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A Characterization of a Modulus of Smoothness in Multidimensional Setting

LAURA ANGELONI

Abstract. – *A classical result of approximation theory states that* $\lim_{\delta \to 0^+} \omega(f, \delta) = 0$, *where* ω *is the modulus of smoothness of* f *defined by means of the variation functional, if and only if* f *is absolutely continuous. Such theorem is crucial in order to obtain results about convergence and order of approximation for linear and nonlinear integral operators in BV-spaces. It was an open problem to extend the above result to the setting of* φ*-variation in the multidimensional frame. In this paper, working with a concept of multidimensional* φ*-variation introduced in* [3], *we prove that an analogous characterization holds for the multidimensional* φ*-modulus of smoothness.*

1. – Introduction.

In [3] a multidimensional concept of φ-variation is introduced and results about convergence and rate of approximation for linear integral operators are established. This new concept of φ-variation extends the (one-dimensional) φ-variation introduced by J. Musielak and W. Orlicz ([25]), following the idea of Tonelli ([31]) for the classical variation of functions of two variables, extended by C. Vinti ([32]) to the general multidimensional setting. Results concerning φ-variation can be found in [36, 25, 17, 30, 20, 22, 9, 26, 14, 21, 8, 28, 7, 29, 10].

A crucial point in order to obtain results about convergence and order of approximation for linear and nonlinear integral operators (see e.g. [10, 2] and, in different settings, [9, 8, 28, 4]) is to require that

$$\lim_{\delta \to 0^+} \omega^{\varphi}(\lambda f, \delta) = 0, \tag{1}$$

for some $\lambda > 0$, where $\omega^{\varphi}(\lambda f, \delta) := \sup_{|t| < \delta} V^{\varphi}[\lambda(\tau_t f - f)]$, $\delta > 0$, is the φ-modulus of smoothness of the funcion f ([23, 7]) and $\tau_t f(\cdot) = f(\cdot - t)$ is the translation operator. This result is the natural reformulation, in terms of φ-variation, of the condition

$$\lim_{\delta \to 0^+} \omega(f, \delta) = 0, \tag{2}$$

where ω is the modulus of continuity with respect to the classical variation (see e.g. [6, 7, 1]). It is well known that (2) is equivalent to the class of absolutely continuous functions and the same result for the classical variation was proved in [6] in the multidimensional frame. Working with φ-variation, in the one-dimensional setting it is possible to prove that if the φ-function φ is such that $\frac{\varphi(u)}{u} \to 0$, as $u \to 0^+$, then (1) holds if and only if f satisfies a condition of local φ-absolute continuity, which is the natural generalization, in BV^φ−spaces, of the classical absolute continuity ([25, 2] and, in the frame of a multiplicative group, [9, 8]).

An analogous characterization of (1) in the multidimensional frame was an open problem (see [3], where condition (1) is assumed and discussed). In this paper we prove that (1) holds if the function f is φ-absolutely continuous in the Tonelli sense ($f \in AC^\varphi_{loc}(\mathbb{R}^N)$), assuming that $\frac{\varphi(u)}{u} \to 0$ as $u \to 0^+$, analogously to the one-dimensional case and to the case of classical variation (i.e., $\varphi(u) = u$), even if, in the instance of BV^φ−spaces, an integral representation does not hold for φ-absolutely continuous functions. Due to this fact, in order to obtain the main theorem it is necessary to establish several preliminary results about the multidimensional φ-variation (Section 3) and to introduce suitable auxiliary functions (Section 4) for which convergence results are established and through which the function f is approximated. Since the converse implication can be immediately deduced by the results in [3], we finally obtain a complete characterization of (1) in terms of φ-absolutely continuous functions, which generalizes, to the multidimensional frame, a classical and important property of variation.

Since our concept of φ-variation is inspired by Tonelli's idea, a fundamental role is played by the variation of the sections of f. For this reason we will extensively work, in the following, with a kind of variation ($V^\varphi_k[f]$) which takes into account of the single k−th direction (see Section 2). In view of these considerations, the crucial step in the passage from the one-dimensional to the N−dimensional setting is completely pointed out already by the 2-dimensional case. This is the reason for which, in order to simplify the notation and with the purpose of more clearness, we will work in detail with functions of two variables, taking into account that all the results can be easily generalized to the case of N variables (see Remark 4). Again for the sake of simplicity we shall work with φ-variation defined through "pythagorean" partitions, namely partitions generated by a grid (see e.g. [11, 33]); anyway, with simple considerations, all the results can be extended to the case of general partitions.

2. – Notations.

Let Φ be the class of all the functions $\varphi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

1. φ is a φ-function, namely φ is a convex function on $\mathbb{R}^+_0$ such that $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$;

2. $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$.

From now on we will assume that $\varphi \in \Phi$.

We will work with the multidimensional φ-variation introduced in [3]. This concept extends to the setting of Musielak and Orlicz φ-variation ([25]) a multidimensional concept of variation introduced by C. Vinti ([32]), following the Tonelli approach ([31]). We recall that this multidimensional version of the classical variation is equivalent to the distributional variation under some properties of approximate continuity (see e.g. [13, 15, 16, 7]).

We now introduce some notations of the multidimensional setting that we shall use in the following.

Given a function $f : \mathbb{R}^N \to \mathbb{R}$ and a vector $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $N \in \mathbb{N}$, if we are interested in its $k-$th coordinate we will write

$$x'_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1},$$

so that

$$\boldsymbol{x} = (x'_k, x_k), \quad f(\boldsymbol{x}) = f(x'_k, x_k).$$

For an ($N-$dimensional) interval $I = \prod_{i=1}^{N} [a_i, b_i]$, $I'_k = [a'_k, b'_k]$ will denote the $(N-1)-$dimensional interval obtained by deleting the $k-$th coordinate from I, so that

$$I = [a'_k, b'_k] \times [a_k, b_k], \quad k = 1, \ldots, N.$$

For $k = 1, \ldots, N$, $g_k(x_k) := f(x'_k, x_k)$ will denote the $k-$th section of f. Let us consider the $(N-1)-$dimensional integrals

$$\Phi_k^{\varphi}(f, I) := \int_{a'_k}^{b'_k} V^{\varphi}_{[a_k, b_k]}[f(x'_k, \cdot)]\, dx'_k,$$

where $V^{\varphi}_{[a_k,b_k]}[f(x'_k, \cdot)]$ is the (one-dimensional) Musielak-Orlicz φ-variation of the $k-$th section of f. The φ-variation was introduced by J. Musielak and W. Orlicz as a generalization of Wiener's quadratic variation ([34]), extended by Young ([35, 36]) to the concept of $p-$variation, $p \geq 1$. We recall that the φ-variation of a function $g : [a, b] \to \mathbb{R}$ is defined as

$$V^{\varphi}_{[a,b]}[g] := \sup_D \sum_{i=1}^{n} \varphi(|g(s_i) - g(s_{i-1})|),$$

where $D = \{s_0 = a, s_1, \ldots, s_n = b\}$ denotes a partition of the interval $[a, b]$ ([25, 23, 7]), and $g : [a, b] \to \mathbb{R}$ is said to be of bounded φ-variation ($g \in BV^{\varphi}([a, b])$) if there exists $\lambda > 0$ such that $V^{\varphi}_{[a,b]}[\lambda g] < +\infty$. We refer to [25] for the properties of the classical one-dimensional φ-variation.

Now, let $\Phi^\varphi(f,I)$ be the euclidean norm of the vector $(\Phi_1^\varphi(f,I),\ldots,\Phi_N^\varphi(f,I))$, namely

$$\Phi^\varphi(f,I) := \left\{ \sum_{j=1}^N [\Phi_j^\varphi(f,I)]^2 \right\}^{\frac{1}{2}},$$

where we put $\Phi^\varphi(f,I)=\infty$ if $\Phi_j^\varphi(f,I)=\infty$ for some $j=1,\ldots,N$.

The multidimensional φ-variation of f on an interval $I\subset\mathbb{R}^N$ is then defined as

$$V^\varphi[f,I] := \sup \sum_{i=1}^m \Phi^\varphi(f,J_i),$$

where the supremum is taken over all the finite families of $N-$dimensional intervals $\{J_1,\ldots,J_m\}$ which form partitions of I.

The φ-variation of f over the whole space $\mathbb{R}^N$ is defined as

$$V^\varphi[f] := \sup_{I\subset\mathbb{R}^N} V^\varphi[f,I],$$

where the supremum is taken over all the intervals $I\subset\mathbb{R}^N$.

In a similar fashion it is possible to define the φ-variation of f over a half-space $J=\mathbb{R}\times\ldots\times(-\infty,a_i]\times\ldots\times\mathbb{R}$ (or $J=\mathbb{R}\times\ldots\times[a_i,+\infty)\times\ldots\times\mathbb{R}$) as

$$V^\varphi[f,J] := \sup_{I\subset J} V^\varphi[f,I],$$

where the supremum is taken over all the intervals $I\subset J$.

By $BV^\varphi(\mathbb{R}^N)$ we will denote the *space of functions of bounded φ-variation* over $\mathbb{R}^N$, i.e.,

$$BV^\varphi(\mathbb{R}^N) = \{f\in L^1(\mathbb{R}^N):\ \exists\lambda>0 \text{ s.t. } V^\varphi[\lambda f]<+\infty\}.$$

We also define, for every $k=1,\ldots,N$, the "separated" variations

$$V_k^\varphi[f,I] := \sup\left\{ \sum_{i=1}^m \Phi_k^\varphi(f,J_i) \right\},$$

where the supremum is taken over all the partitions $\{J_1,\ldots,J_m\}$ of I. $V_k^\varphi[f,I]$ is a kind of variation with respect to just the $k-$th direction, while $V^\varphi[f,I]$ takes into account of all the N directions. Of course, if $N=1$, $V_1^\varphi[f,I]\equiv V^\varphi[f,I]$ coincides with the classical Musielak-Orlicz φ-variation of f. Moreover obviously we have, for every $k=1,\ldots,N$,

$$V_k^\varphi[f,I] \le V^\varphi[f,I] \le \sum_{k=1}^N V_k^\varphi[f,I].$$

In [3] a multidimensional concept of φ-absolute continuity is also introduced: a function $f:\mathbb{R}^N\to\mathbb{R}$ is *locally φ-absolutely continuous* ($f\in AC_{loc}^\varphi(\mathbb{R}^N)$) if it is φ-absolutely continuous in the Tonelli sense, i.e., for any interval $I = \prod_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}^N$ and for every $k = 1, 2, \ldots, N$, the k–th section $g_k : [a_k, b_k] \to \mathbb{R}$ is φ-absolutely continuous for almost every $x'_k \in [a'_k, b'_k]$. We recall that a function $g : [a, b] \to \mathbb{R}$ is φ-absolutely continuous if there exists $\lambda > 0$ such that the following property holds:

for every $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} \varphi(\lambda |g(\beta_i) - g(\alpha_i)|) < \varepsilon,$$

for all finite collections of non-overlapping intervals $[\alpha_i, \beta_i] \subset [a, b]$, $i = 1, \ldots, n$ *such that*

$$\sum_{i=1}^{n} \varphi(\beta_i - \alpha_i) < \delta.$$

By $AC^{\varphi}(\mathbb{R}^N)$ we will denote the space of functions $f \in BV^{\varphi}(\mathbb{R}^N) \cap AC^{\varphi}_{loc}(\mathbb{R}^N)$.

We now introduce the notion of φ-modulus of smoothness, which is the natural generalization, in the frame of BV^{φ}–spaces, of the classical modulus of continuity (see, for example, [24, 7, 3]). The φ-modulus of smoothness of a function $f \in BV^{\varphi}(\mathbb{R}^N)$ will be denoted by $\omega^{\varphi}(f, \delta)$, $\delta > 0$, and it is defined as

$$\omega^{\varphi}(f, \delta) := \sup_{|\boldsymbol{t}| \le \delta} V^{\varphi}[\tau_{\boldsymbol{t}} f - f],$$

where $(\tau_{\boldsymbol{t}} f)(\boldsymbol{s}) := f(\boldsymbol{s} - \boldsymbol{t})$, for every $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{R}^N$, is the translation operator.

Our aim will be to prove that $\omega^{\varphi}(f, \delta) \to 0$, as $\delta \to 0^+$, if and only if $f \in AC^{\varphi}_{loc}(\mathbb{R}^N)$.

For the sake of simplicity from now on we will work in the case $N = 2$, hence in $\mathbb{R}^2$, but all the results can be easily extended to the general case of $\mathbb{R}^N$ (see Remark 4).

Moreover, as pointed out in the Introduction, in order to simplify notations and proofs, we will work in the particular case of φ-variation defined through "pythagorean" partitions, instead of general partitions. Nevertheless, it is easy to prove that the main result of the paper holds also for φ-variation defined as usual with general ("extended") partitions (see Remark 3). Working with pythagorean partitions means that, in the particular case $N = 2$, that we will examine in detail in the following, the φ-variation of $f : \mathbb{R}^2 \to \mathbb{R}$ on an interval $I = [a, \alpha] \times [b, \beta] \subset \mathbb{R}^2$ is defined as

$$V^{\varphi}[f, I] := \sup \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} \Phi^{\varphi}(f, I_{ij}) \right\},$$

where the supremum is taken over all the finite families of intervals $\{I_{11},\dots,I_{mp}\}$ such that $I_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$, where $x_0=a<x_1<\dots<x_m=\alpha$ and $y_0=b<y_1<\dots<y_p=\beta$ are partitions of $[a,\alpha]$ and $[b,\beta]$, respectively. Moreover

$$V_k^{\varphi}[f,I]:=\sup\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_k^{\varphi}(f,I_{ij})\right\},\quad k=1,2,$$

where the supremum is taken over all the (pythagorean) partitions $\{I_{ij}\}$ of I, where $I_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$ and $a=x_0<x_1<\dots<x_m=\alpha$, $b=y_0<y_1<\dots<y_p=\beta$ are partitions of $[a,\alpha]$ and $[b,\beta]$, respectively. We also notice that, by Proposition 1.17 of [25], there holds, for every partition $\{I_{ij}\}$ of I,

$$\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_1^{\varphi}(f,I_{ij})=\sum_{i=1}^{m}\sum_{j=1}^{p}\int_{y_{j-1}}^{y_j}V^{\varphi}_{[x_{i-1},x_i]}[f(\cdot,y)]\,dy\le\int_b^{\beta}V^{\varphi}_{[a,\alpha]}[f(\cdot,y)]\,dy,$$

hence, passing to the supremum over all the partitions of I,

$$V_1^{\varphi}[f,I]\le\int_b^{\beta}V^{\varphi}_{[a,\alpha]}[f(\cdot,y)]\,dy.$$

Since also the converse inequality obviously holds, then we conclude that

$$V_1^{\varphi}[f,I]=\int_b^{\beta}V^{\varphi}_{[a,\alpha]}[f(\cdot,y)]\,dy \tag{3}$$

and, analogously,

$$V_2^{\varphi}[f,I]=\int_a^{\alpha}V^{\varphi}_{[b,\beta]}[f(x,\cdot)]\,dx. \tag{4}$$

3. – Preliminary results about φ-variation.

In order to prove the main results, we first need to establish some preliminary propositions that extend to the multidimensional frame some properties of the classical Musielak-Orlicz φ-variation (see [25]). In particular, we will prove that Propositions 1.12, 1.17 and 1.18 of [25] can be generalized to the setting of multidimensional φ-variation.

PROPOSITION 1. – *Let* $f_1, \ldots, f_n \in BV^{\varphi}(\mathbb{R}^2)$. *Then*

$$V_k^{\varphi}[\lambda(f_1 + \ldots + f_n)] \leq \frac{1}{n}\Big\{V_k^{\varphi}[\lambda n f_1] + \ldots + V_k^{\varphi}[\lambda n f_n]\Big\}, \quad k = 1, 2$$

and

$$V^{\varphi}[\lambda(f_1 + \ldots + f_n)] \leq \frac{1}{n}\Big\{V^{\varphi}[\lambda n f_1] + \ldots + V^{\varphi}[\lambda n f_n]\Big\}, \quad \lambda > 0.$$

PROOF. – For the sake of simplicity we can take $\lambda = 1$, since, in the trivial case, all the inequalities are obviously satisfied for every $\lambda > 0$. Let $I = [a, \alpha] \times [b, \beta] \subset \mathbb{R}^2$ and let $\{I_{ij}\}$, $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, m$, $j = 1, \ldots, p$, be a partition of I, with $x_0 = a < x_1 < \ldots < x_m = \alpha$ and $y_0 = b < y_1 < \ldots < y_p = \beta$. Then, by Proposition 1.12 of [25],

$$V^{\varphi}_{[x_{i-1}, x_i]}[(f_1 + \ldots + f_n)(\cdot, y)] \leq \frac{1}{n}\sum_{\nu=1}^{n} V^{\varphi}_{[x_{i-1}, x_i]}[n f_\nu(\cdot, y)],$$

for a.e. $y \in [b, \beta]$, for every $i = 1, \ldots, m$, and

$$V^{\varphi}_{[y_{j-1}, y_j]}[(f_1 + \ldots + f_n)(x, \cdot)] \leq \frac{1}{n}\sum_{\nu=1}^{n} V^{\varphi}_{[y_{j-1}, y_j]}[n f_\nu(x, \cdot)],$$

for a.e. $x \in [a, \alpha]$, for every $j = 1, \ldots, p$. Hence

$$\begin{aligned}\Phi_1^{\varphi}(f_1 + \ldots + f_n, I_{ij}) &= \int_{y_{j-1}}^{y_j} V^{\varphi}_{[x_{i-1}, x_i]}[(f_1 + \ldots + f_n)(\cdot, y)]\, dy \leq \frac{1}{n}\sum_{\nu=1}^{n} \int_{y_{j-1}}^{y_j} V^{\varphi}_{[x_{i-1}, x_i]}[n f_\nu(\cdot, y)]\, dy \\ &= \frac{1}{n}\sum_{\nu=1}^{n} \Phi_1^{\varphi}(n f_\nu, I_{ij})\end{aligned}$$

and

$$\Phi_2^{\varphi}(f_1 + \ldots + f_n, I_{ij}) \leq \frac{1}{n}\sum_{\nu=1}^{n} \Phi_2^{\varphi}(n f_\nu, I_{ij}).$$

This implies, summing over $i = 1, \ldots, m$, $j = 1, \ldots, p$, and passing to the supremum over all the partitions of I, that

$$V_k^{\varphi}[f_1 + \ldots + f_n, I] \leq \frac{1}{n}\Big\{V_k^{\varphi}[n f_1, I] + \ldots + V_k^{\varphi}[n f_n, I]\Big\}, \quad k = 1, 2.$$

Moreover, by the Minkowski inequality,

$$\Phi^{\varphi}(f_1 + \ldots + f_n, I_{ij}) \leq \frac{1}{n}\sum_{\nu=1}^{n} \Phi^{\varphi}(n f_\nu, I_{ij}),$$

and so, summing again over $i = 1, \ldots, m$, $j = 1, \ldots, p$, and passing to the supremum over all the possible partitions of I,

$$V^{\varphi}[f_1+\ldots+f_n,I]\le\frac{1}{n}\sum_{\nu=1}^{n}V^{\varphi}[nf_{\nu},I].$$

Hence the thesis follows by the arbitrariness of the interval $I\subset\mathbb{R}^2$. □

PROPOSITION 2. – *Let $f\in BV^{\varphi}(\mathbb{R}^N)$ and let $\{I_{ij}\}$ be a fixed partition of an interval $I=[a,\alpha]\times[b,\beta]$, with $I_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$, $i=1,\ldots,m$, $j=1,\ldots,p$, where $x_0=a<x_1<\ldots<x_m=\alpha$ and $y_0=b<y_1<\ldots<y_p=\beta$. Then, if $\lambda>0$ is such that $V^{\varphi}[2\lambda f]<+\infty$,*

(a) *if $f(x_i,y)=0$ for every $i=1,\ldots,m-1$ and almost every $y\in[b,\beta]$,*

$$V_1^{\varphi}[\lambda f,I]\le\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p}V_1^{\varphi}[2\lambda f,I_{ij}];$$

(b) *if $f(x,y_j)=0$ for every $j=1,\ldots,p-1$ and almost every $x\in[a,\alpha]$,*

$$V_2^{\varphi}[\lambda f,I]\le\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p}V_2^{\varphi}[2\lambda f,I_{ij}];$$

(c) *if $f(x_i,y)=0=f(x,y_j)$ for every $i=1,\ldots,m-1$, a.e. $y\in[b,\beta]$ and $j=1,\ldots,p-1$, a.e. $x\in[a,\alpha]$,*

$$V^{\varphi}[\lambda f,I]\le\frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{p}V^{\varphi}[2\lambda f,I_{ij}].$$

PROOF. – Let us prove (*a*). We notice that, by Proposition 1.18 of [25], there holds

$$V_{[a,\alpha]}^{\varphi}[\lambda f(\cdot,y)]\le\frac{1}{2}\sum_{i=1}^{m}V_{[x_{i-1},x_i]}^{\varphi}[2\lambda f(\cdot,y)],$$

for almost every $y\in[b,\beta]$. Hence, by (3),

$$\begin{aligned}V_1^{\varphi}[\lambda f,I]&=\int_b^{\beta}V_{[a,\alpha]}^{\varphi}[\lambda f(\cdot,y)]\,dy\le\frac{1}{2}\int_b^{\beta}\sum_{i=1}^{m}V_{[x_{i-1},x_i]}^{\varphi}[2\lambda f(\cdot,y)]\,dy\\&=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p}\int_{y_{j-1}}^{y_j}V_{[x_{i-1},x_i]}^{\varphi}[2\lambda f(\cdot,y)]\,dy=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p}V_1^{\varphi}[2\lambda f,I_{ij}].\end{aligned}$$

Now (*b*) follows with analogous reasonings, while in order to obtain (*c*) it is sufficient to notice that, by (*a*) and (*b*), if $\{J_{\mu\nu}\}$ is an arbitrary partition of I, where $J_{\mu\nu}=[a_{\mu-1},a_{\mu}]\times[b_{\nu-1},b_{\nu}]$, $\mu=1,\ldots,n$, $\nu=1,\ldots,q$, with $a_0=a<$

$a_1 < \ldots < a_n = \alpha$ and $b_0 = b < b_1 < \ldots < b_q = \beta$, then

$$\begin{aligned}\sum_{\mu=1}^{n}\sum_{\nu=1}^{q} \Phi^{\varphi}[\lambda f, J_{\mu\nu}] &\leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{q}\left\{\Phi_1^{\varphi}[\lambda f, J_{\mu\nu}] + \Phi_2^{\varphi}[\lambda f, J_{\mu\nu}]\right\} \leq V_1^{\varphi}[\lambda f, I] + V_2^{\varphi}[\lambda f, I] \\ &\leq \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p}\left\{\Phi_1^{\varphi}[2\lambda f, I_{ij}] + \Phi_2^{\varphi}[2\lambda f, I_{ij}]\right\} \\ &\leq \frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{p}\left\{\left(\Phi_1^{\varphi}[2\lambda f, I_{ij}]\right)^2 + \left(\Phi_2^{\varphi}[2\lambda f, I_{ij}]\right)^2\right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{p} \Phi^{\varphi}[2\lambda f, I_{ij}] \leq \frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{p} V^{\varphi}[2\lambda f, I_{ij}].\end{aligned}$$

So, passing to the supremum over all the possible partitions of I,

$$V^{\varphi}[\lambda f, I] \leq \frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{p} V^{\varphi}[2\lambda f, I_{ij}]. \qquad \square$$

PROPOSITION 3. – *Let $f \in BV^{\varphi}(\mathbb{R}^2)$ and let $I = [a, \alpha] \times [b, \beta]$. Then for every $a < \bar{a} < \alpha$, $b < \bar{b} < \beta$,*

(*a*) $V_k^{\varphi}[\lambda f, I] \leq \frac{1}{2}\left\{V_k^{\varphi}[2\lambda f, [a, \bar{a}] \times [b, \beta]] + V_k^{\varphi}[2\lambda f, [\bar{a}, \alpha] \times [b, \beta]]\right\}, \quad k = 1, 2,$

and $V^{\varphi}[\lambda f, I] \leq \frac{1}{2}\left\{V^{\varphi}[2\lambda f, [a, \bar{a}] \times [b, \beta]] + V^{\varphi}[2\lambda f, [\bar{a}, \alpha] \times [b, \beta]]\right\}, \ \lambda > 0;$

(*b*) $V_k^{\varphi}[\lambda f, I] \leq \frac{1}{2}\left\{V_k^{\varphi}[2\lambda f, [a, \alpha] \times [b, \bar{b}]] + V_k^{\varphi}[2\lambda f, [a, \alpha] \times [\bar{b}, \beta]]\right\}, \quad k = 1, 2,$

and $V^{\varphi}[\lambda f, I] \leq \frac{1}{2}\left\{V^{\varphi}[2\lambda f, [a, \alpha] \times [b, \bar{b}]] + V^{\varphi}[2\lambda f, [a, \alpha] \times [\bar{b}, \beta]]\right\}, \ \lambda > 0.$

PROOF. – As in Proposition 1, for the sake of simplicity we can take $\lambda = 1$. By (3) and Proposition 1.17 of [25], we have

$$\begin{aligned}V_1^{\varphi}[f, I] &= \int_b^{\beta} V_{[a,\alpha]}^{\varphi}[f(\cdot, y)]\, dy \leq \frac{1}{2}\int_b^{\beta}\left(V_{[a,\bar{a}]}^{\varphi}[2f(\cdot, y)] + V_{[\bar{a},\alpha]}^{\varphi}[2f(\cdot, y)]\right) dy \\ &= \frac{1}{2}\left(V_1^{\varphi}[2f, [a, \bar{a}] \times [b, \beta]] + V_1^{\varphi}[2f, [\bar{a}, \alpha] \times [b, \beta]]\right).\end{aligned}$$

Moreover, by (4), by convexity of φ and since $\varphi(0) = 0$,

$$\begin{aligned}V_2^{\varphi}[f, I] &= \int_a^{\alpha} V_{[b,\beta]}^{\varphi}[f(x, \cdot)]\, dx \leq \frac{1}{2}\left\{\int_a^{\bar{a}} + \int_{\bar{a}}^{\alpha}\right\} V_{[b,\beta]}^{\varphi}[2f(x, \cdot)]\, dx \\ &= \frac{1}{2}\left(V_2^{\varphi}[2f, [a, \bar{a}] \times [b, \beta]] + V_2^{\varphi}[2f, [\bar{a}, \alpha] \times [b, \beta]]\right).\end{aligned}$$

Hence (a) is proved, taking into account that the statement for V^φ follows by the previous relations and by the Minkowski inequality. (b) follows with similar arguments. □

PROPOSITION 4. – *Let $f \in BV^\varphi(\mathbb{R}^2)$ and let $I = [a, \alpha] \times [b, \beta]$. Then for every $a < \bar{a} < \alpha$, $b < \bar{b} < \beta$,*

(a) $V_k^\varphi[\lambda f, [a, \bar{a}] \times [b, \beta]] + V_k^\varphi[\lambda f, [\bar{a}, \alpha] \times [b, \beta]] \leq V_k^\varphi[\lambda f, I]$, $k = 1, 2$, $\lambda > 0$;

(b) $V_k^\varphi[\lambda f, [a, \alpha] \times [b, \bar{b}]] + V_k^\varphi[\lambda f, [a, \alpha] \times [\bar{b}, \beta]] \leq V_k^\varphi[\lambda f, I]$, $k = 1, 2$, $\lambda > 0$.

PROOF. – For the sake of simplicity let us take $\lambda = 1$, as in Proposition 1. In order to prove (a), for $k = 1$ it is sufficient to notice that, by (3) and by Proposition 1.17 of [25],

$$V_1^\varphi[f, [a, \bar{a}] \times [b, \beta]] + V_1^\varphi[f, [\bar{a}, \alpha] \times [b, \beta]] = \int_b^\beta \left(V_{[a,\bar{a}]}^\varphi[f(\cdot, y)] + V_{[\bar{a},\alpha]}^\varphi[f(\cdot, y)] \right) dy$$

$$\leq \int_b^\beta V_{[a,\alpha]}^\varphi[f(\cdot, y)]\, dy = V_1^\varphi[f, I],$$

while for $k = 2$ it is obvious. (b) follows with analogous reasonings. □

REMARK 1. – (a) Propositions 2, 3 and 4 generalize Propositions 1.18 and 1.17 of [25] to the multidimensional frame and such results reproduce, working with φ-variation, the usual additivity with respect to intervals of the classical variation. In the present multidimensional setting, since we will work with the single directions in order to obtain the main theorems, we need to state the results with respect to each section, i.e., for the "separated" variations V_1^φ and V_2^φ.

(b) We remark that Propositions 3 and 4 hold also in the case of unbounded intervals: indeed the proofs can be easily adapted if, for example, $I = ([a, \bar{a}] \times \mathbb{R}) \cup ([\bar{a}, \alpha] \times \mathbb{R})$ or in all the other cases.

In order to prove the main theorems we finally need the following result in the one-dimensional case, which is similar to 1.18 of [25] but in the particular case of step functions. We note that, in order to get the property of subadditivity on intervals for a step function defined as below, it is not necessary to assume that the function vanishes on the nodes of the fixed partition, as it happens in the general case (see also Proposition 2).

PROPOSITION 5. – *Let $v : \mathbb{R} \to \mathbb{R}$ be defined as*

$$v(x) = \begin{cases} \alpha_0, & x < x_0 = a, \\ \alpha_{i-1}, & x_{i-1} \leq x < x_i, \\ \alpha_m, & x \geq x_m = b, \end{cases}$$

where $a = x_0 < x_1 < \ldots < x_m = b$ *is a partition of* $[a, b] \subset \mathbb{R}$ *and* $\alpha_i \in \mathbb{R}$, $i = 0, \ldots, m$. *Then for every* $\lambda > 0$,

$$V^{\varphi}_{[a,b]}[\lambda v] \leq \frac{1}{2} \sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[2\lambda v].$$

PROOF. – Let $\{a_\mu\}$, $\mu = 0, \ldots, n$ be a partition of $[a, b]$ such that $\max_{\mu=1,\ldots,n} \{a_\mu - a_{\mu-1}\} < \min_{i=1,\ldots,m} \{x_i - x_{i-1}\}$. Let A be the set of the indexes μ such that $a_{\mu-1} < x_i < a_\mu$, for some $i = 0, \ldots, m$ and let B be the set of the remaining indexes. If $\mu \in A$ then there holds, for every $\lambda > 0$,

$$\begin{aligned} \varphi(\lambda|v(a_\mu) - v(a_{\mu-1})|) = \varphi(\lambda|\alpha_i - \alpha_{i-1}|) &\leq \frac{1}{2}\varphi(2\lambda|v(x_i) - v(a_{\mu-1})|) + \frac{1}{2}\varphi(2\lambda|v(a_\mu) - v(x_i)|) \\ &\leq \frac{1}{2} V^{\varphi}_{[a_{\mu-1},x_i]}[2\lambda v] + \frac{1}{2} V^{\varphi}_{[x_i,a_\mu]}[2\lambda v], \end{aligned}$$

taking into account that φ is convex and that $\varphi(0) = 0$. Hence

$$\begin{aligned} \sum_{\mu=1}^{n} \varphi(\lambda|v(a_\mu) - v(a_{\mu-1})|) &= \left(\sum_{\mu\in A} + \sum_{\mu \in B} \right) \varphi(\lambda|v(a_\mu) - v(a_{\mu-1})|) \\ &\leq \frac{1}{2} \sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[2\lambda v] \end{aligned}$$

by 1.17 of [25] and so, passing to the supremum over all the partitions of $[a, b]$, we conclude that

$$V^{\varphi}_{[a,b]}[\lambda v] \leq \frac{1}{2} \sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[2\lambda v],$$

for every $\lambda > 0$. □

REMARK 2. – We remark that Proposition 5 holds also if the step function $v : \mathbb{R} \to \mathbb{R}$ is defined as

$$v(x) = \begin{cases} \alpha_0, & x \leq x_0 = a, \\ \alpha_i, & x_{i-1} < x \leq x_i, \\ \alpha_m, & x > x_m = b, \end{cases}$$

with the same notations as before.

4. – Convergence results for the auxiliary step functions.

We will now define two step functions, associated to a function $f \in BV^{\varphi}(\mathbb{R}^2)$, which will be useful in order to obtain the main result.

Let $f \in BV^{\varphi}(\mathbb{R}^2)$ be fixed. Let $I = [a, \alpha] \times [b, \beta]$ be a fixed interval and let $\{I_{ij}\}$ be a partition of I, with $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, m, j = 1, \ldots, p$, where $a = x_0 < x_1 < \ldots < x_m = \alpha$, $b = y_0 < y_1 < \ldots < y_p = \beta$. We define a function $v_1 : \mathbb{R}^2 \to \mathbb{R}$ as follows: in the set $([a, \alpha] \times \mathbb{R}) \cup (\mathbb{R} \times [b, \beta])$ it is defined as

$$v_1(x, y) = \begin{cases} f(x_{i-1}, y), & x_{i-1} \leq x < x_i, \ y \in [b, \beta], \\ f(a, y), & x < a, \ y \in [b, \beta], \\ f(\alpha, y), & x \geq \alpha, \ y \in [b, \beta], \\ f(x_{i-1}, b), & x_{i-1} \leq x < x_i, \ y < b, \\ f(x_{i-1}, \beta), & x_{i-1} \leq x < x_i, \ y > \beta, \\ f(\alpha, b), & x = \alpha, \ y < b, \\ f(\alpha, \beta), & x = \alpha, \ y > \beta, \end{cases}$$

$i = 1, \ldots, m$, while elsewhere the definition of v_1 is constant and extended with continuity. Hence v_1 is defined in such a way that it coincides with f on the segments $(x_i, \cdot)$, $i = 0, \ldots, m$, while the sections $v_1(\cdot, y)$ are constant in each interval $]x_{i-1}, x_i[$, for every $y \in \mathbb{R}$.

We now define another function $v_2 : \mathbb{R}^2 \to \mathbb{R}$, that will play a symmetric role with respect to v_1: in particular, v_2 coincides with f on the segments $(\cdot, y_j)$, $j = 0, \ldots, p$, while the sections $v_2(x, \cdot)$ are constant in each interval $]y_{j-1}, y_j[$, for every $x \in \mathbb{R}$. Hence on $([a, \alpha] \times \mathbb{R}) \cup (\mathbb{R} \times [b, \beta])$ v_2 is defined as

$$v_2(x, y) = \begin{cases} f(x, y_{j-1}), & x \in [a, \alpha], \ y_{j-1} \leq y < y_j, \\ f(x, b), & x \in [a, \alpha], \ y \leq b, \\ f(x, \beta), & x \in [a, \alpha], \ y \geq \beta, \\ f(a, y_{j-1}), & x < a, \ y_{j-1} \leq y < y_j, \\ f(\alpha, y_{j-1}), & x \geq \alpha, \ y_{j-1} \leq y < y_j, \\ f(a, \beta), & x < a, \ y = \beta, \\ f(\alpha, \beta), & x > \alpha, \ y = \beta, \end{cases}$$

$j = 1, \ldots, p$, otherwise it is constant and extended with continuity, so $v_2(x, y) = v_1(x, y)$, for every $(x, y) \notin ([a, \alpha] \times \mathbb{R}) \cup (\mathbb{R} \times [b, \beta])$.

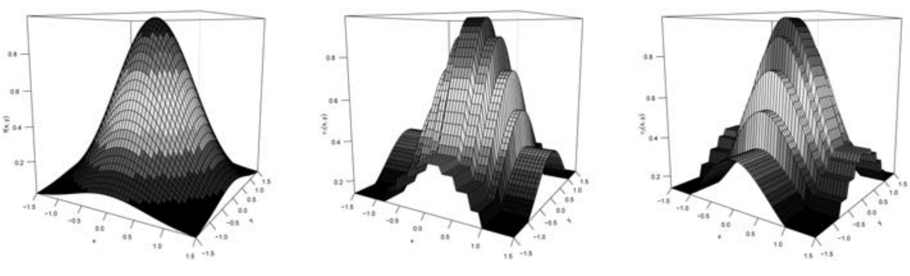

Fig. 1. – An example: $f(x, y) = e^{-(x^2+y^2)}$, $v_1(x, y)$ and $v_2(x, y)$, $I = [-1, 1]^2$.

We will now prove a first convergence result for v_1 and v_2. In the following, if $I = [a, \alpha] \times [b, \beta]$ is a fixed interval in $\mathbb{R}^2$, we will use the notations

$$V^{\varphi}[f, \overline{I^c}] := V^{\varphi}[f, (-\infty, a] \times [b, \beta]] + V^{\varphi}[f, [\alpha, +\infty) \times [b, \beta]] + V^{\varphi}[f, [a, \alpha] \times (-\infty, b]]$$
$$+ V^{\varphi}[f, [a, \alpha] \times [\beta, +\infty)] + V^{\varphi}[f, (-\infty, a] \times (-\infty, b]] + V^{\varphi}[f, [\alpha, +\infty) \times (-\infty, b]]$$
$$+ V^{\varphi}[f, (-\infty, a] \times [\beta, +\infty)] + V^{\varphi}[f, [\alpha, +\infty) \times [\beta, +\infty)],$$

($V_k^{\varphi}[f, \overline{I^c}]$, $k = 1, 2$, will be defined in a similar way) and

$$I_\delta := [a - \delta, \alpha + \delta] \times [b - \delta, \beta + \delta], \quad \delta > 0.$$

PROPOSITION 6. – *Let $f \in AC^{\varphi}(\mathbb{R}^2)$. Then there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exist an interval $I = [a, \alpha] \times [b, \beta]$ and a constant $\delta > 0$ for which, if $\{I_{ij}\}$ is a partition of I, $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, m$, $j = 1, \ldots, p$, such that $\max_{i=1,\ldots,m,\, j=1,\ldots,p} \{x_i - x_{i-1}, y_j - y_{j-1}\} < \delta$, then*

(a) $V^{\varphi}[\lambda f, \overline{I^c}] < \varepsilon$;

(b) $\sum_{i=1}^{m} \sum_{j=1}^{p} V_k^{\varphi}[\lambda f, I_{ij}] < \varepsilon$, $k = 1, 2$;

(c) *the step functions v_1 and v_2 defined as above in correspondence to the partition $\{I_{ij}\}$ are such that*

$$V_1^{\varphi}[\lambda(f - v_1), I_{\tilde{\delta}}] < \frac{\varepsilon}{2}, \quad V_2^{\varphi}[\lambda(f - v_2), I_{\tilde{\delta}}] < \frac{\varepsilon}{2},$$

for some $\widetilde{\delta} > 0$.

PROOF. – Since $f \in AC^{\varphi}(\mathbb{R}^2)$, in particular $f \in BV^{\varphi}(\mathbb{R}^2)$. Let $\overline{\lambda} > 0$ be such that $V^{\varphi}[\overline{\lambda} f] < +\infty$ and let us fix $\varepsilon > 0$. Then there exists an interval $I = [a, \alpha] \times [b, \beta] \subset \mathbb{R}^2$ such that

$$V_k^{\varphi}[\overline{\lambda} f] < V_k^{\varphi}[\overline{\lambda} f, I] + \frac{\varepsilon}{2}, \quad k = 1, 2,$$

and so, by Proposition 4,

$$V_k^{\varphi}[\overline{\lambda} f, \overline{I^c}] + V_k^{\varphi}[\overline{\lambda} f, I] \leq V_k^{\varphi}[\overline{\lambda} f] < V_k^{\varphi}[\overline{\lambda} f, I] + \frac{\varepsilon}{2},$$

which implies that

$$V_k^{\varphi}[\overline{\lambda} f, \overline{I^c}] < \frac{\varepsilon}{2}, \quad k = 1, 2. \tag{5}$$

Hence (a) follows since

$$V^{\varphi}[\overline{\lambda} f, \overline{I^c}] \leq V_1^{\varphi}[\overline{\lambda} f, \overline{I^c}] + V_2^{\varphi}[\overline{\lambda} f, \overline{I^c}] < \varepsilon.$$

We now prove (b). Since $f \in AC^{\varphi}_{loc}(\mathbb{R}^2)$ and $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$ (assumption 2. on $\varphi \in \Phi$), then there is $\widetilde{\lambda} > 0$ such that, in correspondence to ε, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\{I_{ij}\}$ is a fixed partition of I, $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \dots, m, j = 1, \dots, p$ with $\max_{i=1,\dots,m,\, j=1,\dots,p} \{x_i - x_{i-1}, y_j - y_{j-1}\} < \delta$, then

$$\sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[\widetilde{\lambda} f(\cdot, y)] < \frac{\varepsilon}{\beta - b}, \quad \text{a.e. } y \in [b, \beta],$$

$$\sum_{j=1}^{p} V^{\varphi}_{[y_{j-1},y_j]}[\widetilde{\lambda} f(x, \cdot)] < \frac{\varepsilon}{\alpha - a}, \quad \text{a.e. } x \in [\alpha, a],$$

(see 2.1 of [25]).

Hence, by (3),

$$\sum_{i=1}^{m}\sum_{j=1}^{p} V^{\varphi}_1[\widetilde{\lambda} f, I_{ij}] = \sum_{i=1}^{m}\sum_{j=1}^{p} \int_{y_{j-1}}^{y_j} V^{\varphi}_{[x_{i-1},x_i]}[\widetilde{\lambda} f(\cdot, y)]\, dy = \int_b^{\beta} \sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[\widetilde{\lambda} f(\cdot, y)]\, dy < \varepsilon,$$

and, analogously,

$$\sum_{i=1}^{m}\sum_{j=1}^{p} V^{\varphi}_2[\widetilde{\lambda} f, I_{ij}] < \varepsilon.$$

Let us now prove the first relation of (c).

Let $0 < \bar{\delta} < \delta$. By (3) and by 1.18 of [25], we have

$$\begin{aligned} V^{\varphi}_1[\lambda(f - v_1), I_{\bar{\delta}}] &= \int_{b-\bar{\delta}}^{\beta+\bar{\delta}} V^{\varphi}_{[a-\bar{\delta},\alpha+\bar{\delta}]}[\lambda(f - v_1)(\cdot, y)]\, dy \\ &\leq \frac{1}{3} \int_{b-\bar{\delta}}^{\beta+\bar{\delta}} \Big(V^{\varphi}_{[a-\bar{\delta},a]}[3\lambda(f - v_1)(\cdot, y)] + V^{\varphi}_{[a,\alpha]}[3\lambda(f - v_1)(\cdot, y)] \\ &\quad + V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[3\lambda(f - v_1)(\cdot, y)] \Big)\, dy := \frac{1}{3}(S_1 + S_2 + S_3). \end{aligned}$$

About S_1 and S_3, let us notice that, for every $y \in [b, \beta]$,

$$v_1(x, y) = \begin{cases} f(a, y), & x \leq a, \\ f(\alpha, y), & x \geq \alpha, \end{cases}$$

while $v_1(x, y)$ is constant if $y > \beta$ or $y < b$ and $x < a$ or $x > \alpha$. Hence $V^{\varphi}_{[a-\bar{\delta},a]}[\lambda v_1(\cdot, y)] = 0 = V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[\lambda v_1(\cdot, y)]$, for every $y \in [b - \bar{\delta}, \beta + \bar{\delta}]$, $\lambda > 0$. Then, by 1.12 of [25],

$$S_1 + S_3 \le \frac{1}{2} \int\limits_{b-\bar{\delta}}^{\beta+\bar{\delta}} \Big(V^{\varphi}_{[a-\bar{\delta},a]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[a-\bar{\delta},a]}[6\lambda v_1(\cdot,y)] + V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[6\lambda v_1(\cdot,y)]\Big)$$

$$= \frac{1}{2} \int\limits_{b-\bar{\delta}}^{\beta+\bar{\delta}} \Big(V^{\varphi}_{[a-\bar{\delta},a]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[6\lambda f(\cdot,y)]\Big)\, dy$$

$$= \frac{1}{2} \left(\int\limits_{b-\bar{\delta}}^{b} + \int\limits_{b}^{\beta} + \int\limits_{\beta}^{\beta+\bar{\delta}} \right) \Big(V^{\varphi}_{[a-\bar{\delta},a]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[\alpha,\alpha+\bar{\delta}]}[6\lambda f(\cdot,y)]\Big)\, dy$$

$$\le V^{\varphi}_1[6\lambda f, \overline{I^c}] < \frac{\varepsilon}{2},$$

if $6\lambda < \bar{\lambda}$, by (5).

About S_2, as before we can write

$$S_2 \le \frac{1}{2} \int\limits_{b-\bar{\delta}}^{b} \Big(V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[a,\alpha]}[6\lambda v_1(\cdot,y)]\Big)\, dy + \int\limits_{b}^{\beta} V^{\varphi}_{[a,\alpha]}[3\lambda(f - v_1)(\cdot,y)]\, dy$$

$$+ \frac{1}{2} \int\limits_{\beta}^{\beta+\bar{\delta}} \Big(V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot,y)] + V^{\varphi}_{[a,\alpha]}[6\lambda v_1(\cdot,y)]\Big)\, dy$$

$$\le \frac{1}{2} V^{\varphi}_1[6\lambda f, \overline{I^c}] + \frac{1}{2} \left(\int\limits_{b-\bar{\delta}}^{b} + \int\limits_{\beta}^{\beta+\bar{\delta}} \right) V^{\varphi}_{[a,\alpha]}[6\lambda v_1(\cdot,y)]\, dy + V^{\varphi}_1[3\lambda(f - v_1), I].$$

We notice that, by the definition of v_1, $(f - v_1)(x_i, y) = 0$, $i = 0, \ldots, m$, for every $y \in [b, \beta]$. Then, by Proposition 2, for every $\lambda < \frac{\bar{\lambda}}{2}$,

$$\text{(6)} \qquad V^{\varphi}_1[\lambda(f - v_1), I] \le \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{p} V^{\varphi}_1[2\lambda(f - v_1), I_{ij}].$$

Hence, using (6) and then Proposition 1,

$$V^{\varphi}_1[3\lambda(f - v_1), I] \le \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{p} V^{\varphi}_1[6\lambda(f - v_1), I_{ij}]$$

$$\le \frac{1}{4} \sum_{i=1}^{m} \sum_{j=1}^{p} V^{\varphi}_1[12\lambda f, I_{ij}] + \frac{1}{4} \sum_{i=1}^{m} \sum_{j=1}^{p} V^{\varphi}_1[12\lambda v_1, I_{ij}]$$

$$\le \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{p} V^{\varphi}_1[12\lambda f, I_{ij}] < \frac{\varepsilon}{2},$$

by part (b), provided that $0 < \lambda < \frac{\widetilde{\lambda}}{12}$.

Moreover, let us notice that, if $a \leq x_{i-1} \leq x < x_i \leq \alpha$, then

$$v_1(x,y) = \begin{cases} f(x_{i-1}, b), & y \leq b, \\ f(x_{i-1}, \beta), & b \geq \beta, \end{cases}$$

while $v_1(\alpha, y) = \begin{cases} f(\alpha, b), & y \leq b, \\ f(\alpha, \beta), & y \geq \beta, \end{cases}$ and so

$$\int_{b-\bar{\delta}}^{b} V^{\varphi}_{[a,\alpha]}[6\lambda v_1(\cdot, y)]\, dy \leq \bar{\delta} V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot, b)],$$

$$\int_{\beta}^{\beta+\bar{\delta}} V^{\varphi}_{[a,\alpha]}[6\lambda v_1(\cdot, y)]\, dy \leq \bar{\delta} V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot, \beta)].$$

Now, if $V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, b)] \neq 0$ and $V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, \beta)] \neq 0$ and if we consider $0 < \overline{\delta} < \delta$ such that

$$\overline{\delta} < \min\left\{ \frac{\varepsilon}{4V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, b)]}, \frac{\varepsilon}{4V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, \beta)]} \right\}$$

(of course if, for example, $V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, b)] \neq 0$ and $V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, \beta)] = 0$ we just take $\overline{\delta} < \dfrac{\varepsilon}{2V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot, b)]}$, and analogously in the other cases), then in particular

$$\overline{\delta} < \min\left\{ \frac{\varepsilon}{4V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot, b)]}, \frac{\varepsilon}{4V^{\varphi}_{[a,\alpha]}[6\lambda f(\cdot, \beta)]} \right\}$$

for every $\lambda > 0$ such that $6\lambda < \overline{\lambda}$, and so

$$S_2 < \varepsilon.$$

Hence we conclude that

$$V^{\varphi}_1[\lambda(f - v_1), I_{\overline{\delta}}] < \frac{\varepsilon}{2},$$

if $0 < \lambda < \min\left\{ \dfrac{\overline{\lambda}}{6}, \dfrac{\widetilde{\lambda}}{12} \right\}$, that is, the first relation of (c).

The second relation can be proved with similar reasonings to the previous ones. Indeed, it is sufficient to change the variables and to repeat the same arguments replacing the $x-$section with the $y-$section and viceversa. Then it is possible to prove that

$$V^{\varphi}_2[\lambda(f - v_2), I_{\delta'}] < \frac{\varepsilon}{2},$$

if $0 < \lambda < \min\left\{\frac{\overline{\lambda}}{6}, \frac{\widetilde{\lambda}}{12}\right\}$, for $\delta' < \min\left\{\frac{\varepsilon}{4V^{\varphi}_{[b,\beta]}[\overline{\lambda}f(a,\cdot)]}, \frac{\varepsilon}{4V^{\varphi}_{[b,\beta]}[\overline{\lambda}f(\alpha,\cdot)]}\right\}$ (as before of course if $V^{\varphi}_{[b,\beta]}[\overline{\lambda}f(a,\cdot)]V^{\varphi}_{[b,\beta]}[\overline{\lambda}f(\alpha,\cdot)] \neq 0$). In conclusion, we have proved that there exists $\widetilde{\delta} = \min\{\overline{\delta}, \delta'\} > 0$ such that, if $0 < \lambda < \min\left\{\frac{\overline{\lambda}}{6}, \frac{\widetilde{\lambda}}{12}\right\}$,

$$V^{\varphi}_1[\lambda(f - \nu_1), I_{\widetilde{\delta}}] < \frac{\varepsilon}{2}, \quad V^{\varphi}_2[\lambda(f - \nu_2), I_{\widetilde{\delta}}] < \frac{\varepsilon}{2},$$

which concludes the proof of the theorem. □

We now prove a first convergence result for $(\tau_{\boldsymbol{t}}\nu_k - \nu_k)$, $k = 1, 2$, where $\tau_{\boldsymbol{t}}$ is the translation operator, defined as $\tau_{\boldsymbol{t}}f(x, y) = f(x - t_1, y - t_2)$, if $\boldsymbol{t} = (t_1, t_2)$.

THEOREM 1. – *Let $f \in AC^{\varphi}(\mathbb{R}^2)$. Then there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which*

$$V^{\varphi}_1[\lambda(\tau_{\boldsymbol{t}}\nu_1 - \nu_1), I] < \frac{\varepsilon}{2}, \quad V^{\varphi}_2[\lambda(\tau_{\boldsymbol{t}}\nu_2 - \nu_2), I] < \frac{\varepsilon}{2},$$

where $\boldsymbol{t} = (t, 0)$ or $\boldsymbol{t} = (0, t)$, if $-\delta < t < 0$, with ν_1, ν_2 defined as in Proposition 6.

PROOF. – We first prove the result in the case $\boldsymbol{t} = (t, 0)$.

Let us fix $\varepsilon > 0$. By Proposition 6 there exist $\overline{\lambda} > 0$, $I = [a, \alpha] \times [b, \beta]$ and $\overline{\delta} > 0$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{p} V^{\varphi}_k[\overline{\lambda}f, I_{ij}] < \varepsilon, \quad k = 1, 2, \tag{7}$$

for a suitable partition $\{I_{ij}\}$ of I, $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, m$, $j = 1, \ldots, p$ (such that, in particular, $\max_{i=1,\ldots,m,\, j=1,\ldots,p}\{x_i - x_{i-1}, y_j - y_{j-1}\} < \overline{\delta}$). Moreover, $V^{\varphi}[\overline{\lambda}(f - \nu_k), I_{\widetilde{\delta}}] < \frac{\varepsilon}{2}$, $k = 1, 2$, for some $\widetilde{\delta} > 0$.

Let us now consider $0 < \delta < \widetilde{\delta}$ such that $\delta \leq \min_{i=1,\ldots,m,\, j=1,\ldots,p}\{x_i - x_{i-1}, y_j - y_{j-1}\}$.

We first prove that $V^{\varphi}_1[\lambda(\tau_{\boldsymbol{t}}\nu_1 - \nu_1), I] < \frac{\varepsilon}{2}$ if $-\delta < t < 0$.

Let us notice that, for every fixed $y \in [b, \beta]$, the function $\nu_1(\cdot, y)$ is a step function on $[a, \alpha]$ defined exactly as the function ν of Theorem 1 in [2]. Hence, following the proof of Theorem 1 in [2], one can show that, if $-\delta < t < 0$, then, for every $\lambda > 0$, $y \in [b, \beta]$,

$$V^{\varphi}_{[a,\alpha]}[\lambda(\tau_{\boldsymbol{t}}\nu_1 - \nu_1)(\cdot, y)] \leq \frac{1}{2}\sum_{i=1}^{m} V^{\varphi}_{[x_{i-1},x_i]}[4\lambda f(\cdot, y)]. \tag{8}$$

Now, by Proposition 1.17 of [25] and using (8) and (7),

$$V_1^\varphi[\lambda(\tau_{\boldsymbol{t}}v_1 - v_1), I] = \int_b^\beta V^\varphi_{[a,\alpha]}[\lambda(\tau_{\boldsymbol{t}}v_1 - v_1)(\cdot, y)]\, dy \leq \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^p \int_{y_{j-1}}^{y_j} V^\varphi_{[x_{i-1},x_i]}[4\lambda f(\cdot, y)]\, dy$$

$$= \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^p V_1^\varphi[4\lambda f, I_{ij}] < \frac{\varepsilon}{2},$$

if $\lambda > 0$ is such that $4\lambda < \bar{\lambda}$ and $-\delta < t < 0$.

Let us now prove that $V_2^\varphi[\lambda(\tau_{\boldsymbol{t}}v_2 - v_2), I] < \dfrac{\varepsilon}{2}$.

We first notice that, for every fixed $x \in [a, \alpha]$, $v_2(x, \cdot)$ is a step function to which it is possible to apply Proposition 5, hence, for every $\lambda > 0$,

$$V^\varphi_{[b,\beta]}[\lambda v_2(x, \cdot)] \leq \frac{1}{2}\sum_{j=1}^p V^\varphi_{[y_{j-1},y_j]}[2\lambda v_2(x, \cdot)]. \tag{9}$$

Proposition 5 can be applied also to $\tau_{\boldsymbol{t}}v_2(x, \cdot) = v_2(x - t, \cdot)$, for every fixed $\boldsymbol{t} = (t, 0)$, and so

$$V^\varphi_{[b,\beta]}[\lambda\tau_{\boldsymbol{t}}v_2(x, \cdot)] \leq \frac{1}{2}\sum_{j=1}^p V^\varphi_{[y_{j-1},y_j]}[2\lambda\tau_{\boldsymbol{t}}v_2(x, \cdot)]. \tag{10}$$

Now, by the definition of v_2 we have that, for every $x \in [a, \alpha]$,

$$V^\varphi_{[y_{j-1},y_j]}[4\lambda v_2(x, \cdot)] \leq V^\varphi_{[y_{j-1},y_j]}[4\lambda f(x, \cdot)], \tag{11}$$

while if $x \geq \alpha$, $v_2(x, \cdot) = v_2(\alpha, \cdot)$, hence in this case

$$V^\varphi_{[y_{j-1},y_j]}[4\lambda v_2(x, \cdot)] \leq V^\varphi_{[y_{j-1},y_j]}[4\lambda f(\alpha, \cdot)], \tag{12}$$

for every $j = 1, \ldots, p$. Then there holds, by Propositions 1.12 and 1.17 of [25] and by (9)-(12),

$$V_2^\varphi[\lambda(\tau_{\boldsymbol{t}}v_2 - v_2), I] = \int_a^\alpha V^\varphi_{[b,\beta]}[\lambda(\tau_{\boldsymbol{t}}v_2 - v_2)(x, \cdot)]\, dx$$

$$\leq \frac{1}{2}\int_a^\alpha \left(V^\varphi_{[b,\beta]}[2\lambda\tau_{\boldsymbol{t}}v_2(x, \cdot)] + V^\varphi_{[b,\beta]}[2\lambda v_2(x, \cdot)]\right) dx$$

$$\leq \frac{1}{4}\int_a^\alpha \sum_{j=1}^p \left(V^\varphi_{[y_{j-1},y_j]}[4\lambda\tau_{\boldsymbol{t}}v_2(x, \cdot)] + V^\varphi_{[y_{j-1},y_j]}[4\lambda v_2(x, \cdot)]\right) dx$$

$$= \frac{1}{4}\int_{a-t}^{\alpha-t} \sum_{j=1}^p V^\varphi_{[y_{j-1},y_j]}[4\lambda v_2(x, \cdot)]\, dx + \frac{1}{4}\sum_{i=1}^m\sum_{j=1}^p \int_{x_{i-1}}^{x_i} V^\varphi_{[y_{j-1},y_j]}[4\lambda v_2(x, \cdot)]\, dx$$

$$\leq \frac{1}{4}\int_{a-t}^{\alpha} \sum_{j=1}^p V^\varphi_{[y_{j-1},y_j]}[4\lambda f(x, \cdot)]\, dx + \frac{1}{4}\int_\alpha^{\alpha-t} \sum_{j=1}^p V^\varphi_{[y_{j-1},y_j]}[4\lambda f(\alpha, \cdot)]\, dx$$

$$+ \frac{1}{4}\sum_{i=1}^m\sum_{j=1}^p \int_{x_{i-1}}^{x_i} V^\varphi_{[y_{j-1},y_j]}[4\lambda f(x, \cdot)]\, dx =: \frac{1}{4}(S_1 + S_2 + S_3).$$

About S_1 and S_3 we have, by (7) and taking into account that φ is convex and $\varphi(0) = 0$,

$$S_3 = \sum_{i=1}^{m}\sum_{j=1}^{p} V_2^{\varphi}[4\lambda f, I_{ij}] \leq \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{p} V_2^{\varphi}[8\lambda f, I_{ij}] < \frac{\varepsilon}{2},$$

if $8\lambda < \overline{\lambda}$ and, similarly,

$$S_1 \leq \int_a^{\alpha} \sum_{j=1}^{p} V^{\varphi}_{[y_{j-1}y_j]}[4\lambda f(x,\cdot)]\,dx = \sum_{i=1}^{m}\sum_{j=1}^{p} \int_{x_{i-1}}^{x_i} V^{\varphi}_{[y_{j-1},y_j]}[4\lambda f(x,\cdot)]\,dx$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{p} V_2^{\varphi}[4\lambda f, I_{ij}] < \frac{\varepsilon}{2}.$$

About S_2, notice that, if $-\delta < t < 0$, then by 1.17 of [25], $S_2 \leq \delta V^{\varphi}_{[b,\beta]}[4\lambda f(\alpha,\cdot)]$.

Let us now recall that δ is such that, in particular,

$$\delta < \frac{\varepsilon}{V^{\varphi}_{[b,\beta]}[\overline{\lambda} f(\alpha,\cdot)]}$$

(see Proposition 6), therefore, if $4\lambda < \overline{\lambda}$, then

$$\delta < \frac{\varepsilon}{V^{\varphi}_{[b,\beta]}[4\lambda f(\alpha,\cdot)]}$$

(of course here we consider the case $V^{\varphi}_{[b,\beta]}[\overline{\lambda} f(\alpha,\cdot)] \neq 0$, since the other case is obvious). Hence, if $0 < \lambda < \dfrac{\overline{\lambda}}{8}$ and $-\delta < t < 0$, we conclude that

$$V_2^{\varphi}[\lambda(\tau_{\boldsymbol{t}} v_2 - v_2), I] < \frac{\varepsilon}{2},$$

and so the thesis follows.

The proof in the case $\boldsymbol{t} = (0,t)$ follows with analogous reasonings. We just remark that, in this case, one has to use the fact that, by Proposition 6, $0 < \delta < \dfrac{\varepsilon}{V^{\varphi}_{[a,\alpha]}[\overline{\lambda} f(\cdot,\beta)]} < \dfrac{\varepsilon}{V^{\varphi}_{[a,\alpha]}[4\lambda f(\cdot,\beta)]}$, if $0 < \lambda < \dfrac{\overline{\lambda}}{4}$. □

If we use a slight modification of the auxiliary step functions, it is possible to repeat all the previous results and to get the convergence result of Theorem 1 for positive values of t, i.e., for $0 < t < \delta$. In particular, we have to use $\widetilde{v}_1$ instead of v_1 and $\widetilde{v}_2$ instead of v_2, where $\widetilde{v}_1 : \mathbb{R}^2 \to \mathbb{R}$ is defined in the set $([a,\alpha] \times \mathbb{R}) \cup (\mathbb{R} \times [b,\beta])$ as

$$\widetilde{v}_1(x,y) = \begin{cases} f(x_i,y), & x_{i-1}<x \le x_i,\ y \in [b,\beta], \\ f(a,y), & x \le a,\ y \in [b,\beta], \\ f(\alpha,y), & x > \alpha,\ y \in [b,\beta], \\ f(x_i,b), & x_{i-1}<x \le x_i,\ y<b, \\ f(x_i,\beta), & x_{i-1}<x \le x_i,\ y > \beta, \\ f(a,b), & x = a,\ y<b, \\ f(a,\beta), & x = a,\ y > \beta, \end{cases}$$

$i = 1, \ldots, m$, while $\widetilde{v}_1(x,y) = v_1(x,y)$ otherwise. Hence $\widetilde{v}_1$ and v_1 are defined in a similar way, except for the fact that $\widetilde{v}_1(\cdot, y)$ is continuous from the left in the points (x_i, y), for every fixed $y \in [b, \beta]$, while v_1 is continuous from the right. In a similar fashion $\widetilde{v}_2 : \mathbb{R}^2 \to \mathbb{R}$ is defined analogously to v_2, but in such a way that it is continuous from the left in the points (x, y_j), $j = 0, \ldots, p$, for every fixed $x \in [a, \alpha]$.

Since Proposition 6 can be proved also replacing v_1 and v_2 with $\widetilde{v}_1$ and $\widetilde{v}_2$, respectively, with similar reasonings to Theorem 1 it is possible to prove the following:

THEOREM 2. – *Let $f \in AC^{\varphi}(\mathbb{R}^2)$. Then there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which*

$$V_1^{\varphi}[\lambda(\tau_{\boldsymbol{t}}\widetilde{v}_1 - \widetilde{v}_1), I] < \frac{\varepsilon}{2}, \quad V_2^{\varphi}[\lambda(\tau_{\boldsymbol{t}}\widetilde{v}_2 - \widetilde{v}_2), I] < \frac{\varepsilon}{2},$$

where $\boldsymbol{t} = (t, 0)$ or $\boldsymbol{t} = (0, t)$, if $0 < t < \delta$.

We now extend the previous results to the set $I_{\frac{\delta}{2}} = \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times \left[b - \frac{\delta}{2}, \beta + \frac{\delta}{2}\right]$.

THEOREM 3. – *Let $f \in AC^{\varphi}(\mathbb{R}^2)$. Then there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which*

$$V_1^{\varphi}[\lambda(\tau_{\boldsymbol{t}} v_1 - v_1), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2}, \tag{13}$$

$$V_2^{\varphi}[\lambda(\tau_{\boldsymbol{t}} v_2 - v_2), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2}, \tag{14}$$

where $\boldsymbol{t} = (t, 0)$ or $\boldsymbol{t} = (0, t)$ and $-\delta < t < 0$, with v_1, v_2 defined as in Proposition 6.

PROOF. – We will prove the result in the case $\boldsymbol{t} = (t, 0)$ since, as before, the case $\boldsymbol{t} = (0, t)$ follows with similar reasonings.

Let us fix $\varepsilon > 0$. By Theorem 1 there exists $\hat{\lambda} > 0$ such that, in correspondence to $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which $V_1^{\varphi}[\hat{\lambda}(\tau_{\boldsymbol{t}} v_1 - v_1), I] < \frac{\varepsilon}{2}$ and $V_2^{\varphi}[\hat{\lambda}(\tau_{\boldsymbol{t}} v_2 - v_2), I] < \frac{\varepsilon}{2}$, if $-\delta < t < 0$, where v_1 and v_2 are defined as in Proposition 6, hence associated to a suitable partition $\{I_{ij}\}$ of I. We also recall that, by Proposition 6, $\sum_{i=1}^{m} \sum_{j=1}^{p} V_k^{\varphi}[\hat{\lambda} f, I_{ij}] < \varepsilon$, $k = 1, 2$, and that δ is such that, in particular, $\delta < \min_{i=1,\ldots,m,\, j=1,\ldots,p} \{x_i - x_{i-1}, y_j - y_{j-1}\}$.

Let us prove (14).

We first study the set $\left[a - \frac{\delta}{2}, a\right] \times [b, \beta]$. Let us notice that $v_2(x, y) = v_2(a, y) = f(a, y_{j-1})$, for every $x \in \left[a - \frac{\delta}{2}, a\right]$, $y \in [y_{j-1}, y_j[$, $j = 1, \ldots, p$. Hence, with similar reasonings to Theorem 1 one can prove that

$$
\begin{aligned}
V_2^{\varphi}&\left[\lambda(\tau_{\boldsymbol{t}} v_2 - v_2), \left[a - \frac{\delta}{2}, a\right] \times [b, \beta]\right] = \int_{a-\frac{\delta}{2}}^{a} V_{[b,\beta]}^{\varphi}[\lambda(\tau_{\boldsymbol{t}} v_2 - v_2)(x, \cdot)]\, dx \\
&\leq \frac{1}{4} \int_{a-\frac{\delta}{2}}^{a} \sum_{j=1}^{p} \left(V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda \tau_{\boldsymbol{t}} v_2(x, \cdot)] + V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda v_2(x, \cdot)]\right) dx \\
&= \frac{1}{4} \int_{a-\frac{\delta}{2}-t}^{a-t} \sum_{j=1}^{p} V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda v_2(x, \cdot)]\, dx + \frac{1}{4} \int_{a-\frac{\delta}{2}}^{a} \sum_{j=1}^{p} V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda v_2(x, \cdot)]\, dx \\
&\leq \frac{1}{4} \int_{a-\frac{\delta}{2}}^{a} \sum_{j=1}^{p} V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda f(a, \cdot)]\, dx + \frac{1}{4} \int_{a}^{x_1} \sum_{j=1}^{p} V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda f(x, \cdot)]\, dx \\
&+ \frac{1}{4} \int_{a-\frac{\delta}{2}}^{a} \sum_{j=1}^{p} V_{[y_{j-1}, y_j]}^{\varphi}[4\lambda f(a, \cdot)]\, dx \\
&\leq \frac{1}{4} \sum_{j=1}^{p} V_2^{\varphi}[4\lambda f, I_{1j}] + \frac{\delta}{4} V_{[b,\beta]}^{\varphi}[4\lambda f(a, \cdot)] < \frac{1}{4} \sum_{i=1}^{m} \sum_{j=1}^{p} V_2^{\varphi}[4\lambda f, I_{ij}] + \frac{\varepsilon}{4},
\end{aligned}
$$

recalling that $\delta < \dfrac{\varepsilon}{V_{[b,\beta]}^{\varphi}[\overline{\lambda} f(a, \cdot)]} < \dfrac{\varepsilon}{V_{[b,\beta]}^{\varphi}[4\lambda f(a, \cdot)]}$, for $4\lambda < \hat{\lambda} < \overline{\lambda}$ ($\overline{\lambda}$ is such that $V^{\varphi}[\overline{\lambda} f] < +\infty$), if $V_{[b,\beta]}^{\varphi}[\overline{\lambda} f(a, \cdot)] \neq 0$. Since, by Proposition 6, $\sum_{i=1}^{m} \sum_{j=1}^{p} V_2^{\varphi}[4\lambda f, I_{ij}] < \varepsilon$ if $4\lambda < \hat{\lambda}$, then

$$
V_2^{\varphi}\left[\lambda(\tau_{\boldsymbol{t}} v_2 - v_2), \left[a - \frac{\delta}{2}, a\right] \times [b, \beta]\right] < \frac{\varepsilon}{2}. \tag{15}
$$

About the set $\left[\alpha,\alpha+\frac{\delta}{2}\right]\times\left[b-\frac{\delta}{2},\beta+\frac{\delta}{2}\right]$, it is sufficient to notice that, if $x\in\left[\alpha,\alpha+\frac{\delta}{2}\right]$, then $x-t>\alpha$ and so, for every $y\in\left[b-\frac{\delta}{2},\beta+\frac{\delta}{2}\right]$, $\tau_t v_2(x,y)=v_2(x,y)$, which obviously implies that $V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),\left[\alpha,\alpha+\frac{\delta}{2}\right]\times\left[b-\frac{\delta}{2},\beta+\frac{\delta}{2}\right]\right]=0$.

Let us now study the sets $[a,\alpha]\times\left[b-\frac{\delta}{2},b\right]$ and $[a,\alpha]\times\left[\beta,\beta+\frac{\delta}{2}\right]$. Here $(\tau_t v_2-v_2)$ is defined exactly as in $[a,\alpha]\times[b,y_1[$ and $[a,\alpha]\times\{\beta\}$, respectively. Hence, by Theorem 1,

$$V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),[a,\alpha]\times\left[b-\frac{\delta}{2},b\right]\right]\le V_2^{\varphi}[\lambda(\tau_t v_2-v_2),I]<\frac{\varepsilon}{2}, \tag{16}$$

and

$$V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),[a,\alpha]\times\left[\beta,\beta+\frac{\delta}{2}\right]\right]<\frac{\varepsilon}{2}, \tag{17}$$

if $\lambda<\hat{\lambda}$. Finally, if $(x,y)\in\left[a-\frac{\delta}{2},a\right]\times\left[b-\frac{\delta}{2},b\right]\cup\left[a-\frac{\delta}{2},a\right]\times\left[\beta,\beta+\frac{\delta}{2}\right]$ both $v_2(x,\cdot)$ and $\tau_t v_2(x,\cdot)$ are constant with respect to the second variable, hence obviously

$$\begin{aligned}V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),\left[a-\frac{\delta}{2},a\right]\times\left[b-\frac{\delta}{2},b\right]\right]&=0\\ &=V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),\left[a-\frac{\delta}{2},a\right]\times\left[\beta,\beta+\frac{\delta}{2}\right]\right].\end{aligned} \tag{18}$$

Now, using Proposition 3,

$$\begin{aligned}V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),I_{\frac{\delta}{2}}\right]&=V_2^{\varphi}\left[\lambda(\tau_t v_2-v_2),\left[a-\frac{\delta}{2},\alpha\right]\times\left[b-\frac{\delta}{2},\beta+\frac{\delta}{2}\right]\right]\\ &\le\frac{1}{2}V_2^{\varphi}\left[2\lambda(\tau_t v_2-v_2),\left[a-\frac{\delta}{2},a\right]\times[b,\beta]\right]\\ &+\frac{1}{4}V_2^{\varphi}\left[4\lambda(\tau_t v_2-v_2),[a,\alpha]\times\left[b-\frac{\delta}{2},b\right]\right]+\frac{1}{8}V_2^{\varphi}[8\lambda(\tau_t v_2-v_2),I]\\ &+\frac{1}{8}V_2^{\varphi}\left[8\lambda(\tau_t v_2-v_2),[a,\alpha]\times\left[\beta,\beta+\frac{\delta}{2}\right]\right]<\frac{\varepsilon}{2},\end{aligned}$$

by (15)-(18), if $8\lambda<\hat{\lambda}$.

We now prove (13).

Let us first notice that, if $(x,y) \in \left[a-\frac{\delta}{2}, a\right] \times \left[b-\frac{\delta}{2}, \beta+\frac{\delta}{2}\right]$, $v_1(x,y) = f(a,y) = \tau_t v_1(x,y)$, if $-\delta < t < 0$, hence

$$V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), \left[a-\frac{\delta}{2}, a\right] \times \left[b-\frac{\delta}{2}, \beta+\frac{\delta}{2}\right]\right] = 0.$$

For similar reasons, there also holds

$$V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), \left[\alpha, \alpha+\frac{\delta}{2}\right] \times \left[b-\frac{\delta}{2}, \beta+\frac{\delta}{2}\right]\right] = 0.$$

If $(x,y) \in [a,\alpha] \times \left[b-\frac{\delta}{2}, b\right]$, then $(\tau_t v_1 - v_1)(x,y) = (\tau_t v_1 - v_1)(x,b)$. Hence, as in Theorem 1, and using in particular (8), one can prove that

$$\begin{aligned} V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), [a,\alpha] \times \left[b-\frac{\delta}{2}, b\right]\right] &= \int_{b-\frac{\delta}{2}}^{b} V_{[a,\alpha]}^{\varphi}[\lambda(\tau_t v_1 - v_1)(\cdot, b)]\, dy \\ &= \frac{\delta}{2} V_{[a,\alpha]}^{\varphi}[\lambda(\tau_t v_1 - v_1)(\cdot, b)] \\ &\le \frac{\delta}{4} \sum_{i=1}^{m} V_{[x_{i-1},x_i]}^{\varphi}[4\lambda f(\cdot,b)] \le \frac{\delta}{4} V_{[a,\alpha]}^{\varphi}[4\lambda f(\cdot,b)] < \frac{\varepsilon}{2}, \end{aligned}$$

recalling that (see Proposition 6) $\delta < \dfrac{\varepsilon}{V_{[a,\alpha]}^{\varphi}[\overline{\lambda} f(\cdot,b)]} < \dfrac{2\varepsilon}{V_{[a,\alpha]}^{\varphi}[4\lambda f(\cdot,b)]}$, for $4\lambda < \hat{\lambda} < \overline{\lambda}$, of course if $V_{[a,\alpha]}^{\varphi}[\overline{\lambda} f(\cdot,b)] \neq 0$.

In a similar way it is possible to prove that

$$V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), [a,\alpha] \times \left[\beta, \beta+\frac{\delta}{2}\right]\right] < \frac{\varepsilon}{2}.$$

In conclusion, by Proposition 3 and Theorem 1,

$$\begin{aligned} V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), I_{\frac{\delta}{2}}\right] &= V_1^{\varphi}\left[\lambda(\tau_t v_1 - v_1), [a,\alpha] \times \left[b-\frac{\delta}{2}, \beta+\frac{\delta}{2}\right]\right] \\ &\le \frac{1}{2} V_1^{\varphi}\left[2\lambda(\tau_t v_1 - v_1), [a,\alpha] \times \left[b-\frac{\delta}{2}, b\right]\right] \\ &\quad + \frac{1}{4} V_1^{\varphi}[4\lambda(\tau_t v_1 - v_1), I] + \frac{1}{4} V_1^{\varphi}\left[4\lambda(\tau_t v_1 - v_1), [a,\alpha] \times \left[\beta, \beta+\frac{\delta}{2}\right]\right] < \frac{\varepsilon}{2}, \end{aligned}$$

if $4\lambda < \hat{\lambda}$. □

Using the step functions $\widetilde{v}_1$ and $\widetilde{v}_2$ it is possible to obtain the previous convergence result on $I_{\frac{\delta}{2}}$ also for positive values of t. In particular, with similar reasonings to Theorem 3 it is possible to obtain the following:

THEOREM 4. – *Let* $f \in AC^{\varphi}(\mathbb{R}^2)$. *Then there exists* $\lambda > 0$ *such that, for every* $\varepsilon > 0$, *there exist* $I = [a, \alpha] \times [b, \beta]$ *and* $\delta > 0$ *for which*

$$V_1^{\varphi}[\lambda(\tau_{\boldsymbol{t}}\widetilde{v}_1 - \widetilde{v}_1), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2}, \quad V_2^{\varphi}[\lambda(\tau_{\boldsymbol{t}}\widetilde{v}_2 - \widetilde{v}_2), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2},$$

where $\boldsymbol{t} = (t, 0)$ *or* $\boldsymbol{t} = (0, t)$ *and* $0 < t < \delta$.

5. – The main result.

We are now ready to prove the main result which furnishes a characterization about the φ-modulus of smoothness of f.

THEOREM 5. – *Let* $f \in BV^{\varphi}(\mathbb{R}^2)$. *Then there exists* $\lambda > 0$ *such that*

$$\lim_{\delta \to 0^+} \omega^{\varphi}(\lambda f, \delta) = 0,$$

if and only if $f \in AC^{\varphi}_{loc}(\mathbb{R}^2)$.

PROOF. – Let $f \in BV^{\varphi}(\mathbb{R}^2)$. We first assume that $f \in AC^{\varphi}_{loc}(\mathbb{R}^2)$ and we will prove that

$$\lim_{|\boldsymbol{t}| \to 0} V^{\varphi}[\lambda(\tau_{\boldsymbol{t}} f - f)] = 0, \tag{19}$$

for some $\lambda > 0$, from which the sufficient condition obviously follows.

We first prove (19) in the particular case $\boldsymbol{t} = (t, 0)$, with $t < 0$.

Since $f \in AC^{\varphi}(\mathbb{R}^2)$, by Theorem 3 there is $\overline{\lambda} > 0$ such that, for every fixed $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which

$$V_k^{\varphi}[\overline{\lambda}(\tau_{\boldsymbol{t}} v_k - v_k), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2}, \quad k = 1, 2, \tag{20}$$

and $(a) - (c)$ of Proposition 6 hold. By Proposition 3 and Proposition 1 there holds

$$\begin{aligned}
V^{\varphi}[\lambda(\tau_{\boldsymbol{t}} f - f)] \leq & \frac{1}{2} V^{\varphi}[2\lambda(\tau_{\boldsymbol{t}} f - f), \mathbb{R} \times (-\infty, b]] \\
& + \frac{1}{4} V^{\varphi}[4\lambda(\tau_{\boldsymbol{t}} f - f), \mathbb{R} \times [\beta, +\infty)] \\
& + \frac{1}{8} V^{\varphi}\left[8\lambda(\tau_{\boldsymbol{t}} f - f), \left(-\infty, a - \frac{\delta}{2}\right] \times [b, \beta]\right] \\
& + \frac{1}{16} V^{\varphi}\left[16\lambda(\tau_{\boldsymbol{t}} f - f), \left[\alpha + \frac{\delta}{2}, +\infty\right) \times [b, \beta]\right] \\
& + \frac{1}{16} V^{\varphi}\left[16\lambda(\tau_{\boldsymbol{t}} f - f), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right]
\end{aligned}$$

$$\leq \frac{1}{4} V^{\varphi}[4\lambda\tau_t f, \mathbb{R} \times (-\infty, b]] + \frac{1}{4} V^{\varphi}[4\lambda f, \mathbb{R} \times (-\infty, b]]$$

$$+ \frac{1}{8} V^{\varphi}[8\lambda\tau_t f, \mathbb{R} \times [\beta, +\infty)] + \frac{1}{8} V^{\varphi}[8\lambda f, \mathbb{R} \times [\beta, +\infty)]$$

$$+ \frac{1}{16} V^{\varphi}\left[16\lambda\tau_t f, \left(-\infty, a - \frac{\delta}{2}\right] \times [b, \beta]\right]$$

$$+ \frac{1}{16} V^{\varphi}\left[16\lambda f, \left(-\infty, a - \frac{\delta}{2}\right] \times [b, \beta]\right]$$

$$+ \frac{1}{32} V^{\varphi}\left[32\lambda\tau_t f, \left[\alpha + \frac{\delta}{2}, +\infty\right) \times [b, \beta]\right]$$

$$+ \frac{1}{32} V^{\varphi}\left[32\lambda f, \left[\alpha + \frac{\delta}{2}, +\infty\right) \times [b, \beta]\right]$$

$$+ \frac{1}{16} V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right].$$

Now, if $(t, 0)$ is such that $-\frac{\delta}{2} < t < 0$, then $x - t < a$ if $x < a - \frac{\delta}{2}$ and $x - t > \alpha$ if $x > \alpha + \frac{\delta}{2}$, while $a - \delta < x - t < \alpha + \delta$ if $a - \frac{\delta}{2} < x < \alpha + \frac{\delta}{2}$. Hence

$$V^{\varphi}\left[16\lambda\tau_t f, \left(-\infty, a - \frac{\delta}{2}\right] \times [b, \beta]\right] \leq V^{\varphi}[16\lambda f, (-\infty, a] \times [b, \beta]], \tag{21}$$

$$V^{\varphi}\left[32\lambda\tau_t f, \left[\alpha + \frac{\delta}{2}, +\infty\right) \times [b, \beta]\right] \leq V^{\varphi}[32\lambda f, [\alpha, +\infty) \times [b, \beta]], \tag{22}$$

and so

$$V^{\varphi}[\lambda(\tau_t f - f)] \leq \frac{1}{2} V^{\varphi}[4\lambda f, \mathbb{R} \times (-\infty, b]] + \frac{1}{4} V^{\varphi}[8\lambda f, \mathbb{R} \times [\beta, +\infty)]$$

$$+ \frac{1}{8} V^{\varphi}[16\lambda f, (-\infty, a] \times [b, \beta]] + \frac{1}{16} V^{\varphi}[32\lambda f, [\alpha, +\infty) \times [b, \beta]]$$

$$+ \frac{1}{16} V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right]$$

$$< \frac{15}{16}\varepsilon + \frac{1}{16} V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right],$$

if $32\lambda < \overline{\lambda}$, by (a) of Proposition 6.

About $V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right]$, obviously there holds

$$V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right] \le V_1^{\varphi}\left[16\lambda(\tau_t f - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]$$

$$+ V_2^{\varphi}\left[16\lambda(\tau_t f - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]$$

$$=: J_1 + J_2.$$

Concerning J_1, using Proposition 1 we have

$$J_1 \le \frac{1}{3}\left\{V_1^{\varphi}\left[48\lambda(\tau_t f - \tau_t v_1), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right.$$

$$+ V_1^{\varphi}\left[48\lambda(\tau_t v_1 - v_1), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]$$

$$\left.+ V_1^{\varphi}\left[48\lambda(v_1 - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right\}$$

$$\le \frac{1}{3}\{2V_1^{\varphi}[48\lambda(v_1 - f), I_\delta] + V_1^{\varphi}[48\lambda(\tau_t v_1 - v_1), I_{\frac{\delta}{2}}]\} < \frac{\varepsilon}{2},$$

by (c) of Proposition 6 and by (20), if $\lambda > 0$ is such that $48\lambda < \overline{\lambda}$ and $-\frac{\delta}{2} < t < 0$.

For the same reasons,

$$J_2 \le \frac{1}{3}\left\{V_2^{\varphi}\left[48\lambda(\tau_t f - \tau_t v_2), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right.$$

$$+ V_2^{\varphi}\left[48\lambda(\tau_t v_2 - v_2), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right].$$

$$\left.+ V_2^{\varphi}\left[48\lambda(v_2 - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right\}$$

$$\le \frac{1}{3}\{2V_2^{\varphi}[48\lambda(v_2 - f), I_\delta] + V_2^{\varphi}[48\lambda(\tau_t v_2 - v_2), I_{\frac{\delta}{2}}]\} < \frac{\varepsilon}{2}.$$

Hence we conclude that

$$V^{\varphi}\left[16\lambda(\tau_t f - f), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right] < \varepsilon$$

and so

$$V^{\varphi}[\lambda(\tau_t f - f)] < \varepsilon.$$

This concludes the proof of (19) in the case $\boldsymbol{t} = (t, 0)$. The case $\boldsymbol{t} = (0, t)$, $t < 0$, is similar.

If $\boldsymbol{t} = (t, 0)$ with $t > 0$ (analogously if $\boldsymbol{t} = (0, t)$, $t > 0$) we can repeat all the proof with $\widetilde{v}_1$ and $\widetilde{v}_2$ instead of v_1 and v_2 and using Theorems 2 and 4 instead of Theorems 1 and 3, respectively.

Finally, in order to treat the general case $\boldsymbol{t} = (t_1, t_2)$, it is sufficient to notice that, by Proposition 1,

$$V^\varphi[\lambda(\tau_{\boldsymbol{t}} f - f)] \leq \frac{1}{2}\{V^\varphi[2\lambda(\tau_{(t_1,t_2)} f - \tau_{(0,t_2)} f)] + V^\varphi[2\lambda(\tau_{(0,t_2)} f - f)]\}$$

where in the two terms of the right-hand side we have two translations of the kind $\tau_{(t,0)} f$ and $\tau_{(0,t)} f$, respectively, which can be separately treated as indicated above, just replacing $\frac{\delta}{2}$ with $\frac{\delta}{4}$.

Let us now prove the necessary condition. Let $f \in BV^\varphi(\mathbb{R}^2)$ and assume that there exists $\lambda > 0$ such that $\lim_{\delta \to 0^+} \omega^\varphi(\lambda f, \delta) = 0$. Let us consider a family of approximating integral operators of the form

$$(T_w f)(\boldsymbol{s}) := \int_{\mathbb{R}^2} \rho_w(\boldsymbol{t}) f(\boldsymbol{s} - \boldsymbol{t}) d\boldsymbol{t}, \quad w > 0, \ \boldsymbol{s} \in \mathbb{R}^2,$$

where $\{\rho_w\}_{w>0}$ is a net of mollifiers. Then obviously $\{\rho_w\}_{w>0}$ satisfy all the assumptions of Theorem 3.3 of [3], and so there exists $\mu > 0$ such that $\lim_{w \to +\infty} V^\varphi[\mu(T_w f - f)] \to 0$, as $w \to +\infty$, namely the operators $\{T_w f\}_{w>0}$ converge in φ-variation to f. Now, $T_w f \in AC^\varphi(\mathbb{R}^2)$, by Proposition 4.2 of [3], and so, being $AC^\varphi(\mathbb{R}^2)$ a closed subspace of $BV^\varphi(\mathbb{R}^2)$ with respect to the φ-variation functional (Theorem 4.3 of [3]), $f \in AC^\varphi(\mathbb{R}^2)$, which concludes the proof of the theorem. □

REMARK 3. – As pointed out in the Introduction, Theorem 5 holds also in the case of φ-variation defined through general partitions. We recall that it is possible to define the variation (and hence also the φ-variation) using “pythagorean” or “extended” partitions (see e.g. [11, 33]). In particular, if $I = [a, \alpha] \times [b, \beta] \subset \mathbb{R}^2$, a partition $\{J_1, \ldots, J_m\}$ of I is said to be “pythagorean” if the subsets J_i $(i = 1, \ldots, m)$ are obtained as cartesian product of two partitions of $[a, \alpha]$ and $[b, \beta]$, while it is “extended” if the subsets J_i are rectangular, but without the previous constraint. If we denote by $V^\varphi_{k,p}[f, I]$ and $V^\varphi_{k,e}[f, I]$, $k = 1, 2$, the separated φ-variations of f defined through pythagorean or extended partitions, respectively, then obviously

$$V^\varphi_{k,p}[f, I] \leq V^\varphi_{k,e}[f, I], \quad k = 1, 2.$$

On the other side, using simple geometric considerations, it is not difficult to prove that also the converse inequality holds, hence

$$V^{\varphi}_{k,p}[f,I] = V^{\varphi}_{k,e}[f,I], \quad k = 1,2.$$

Now, Propositions 1 and 3 and (a) of Proposition 6 obviously hold also for the φ-variation $V^{\varphi}_{e}[f]$ defined through extended partitions, hence the proof of Theorem 5 can be repeated, taking into account that $J_k := V^{\varphi}_{k,e}[f,I] = V^{\varphi}_{k,p}[f,I]$, $k = 1,2$, and using all the previous results given in the case of pythagorean partitions.

REMARK 4. – All the previous results can be proved in the general frame of $\mathbb{R}^N$. In particular, in the more general case it is necessary to define N auxiliary step functions $\nu_1, \ldots, \nu_N$, similarly to ν_1 and ν_2. For example, if $I = \prod_{i=1}^{N} [a_i, b_i]$ and $\{^{(0)}x_i, \ldots, ^{(n_i)}x_i\}$ is a partition of $[a_i, b_i]$, for every $i = 1, \ldots, N$, then ν_i is defined in such a way that it is equal to f on the segments $(^{(k)}x_i, \cdot)$, $k = 1, \ldots, n_i$, and constant on the $(N-1)-$dimensional intervals $]^{(k-1)}x'_i, ^{(k)}x'_i[$.

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