) entails

$$\partial_t\int \xi_\parallel \cdot u_\parallel dx = \int \nabla_{x_\parallel}\xi_\parallel : u_\parallel^{\otimes 2} dx + \int u_\perp u_\parallel \cdot \partial_{x_\perp}\xi_\parallel dx + \nu\int u_\parallel \cdot \partial_{x_\perp}^2\xi_\parallel dx \tag{51}$$

in the viscous case $q = 4$, and

$$\partial_t\int \xi_\parallel \cdot u_\parallel dx = \int \nabla_{x_\parallel}\xi_\parallel : u_\parallel^{\otimes 2} dx + \int u_\perp\partial_{x_\perp}\xi_\parallel \cdot u_\parallel dx \tag{52}$$

in the inviscid case $q > 4$.

If a vector-valued distribution $T = (T_1, T_2, 0)$ satisfies

$$\langle\!\langle T_1, \xi_1 \rangle\!\rangle + \langle\!\langle T_2, \xi_2 \rangle\!\rangle = 0$$

for each test vector-field $\xi = (\xi_1, \xi_2, \xi_3)$ satisfying $\mathrm{div}\xi = 0$, there exists a real-valued distribution π such that

$$T = \nabla\pi\,, \qquad \text{with } \partial_{x_\perp}\pi = 0\,.$$

Equivalently π is constant in the variable x_3. This shows that (51) and (52) are the weak formulations of the Prandtl equations (6) and the hydrostatic equations (7) respectively. □

Before going further, some remarks are in order. Because of the von Karman relation ($\mathrm{Kn} = \mathrm{Ma}/\mathrm{Re}$) mentioned above, in a gas flow where the Knudsen number is small and the Mach number of order unity, the Reynolds number is large, so that a viscous boundary layer appears near any solid boundary immersed in the flow. Therefore, any systematic study of the asymptotic expansion of the solution of the Boltzmann equation in the small Knudsen regime must include a derivation of the compressible analogue of the Prandtl boundary layer theory. The reader interested in this question is referred to [39], where the analysis is based upon asymptotic expansions in the form of power series in terms of the square root $\sqrt{\mathrm{Kn}}$ of the Knudsen number. Prandtl's classical, incompressible viscous boundary layer theory can also be derived [1] from the Boltzmann equation by the same type of asymptotic expansions. However our interest in the proof presented above was to demonstrate the power of this idea suggested by [7] of using the moment method of [4] with the symmetry $v \mapsto (v_\parallel, v_\perp)$.

5. – The boundary conditions.

In the discussion above, we have left aside the question of boundary conditions, which is especially unfortunate when dealing with such matters as the Prandtl equations. A few remarks on this important issue are in order.

The usual setting for the Prandtl equation is as follows: $x_\parallel \in \boldsymbol{R}$ or $\boldsymbol{R}^2$ according to whether the flow is 2- or 3-dimensional, while $x_\perp > 0$. (More intricate geometries are also considered, where $x_\parallel$ typically belongs to a curve Γ or a surface Σ, while $x_\perp > 0$ represents the distance from the point $x = (x_\parallel, x_\perp)$ to Γ or Σ.) The Prandtl equations govern the viscous boundary layer of an incompressible flow past a immersed rigid body, whenever viscous effects remain confined near the surface of the body. Typically, the velocity field of a viscous fluid satisfies the Dirichlet boundary condition on the surface of an immersed body, while it (almost) satisfies the Euler equations far away from the surface of the body. Since the Dirichlet boundary condition is overspecified for the solution of Euler's equations, the Prandtl boundary layer equations provide the transition between the Dirichlet boundary condition and the appropriate boundary condition for the Euler equations.

More precisely, the Prandtl equations (6) are supplemented with the following conditions:

$$\begin{cases} u_{\|}\big|_{x_\perp=0} = 0\,, \quad u_{\|}\big|_{x_\perp=+\infty} = U\big|_{x_\perp=0}\,, \quad p = P\big|_{x_\perp=0}\,, \\ u_{\|}\big|_{t=0} = u_{\|}^{in}\,, \end{cases} \tag{53}$$

where U and P are respectively the Euler velocity and pressure fields in the half-space $x_\perp > 0$. Notice that the longitudinal derivative of the pressure is a source term in the Prandtl equations (6), while $U_\perp\big|_{x_\perp=0} = 0$, which is the usual boundary condition for the Euler equations. Of course the initial condition should be compatible with the boundary conditions, which means that

$$u_{\|}^{in}\big|_{x_\perp=0} = 0\,, \quad u_{\|}^{in}\big|_{x_\perp=+\infty} = U\big|_{x_\perp=0}\,.$$

For instance, the Euler solution could be the trivial null solution $U = 0$ and $P = 0$ (see [22] for instance.)

The corresponding initial and boundary conditions for the scaled Boltzmann equation (29) with $q = 4$ can be chosen as follows. As initial condition for (29), we choose

$$F_\varepsilon\big|_{t=0} = \mathcal{M}_{(1,\varepsilon^2 u_{\|}^{in},1)}\,. \tag{54}$$

Next, we work in the class of solutions whose relative entropy satisfies

$$H(F_\varepsilon|M) := \iint \left[F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M\right] dxdv = O(\varepsilon^4)\,, \tag{55}$$

in accordance with the P.-L. Lions' theory of renormalized solutions of the Boltzmann equation converging to some Maxwellian state at infinity. For F_ε of the form $F_\varepsilon = M(1 + \varepsilon^2 g_+ + O(\varepsilon^3))$, this condition reduces to

$$\iint g_+^2 M dxdv < +\infty$$

as $\varepsilon \to 0$, which implies in particular that

$$\int |u_{\|}|^2 dx < +\infty\,,$$

and this is consistent with the condition that $u_{\|}$ vanishes at infinity. Finally, one can impose a diffuse reflection condition at $x_\perp = 0$:

$$F_\varepsilon\big|_{x_\perp=0, v_\perp>0} = \rho_\varepsilon M \tag{56}$$

with $\rho_\varepsilon \equiv \rho_\varepsilon(t, x_\parallel)$ defined by the null-flux condition transverse to the boundary

$$\int_{\boldsymbol{R}^3} v_\perp F_\varepsilon dv = 0\,,$$

i.e.

$$\rho_\varepsilon(t, x_\parallel) = \frac{\int_{v_\perp < 0} |v_\perp| F_\varepsilon(t, (x_\parallel, 0), v) dv}{\int_{v_\perp < 0} |v_\perp| M dv}\,.$$

Now for the hydrostatic equations (7). Following the discussion in [27, 11, 12], we assume that the transverse variable $0 < x_\perp < 1$, while the longitudinal variable $x_\parallel \in \boldsymbol{T}^1$ or $\boldsymbol{T}^2$ – or equivalently, $x_\parallel \in \boldsymbol{R}$ or $\boldsymbol{R}^2$ and (u, p) is periodic in the variable $x_\parallel$. The boundary conditions for (7) are thus

$$u_\perp\big|_{x_\perp = 0} = u_\perp\big|_{x_\perp = 1} = 0\,. \tag{57}$$

This condition means that the fluid does not penetrate the boundaries $x_\perp = 0$ and $x_\perp = 1$. The corresponding boundary condition for the Boltzmann equation (29) is the specular reflection condition

$$\begin{aligned} F_\varepsilon(t, (x_\parallel, 0), (v_\parallel, v_\perp)) &= F_\varepsilon(t, (x_\parallel, 0), (v_\parallel, -v_\perp))\,, \\ F_\varepsilon(t, (x_\parallel, 1), (v_\parallel, v_\perp)) &= F_\varepsilon(t, (x_\parallel, 1), (v_\parallel, -v_\perp))\,. \end{aligned} \tag{58}$$

The initial condition is (54) as before.

6. – The convergence problem.

As explained above, the results presented in this paper are formal. The problem of obtaining estimates showing that the relative density fluctuations satisfy

$$\langle v_\parallel g_{\varepsilon,\parallel} \rangle \to u_\parallel \quad \text{ and } \langle v_\perp g_{\varepsilon,\perp} \rangle \to u_\perp$$

as $\varepsilon \to 0$, where $(u_\parallel, u_\perp)$ are solutions of either the Prandtl equations (6) if $q = 4$, or of the hydrostatic equations (7) if $q > 4$, remains open at the time of this writing.

A first difficulty is that the Prandtl equations may fail to have solutions defined for all times, even for very smooth initial data. See [22], where a class of solutions of the Prandtl equations with finite time blow-up are constructed for compactly supported, C^∞ initial data. On the contrary, the Prandtl equations are well-posed in appropriate classes of analytic functions: see [14, 36]. This situation is not very satisfying: usually, analytic solutions obtained by some variant of the Cauchy-Kovalevska theorem are of limited physical interest.

Another difficulty is that there exist flows for which the viscous boundary layer will detach from the boundary, which invalidates the scenario of a flow with viscous effects concentrated on a thin layer and inviscid far from the boundary. See [28] for a mathematical analysis of this phenomenon.

The situation is somewhat better for the hydrostatic system: see [11, 27, 12] for a very interesting existence and uniqueness result in the two dimensional case, along with a derivation of the hydrostatic equations (7) from the incompressible Euler equations, assuming that the longitudinal velocity profile is convex in the transverse variable.

All the solutions of either the Prandtl or the hydrostatic system obtained in these works are classical solutions. This suggests the problem of deriving these solutions from the scaled Boltzmann equation by the relative entropy method[2] [10, 32, 35].

Let us briefly recall the principle of that method. We recall that, given two distribution functions $F \equiv F(x,v)$ and $G(x,v)$ with $F \geq 0$ and $G > 0$ a.e., the relative entropy of F with respect to G is

$$H(F|G) := \iint \left(F \ln\left(\frac{F}{G}\right) - F + G \right) dxdv\,.$$

Notice that the integrand is a.e. nonnegative, so that $H(F|G) \geq 0$. Besides, $H(F|G) = 0$ if and only if $F = G$ a.e., so that $H(F|G)$ can be thought of as measuring the distance between F and G. For instance, given two incompressible velocity fields $u \equiv u(x)$ and $U \equiv U(x)$,

$$\begin{aligned} H(\mathcal{M}_{(1,\varepsilon^2 U,1)}|\mathcal{M}_{(1,\varepsilon^2 u;1)}) &= \iint \mathcal{M}_{(1,\varepsilon^2 U,1)} \frac{1}{2}(|v-\varepsilon u|^2 - |v-\varepsilon U|^2) dxdv \\ &= \frac{1}{2}\varepsilon^4 \int |u-U|^2 dx \end{aligned}$$

for each $\varepsilon > 0$. This suggests that the appropriate quantity measuring the distance between some solution F_ε of the scaled Boltzmann equation and the hydrodynamic state defined by the incompressible velocity field u is

$$Z_\varepsilon(t) := \frac{1}{\varepsilon^4} H(F_\varepsilon(t,\cdot,\cdot)|\mathcal{M}_{(1,\varepsilon^2 u(t,\cdot),1)})\,.$$

(^{2}) It seems that the key idea of this method goes back to the results of Leray [29], and later of Dafermos [18] and DiPerna [20], on the uniqueness of classical solutions in appropriate classes of weak solutions of the incompressible Navier-Stokes equations, or of hyperbolic systems of conservation laws.

When the velocity field u is smooth, even if the solution of the Boltzmann equation is a weak, or even renormalized solution in the sense of DiPerna-Lions [21], it is usually possible to prove that Z_ε satisfies a Gronwall inequality of the form

$$Z_\varepsilon(t) \leq Z_\varepsilon(0) + o(1) + C\int_0^t Z_\varepsilon(s)ds$$

where $C = O(\|\nabla_x u\|_{L^\infty})$. Then, one concludes that $Z_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ uniformly in $t \in [0,T]$, provided that the initial data $F_\varepsilon\big|_{t=0}$ and $u\big|_{t=0}$ satisfy $Z_\varepsilon(0) \to 0$ – in other words, provided that the initial data is "well-prepared".

There are two obstructions in using this method in the context of thin layers of fluids studied in this paper.

First, since the total velocity field that is the solution of either (6) or (7) is of the form $(u_\parallel, \varepsilon u_\perp)$, applying the relative entropy method leads to considering the quantity

$$Z_\varepsilon(t) := \frac{1}{\varepsilon^4} H(F_\varepsilon(t,\cdot,\cdot)|\mathcal{M}_{(1,(\varepsilon^2 u_\parallel, \varepsilon^3 u_\perp)(t,\cdot),1)})\,.$$

In the limit as $\varepsilon \to 0$, the leading order of that quantity will capture the distance between $u_{F_\varepsilon,\parallel}$ and $u_\parallel$; in view of the scaling (22), the distance between $u_{F_\varepsilon,\perp}$ and $u_\perp$ is of higher order and will not be controlled by Z_ε.

The second obstruction in applying the relative entropy method to the hydrodynamic limit of the scaled Boltzmann equation (29) leading to the hydrostatic system (7) is suggested by the existing stability theory for this system. Indeed, we recall that derivations of (7) from the incompressible Euler equations in space dimension 2 make use of a certain type of functionals involving not only the velocity field, but also the vorticity, in a way that is reminiscent of the stability criterion of Arnold [2] for equilibrium solutions of the 2-dimensional, incompressible Euler equations. Typically, these functionals are of the type

$$H_\Phi(U) - H_\Phi(u) - DH_\Phi(u)(U-u)$$

where H_Φ is of the form

$$H_\Phi(U) = \int \frac{1}{2}|U|^2 dx + \int \Phi(\Omega)dx$$

where $U = (U_\parallel, \varepsilon U_\perp)$ and $\Omega = \partial_{x_\perp} U_\parallel - \varepsilon \partial_{x_\parallel} U_\perp$ is the scalar vorticity field, and where Φ is a carefully chosen scalar function, depending on the solution $(u_\parallel, \varepsilon u_\perp)$ whose stability one wishes to study. (Using this type of functional in order to study the stability of certain equilibrium flows of the incompressible Euler equations in space dimension 2 is the key idea in Arnold's stability theory [2].)

If one analyzes carefully the argument in either Arnold's original contribution, or in the work of Brenier and Grenier [12, 27] for the specific case of the hydrostatic system (7), it seems highly dubious that the derivation of (7) from (29) can be obtained with quantities involving only the relative entropy Z_ε. (Indeed, if such a derivation was possible, it would most certainly entail stability arguments for (7) involving only the kinetic energy of the velocity field and not the vorticity as in Arnold's analysis.)

On the other hand, whether there exists a functional depending on the distribution function F_ε, adapted to the dynamics of the Boltzmann equation (29) and such that its leading order contribution as $\varepsilon \to 0$ is of the form H_Φ seems to be unknown at the time of this writing.

Finally we insist that the difficulties mentioned above are not specific to the case of hydrodynamic limits of the Boltzmann equations for thin layers of fluids. The first obstruction mentioned above, i.e. the fact that the relative entropy method fails to capture all the components of the velocity field, or more generally, all the components of the number density fluctuations contributing to the limiting fluid dynamical model is also encountered in other contexts – for instance in the fluid dynamic limit theory for the Boltzmann equation leading to incompressible Navier-Stokes equations taking into account viscous heating terms, see [7], or ghost effects studied by Sone, Aoki and the Kyoto school, and reported in [37].

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