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A Time Regularity Result for Forward-Backward Parabolic Equations

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Abstract. – *We give a time regularity result for an abstract mixed type equation whose toy model may be* $(ru)_t - \Delta u = f$ *where* r *is a coefficient whose sign may be positive, null and negative.*

1. – Introduction.

In the paper [2] an existence result for a mixed type evolution equation is given. This result is slightly generalized in [1] to Banach spaces depending on a parameter.

Consider the following family of evolution triplets

$$V(t) \subset H(t) \subset V(t)' \qquad t \in [0, T]$$

where $H(t)$ is a Hilbert space, $V(t)$ a reflexive Banach space which continuously and densely embeds in $H(t)$ and $V(t)'$ the dual space of $V(t)$.

Moreover we will suppose the existence of a set U such that

$$U \subset V(t) \quad \text{dense in } V(t) \text{ for a.e. } t \in [0, T] \tag{1}$$

We define $\mathcal{U}$ the set of polynomials $v(t) = \sum_{k=0}^{N} u_k t^k$ with $u_k \in U$, $N \in \boldsymbol{N}$ and suppose that the functions

$$t \mapsto \|v(t)\|_{V(t)}, \quad t \mapsto \|v(t)\|_{H(t)}, \quad t \mapsto \|v(t)\|_{V(t)'}, \quad t \in [0, T],$$

are measurable for every $v \in \mathcal{U}$. We denote respectively by $\mathcal{V}$ and $\mathcal{H}$ the spaces defines as the closure of $\mathcal{U}$ with respect to the following norms

$$\|v\|_{\mathcal{V}}^2 := \int_0^T \|v(t)\|_{V(t)}^2 dt \qquad \text{and} \qquad \|v\|_{\mathcal{H}}^2 := \int_0^T \|v(t)\|_{H(t)}^2 dt$$

and by $\mathcal{V}'$ the dual space of $\mathcal{V}$.

DEFINITION 1.1. – *Consider* $S : [0, T] \longrightarrow \mathcal{L}(H(t))$, *being* $\mathcal{L}(H(t))$ *the set of linear and bounded operators from* $H(t)$ *in itself. We say that* S *belongs to the* *class* $\mathcal{E}(C_1, C_2)$*,* $C_1, C_2 > 0$*, if it satisfies what follows for every* $u, v \in U$*:*

- $S(t)$ *is self-adjoint and* $\|S(t)\|_{\mathcal{L}(H(t))} \leqslant C_1$ *for every* $t \in [0, T]$,
- $t \mapsto \big(S(t)u, v\big)_{H(t)}$ *is absolutely continuos on* $[0, T]$,
- $\left| \dfrac{d}{dt} \big(S(t)u, v\big)_{H(t)} \right| \leqslant C_2 \, \|u\|_{V(t)} \|v\|_{V(t)}$ *for a.e.* $t \in [0, T]$.

Consider $R \in \mathcal{E}(C_1, C_2)$; we can define an operator $\mathcal{R}$ by

$$\mathcal{R} : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{by} \qquad \mathcal{R}u(t) := R(t)u(t) \tag{2}$$

which turns out to be linear and bounded by the constant C_1.

Now define the Banach space

$$\mathcal{W}_{\mathcal{R}} = \big\{u \in \mathcal{V} \mid (\mathcal{R}u)' \in \mathcal{V}'\big\}, \qquad \|u\|_{\mathcal{W}_{\mathcal{R}}} := \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} . \tag{3}$$

where $(\mathcal{R}u)'$ denotes the derivative of $\mathcal{R}u$ with respect to the variable t. One can prove that if $u \in \mathcal{W}_{\mathcal{R}}$ the function

$$t \mapsto (R(t)u(t), u(t))_{H(t)}$$

turns out to be continuous in $[0, T]$. If we consider, for each t, $H_+(t)$ and $H_-(t)$ respectively the positive and negative part of the spectrum of $R(t)$, by $R_+(t)$ we denote the restriction of $R(t)$ to $H_+(t)$, by $R_-(t)$ we denote the restriction of $-R(t)$ to $H_-(t)$, $\tilde{H}_+(t)$ (respect. $\tilde{H}_-(t)$) the completion of $H_+(t)$ (of $H_-(t)$) with respect to the norm $\|R_+(t)^{1/2} w\|_{H(t)}$ (to the norm $\|R_-(t)^{1/2} w\|_{H(t)}$), we define by $P_+(t)$ and $P_-(t)$ the orthogonal projections defined in $\tilde{H}(t)$ and valued respectively in $H_+(t)$ and $H_-(t)$, where $\tilde{H}(t) := \tilde{H}_+(t) \oplus \operatorname{Ker} R(t) \oplus \tilde{H}_-(t)$. We can also define a family of equibounded operators

$$R' : [0, T] \to \mathcal{L}(V(t), V(t)') \quad \text{by} \quad \langle R'(t)u, v\rangle_{V(t)' \times V(t)} := \frac{d}{dt} \big(R(t)u, v\big)_{H(t)}$$

and, by the density of $\mathcal{U}$ in $\mathcal{V}$, an operator

$$\mathcal{R}' : \mathcal{V} \to \mathcal{V}' \quad \text{by} \quad \big\langle \mathcal{R}'u, v\big\rangle_{\mathcal{V}' \times \mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t)\rangle_{V(t)' \times V(t)} dt$$

which turns out to be linear and bounded by C_2. Consider a family of operators

$$A(t) : V(t) \to V(t)', \ t \in [0, T], \qquad t \mapsto \langle A(t)u, v\rangle_{V(t)' \times V(t)}$$

measurable and define an operator $\mathcal{A}$ as follows

$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}', \qquad \mathcal{A}u(t) = A(t)u(t) \qquad 0 \leqslant t \leqslant T .$$

Suppose the operators $\mathcal{R}'$ and $\mathcal{A}$ are linear and bounded and moreover

$$\left\langle \mathcal{A}v + \frac{1}{2}\mathcal{R}'v, v \right\rangle_{\mathcal{V}' \times \mathcal{V}} \geqslant a\|v\|^2_{\mathcal{V}}, \tag{4}$$

for some positive constant a and every $v \in \mathcal{V}$. Now suppose you are given $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$. Consider the problem

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi \end{cases} \tag{5}$$

Then the following existence result holds.

THEOREM 1.2. – *Given $R \in \mathcal{E}$ and $\mathcal{A}$ such that* (4) *holds, problem* (5) *admits a unique solution in $\mathcal{W}_\mathcal{R}$.*

In [2] a time regularity result is given, but only for non-negative $\mathcal{R}$. In this note we extend it to a more general setting and to $\mathcal{R}$ with variable sign.

2. – The result.

Given A as above suppose that $[0,T] \ni t \mapsto \langle A(t)u, v\rangle_{V'(t),V(t)}$ is absolutely continuous and

$$\left| \frac{d}{dt} \langle A(t)u, v\rangle_{V'(t),V(t)} \right| \leqslant C_3 \|u\|_{V(t)} \|v\|_{V(t)} \tag{6}$$

for every $u, v \in U$ and for a.e. $t \in [0,T]$. In this way one can define $A' : [0,T] \longrightarrow \mathcal{L}(V(t), V(t))$ and $\mathcal{A}' : \mathcal{V} \to \mathcal{V}$.

THEOREM 2.1. – *Denote by u the solution of* (5)*. Assume $R, R' \in \mathcal{E}(C_1, C_2)$ and* (6) *holds. Instead of* (4) *assume*

$$\langle [R'(t) + A(t)]w, w\rangle_{V(t)',V(t)} \geqslant a\|w\|^2_{V(t)}$$

for every $t \in [0,T]$ and $w \in U$. Assume $f' \in \mathcal{V}'$ and the existence of $u_0 \in V(0)$, $u_T \in V(T)$ such that $P_+(0)u_0 = \varphi$ in such a way that $f(0) - A(0)u_0 - R'(0)u_0 \in$ Im $R(0)$ *and $f(T) - A(T)u_T - R'(T)u_T \in$* Im $R(T)$*. Then $u' \in \mathcal{V}$.*

COROLLARY 2.2. – *If $u, u' \in \mathcal{V}$ the function $[0,T] \ni t \mapsto \|u(t)\|_{V(t)}$ is continuous.*

REMARK 2.3. – If $V(t) \equiv V$ for every t theorem and corollary above reduce to say that $u \in H^1(0,T;V) \subset C^0([0,T];V)$.

REMARK 2.4. – If R is defined by a function r, i.e. $\mathcal{R}u = r(x,t)u(x,t)$, to require that $R, R' \in \mathcal{E}$ implies that r admits a weak derivative with respect to time, so

$(R'(t)u(t))(x) = \partial_t r(x,t)u(x,t)$. In particular assumptions of Theorem 2.1 are satisfied if $A(t)$ is strictly monotone and $\partial_t r \geqslant 0$.

PROOF. – Consider u to be the solution of the problem

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi \end{cases}$$

Deriving the equation one gets

$$(\mathcal{R}'u)' + (\mathcal{R}u')' + \mathcal{A}'u + \mathcal{A}u' = f'$$

and since $\mathcal{R}u'(t) = f(t) - \mathcal{A}u(t) - \mathcal{R}'u(t)$ we consider the problem

$$(7) \qquad \begin{cases} (\mathcal{R}v)' + \mathcal{A}v = f' - \mathcal{A}'u - (\mathcal{R}'u)' \\ P_+(0)v(0) = R_+(0)^{-1}\big[P_+(0)\big(f(0) - A(0)u_0 - R'(0)u_0\big)\big] \\ P_-(T)v(T) = R_-(T)^{-1}\big[P_-(T)\big(f(T) - A(T)u_T - R'(T)u_T\big)\big] \end{cases}$$

with

$$(8) \qquad \begin{aligned} & u_0 \in V(0),\ u_T \in V(T), \qquad P_+(0)u_0 = \varphi\,,\ P_-(T)u_T = \psi\,, \\ & P_+(0)[f(0) - A(0)u_0 - R'(0)u_0] \in \operatorname{Im} R_+(0)\,, \\ & P_-(T)[f(T) - A(T)u_T - R'(T)u_T] \in \operatorname{Im} R_-(T)\,. \end{aligned}$$

Denote by v the solution of (12) and consider

$$w(u_0;t) = w(t) = u_0 + \int_0^t v(s)\,ds\,.$$

Integrating the equation above one obtains

$$\mathcal{R}v(t) = f(t) - f(0) - \int_0^t \mathcal{A}'u(s)ds - \int_0^t (\mathcal{R}'u)'(s)\,ds + \mathcal{R}v(0) - \int_0^t \mathcal{A}v(s)ds$$

Using $\mathcal{R}v(t) = (\mathcal{R}w)'(t) - (\mathcal{R}'w)(t)$ we first get

$$\begin{aligned} (\mathcal{R}w)'(t) - (\mathcal{R}'w)(t) = {} & f(t) - f(0) + \\ & - \int_0^t \mathcal{A}'u(s)ds - \int_0^t (\mathcal{R}'u)'(s)\,ds + (\mathcal{R}w)'(0) - (\mathcal{R}'w)(0) - \int_0^t \mathcal{A}v(s)ds\,; \end{aligned}$$

then using

$$\int_0^t \mathcal{A}v(s)ds = \int_0^t \mathcal{A}w'(s)ds = \mathcal{A}w(t) - \mathcal{A}w(0) - \int_0^t \mathcal{A}'w(s)ds$$

we get

$$\begin{aligned}
(\mathcal{R}(w-u))'(t) + \mathcal{A}(w-u)(t) &= (\mathcal{R}w)'(t) + \mathcal{A}w(t) - f(t) = \\
&= (\mathcal{R}'w)(t) + f(t) - f(0) - \int_0^t \mathcal{A}'u(s)ds - \int_0^t (\mathcal{R}'u)'(s)\,ds + (\mathcal{R}w)'(0) \\
&\qquad - (\mathcal{R}'w)(0) - \mathcal{A}w(t) + \mathcal{A}w(0) + \int_0^t \mathcal{A}'w(s)ds + \mathcal{A}w(t) - f(t) \\
&= \int_0^t (\mathcal{R}'w)'(s)ds - f(0) - \int_0^t \mathcal{A}'u(s)ds - \int_0^t (\mathcal{R}'u)'(s)\,ds + (\mathcal{R}w)'(0) \\
&\qquad + \mathcal{A}w(0) + \int_0^t \mathcal{A}'w(s)ds \\
&= \int_0^t [(\mathcal{R}'w)'(s) - (\mathcal{R}'u)'(s)]ds + \int_0^t [\mathcal{A}'w(s) - \mathcal{A}'u(s)]ds \\
&\qquad + R(0)v(0) + R'(0)u_0 - f(0) + A(0)u_0\,.
\end{aligned}$$

The quantity $(\mathcal{R}v)(0) - f(0) + \mathcal{R}'u(0) + \mathcal{A}w(0)$ is nothing else but $R(0)v(0) + R'(0)u_0 - f(0) + A(0)u_0$. We can choose u_0 in such a way that

$$R(0)v(0) + R'(0)u_0 - f(0) + A(0)u_0 = 0\,.$$

We can do that modifying u_0 if necessary taking $u_0 \in V(0)$ the be the solution of the following problem (we recall that $A(0) + R'(0)$ is corcive and bounded in $V(0)$)

$$A(0)z + R'(0)z = f(0) - R(0)v(0)\,, \qquad z \in V\,.$$

Consequently, in particular, $P_+(0)[f(0) - A(0)u_0 - R'(0)u_0] = R_+(0)v(0)$ and the initial condition in (12) still holds.

In this way we obtain that the function $w - u$ solves the problem

$$\begin{cases} (\mathcal{R}y)' + \mathcal{A}y = h \\ P_+(0)y(0) = 0 \\ P_-(0)y(T) = P_-(T)[u_0 + \int_0^T v(s)ds - u_T] \end{cases} \tag{9}$$

where

$$h = \int_0^t [(\mathcal{R}'w)'(s) - (\mathcal{R}'u)'(s)]ds + \int_0^t [\mathcal{A}'w(s) - \mathcal{A}'u(s)]ds \,. \tag{10}$$

Now consider the function

$$\tilde{w}(t) = \tilde{w}(u_T;t) := u_T + \int_T^t v(s)\,ds = \left[u_T - \int_0^T v(s)ds\right] + \int_0^t v(s)ds\,,$$

Since $\mathcal{R}v(t) = (\mathcal{R}\tilde{w})'(t) - (\mathcal{R}'\tilde{w})(t)$ we get, as done for w, but integrating between T and $t \in (0,T)$,

$$\begin{aligned}(\mathcal{R}\tilde{w})'(t) - (\mathcal{R}'\tilde{w})(t) = &f(t) - f(T) + \\ &- \int_T^t \mathcal{A}'u(s)ds - \int_T^t (\mathcal{R}'u)'(s)ds - \int_T^t \mathcal{A}v(s)ds + (\mathcal{R}\tilde{w})'(T) - (\mathcal{R}'\tilde{w})(T)\end{aligned}$$

then using

$$\int_T^t \mathcal{A}v(s)ds = \int_T^t \mathcal{A}\tilde{w}'(s)ds = \mathcal{A}\tilde{w}(t) - \mathcal{A}\tilde{w}(T) - \int_T^t \mathcal{A}'\tilde{w}(s)ds$$

we get similarly as before for w

$$\begin{aligned}(\mathcal{R}\tilde{w})'(t) + \mathcal{A}\tilde{w}(t) &- (\mathcal{R}u)'(t) - \mathcal{A}u(t) \\ &= \int_T^t [(\mathcal{R}'\tilde{w})'(s) - (\mathcal{R}'u)'(s)]ds + \int_T^t [\mathcal{A}'\tilde{w}(s) - \mathcal{A}'u(s)]ds \\ &\qquad + R(T)v(T) - f(T) + R'(T)u_T + A(T)u_T\end{aligned}$$

As before, we can choose u_T in such a way $R(T)v(T) - f(T) + R'(T)u_T + A(T)u_T = 0$. Therefore the function $\tilde{w} - u$ solves the problem

$$\begin{cases}(\mathcal{R}y)' + \mathcal{A}y = \tilde{h} \\ P_+(0)y(0) = P_-(0)\left[u_T + \int_T^0 v(s)ds - u_0\right] \\ P_-(0)y(T) = 0\end{cases}$$

where

$$\tilde{h}(t) = \int_T^t [(\mathcal{R}'\tilde{w})'(s) - (\mathcal{R}'u)'(s)]ds + \int_T^t [\mathcal{A}'\tilde{w}(s) - \mathcal{A}'u(s)]ds \,.$$

Now define the function $z := w - \tilde{w}$ and notice that

$$w(u_0, t) - \tilde{w}(u_T, t) = u_0 - u_T + \int_0^T v(s)ds$$

is independent of t and in particular z is the solution of the evolutionary (but *elliptic*) problem

$$(\mathcal{R}z)' + \mathcal{A}z = \mathcal{R}'z + \mathcal{A}z = g := h - \tilde{h} \tag{11}$$

where

$$\begin{aligned} g(t) = & \int_0^t [(\mathcal{R}'w)'(s) - (\mathcal{R}'u)'(s)]ds + \int_0^t [\mathcal{A}'w(s) - \mathcal{A}'u(s)]ds \\ & + \int_t^T [(\mathcal{R}'\tilde{w})'(s) - (\mathcal{R}'u)'(s)]ds + \int_t^T [\mathcal{A}'\tilde{w}(s) - \mathcal{A}'u(s)]ds \\ = & \int_0^t [(\mathcal{R}'w)'(s) - (\mathcal{R}'\tilde{w})'(s)]ds + \int_0^T [(\mathcal{R}'\tilde{w})'(s) - (\mathcal{R}'u)'(s)]ds \\ & + \int_0^t [\mathcal{A}'w(s) - \mathcal{A}'\tilde{w}(s)]ds + \int_0^T [\mathcal{A}'\tilde{w}(s) - \mathcal{A}'u(s)]ds. \end{aligned}$$

We denote by y the term, independent of t, $\int_0^T [(\mathcal{R}'\tilde{w})(s) - (\mathcal{R}'u)(s)]ds + \int_0^T [\mathcal{A}'\tilde{w}(s) - \mathcal{A}'u(s)]ds$. Estimating g, since $R' \in \mathcal{E}(C_1, C_2)$ and $\|A'(t)\| \leqslant C_3$, one has

$$\begin{aligned} \|g(t)\|_{V(t)'} \leqslant & \int_0^T \|(\mathcal{R}'w)'(s) - (\mathcal{R}'\tilde{w})'(s)\|_{V(s)} ds + \int_0^T C_3 \|w(s) - \tilde{w}(s)\|_{V(s)} ds \\ & + \int_0^T \|(\mathcal{R}'\tilde{w})'(s) - (\mathcal{R}'u)'(s)\|_{V(s)} ds + \int_0^T C_3 \|\tilde{w}(s) - u(s)\|_{V(s)} ds \\ \leqslant & \max\{1, C_3\}\, T^{1/2} \left[\|\tilde{w} - w\|_{\mathcal{W}_{\mathcal{R}'}} + \|\tilde{w} - u\|_{\mathcal{W}_{\mathcal{R}'}} \right]. \end{aligned}$$

By the continuity of g we get

$$\max_{t \in [0,T]} \|g(t)\|_{V(t)'} \leqslant c\, T^{1/2}$$

which gives $g(0) = 0$. By the regularity of the coefficients equation (11) is in fact

$$R'(t)z + A(t)z = g(t) \quad \text{in } V(t) \qquad \text{for } \textit{every } t \in [0, T].$$

Thus we get that the solution of the equation with $t = 0$ is only $z = 0$. This implies that $u_T - u_0 - \int_0^T v(s)ds = 0$ and then, in particular, $w - u$ solves the problem (9) with $P_-(0)y(T) = 0$. Now proceeding as in the proof of Theorem 3.11 in [2] we come to

$$\|w - u\|_{\mathcal{V}} \leqslant c\,T^{1/2}\|w - u\|_{\mathcal{V}}$$

and again, since this holds independently of T, ones gets that $w - u = 0$ and then the thesis. □

COROLLARY 2.5. – *Under the same hypotheses as in Theorem* 2.1, *suppose for simplicity that* $f(0) - A(0)\varphi - R'(0)\varphi \in \operatorname{Im} R(0)$ *and* $f(T) - A(T)u_T - R'(T)\psi \in \operatorname{Im} R(T)$, *the solution* u *of* (5) *satisfies*

$$(\mathcal{R}u')' \in \mathcal{V}'$$

and moreover there exists a positive constant c *depending* (*only*) *on* a^{-1}, $\|\mathcal{A}\|$, $\|\mathcal{R}'\|$, $\|\mathcal{A}'\|$, *such that*

$$\begin{aligned}
&\|u\|_{\mathcal{W}_\mathcal{R}} + \|u'\|_{\mathcal{W}_\mathcal{R}} + \max_{t\in[0,T]}\|u(t)\|_{V(t)} \\
&\quad \leqslant c\,\Big[\|f\|_{\mathcal{V}'} + \|R_+^{1/2}(0)(u(0) - v(0))\|_{H_+(0)} + \|R_-^{1/2}(T)(u(T) - v(T))\|_{H_-(T)}\Big] \\
&\qquad + \|f'\|_{\mathcal{V}'} + \|R_+^{-1/2}(0)\big[P_+(0)\big(f(0) - A(0)\varphi - R'(0)\varphi\big)\big]\|_{H_+(0)} \\
&\qquad + \|R_-^{-1/2}(T)\big[P_-(T)\big(f(T) - A(T)\psi - R'(T)\psi\big)\big]\|_{H_-(T)}\Big]
\end{aligned}$$

PROOF. – By Theorem 3.5 in [1] we have that $u, v \in \mathcal{W}_\mathcal{R}$ satisfy

$$\begin{aligned}
\|u - v\|_{\mathcal{W}_\mathcal{R}} \leqslant \|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{V}'} + c\,\Big[\|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{V}'} \\
+ \|R_+^{1/2}(0)(u(0) - v(0))\|_{H_+(0)} + \|R_-^{1/2}(T)(u(T) - v(T))\|_{H_-(T)}^{2/p}\Big]
\end{aligned}$$

where c depends only on a^{-1} and $\|\mathcal{A}\|$. If u denotes the solution of (5), u solves also

$$\text{(12)}\qquad \begin{cases}
(\mathcal{R}v)' + \mathcal{A}v = f' - \mathcal{A}'u - (\mathcal{R}'u)' \\
P_+(0)v(0) = R_+(0)^{-1}\big[P_+(0)\big(f(0) - A(0)u_0 - R'(0)u_0\big)\big] \\
P_-(T)v(T) = R_-(T)^{-1}\big[P_-(T)\big(f(T) - A(T)u_T - R'(T)u_T\big)\big]
\end{cases}$$

and consequently in particular we get

$$\begin{aligned}
\|u'\|_{\mathcal{W}_\mathcal{R}} \leqslant \|f' - \mathcal{A}'u - (\mathcal{R}'u)'\|_{\mathcal{V}'} + c\,\Big[\|f' - \mathcal{A}'u - (\mathcal{R}'u)'\|_{\mathcal{V}'} \\
+ \|R_+^{-1/2}(0)\big[P_+(0)\big(f(0) - A(0)\varphi - R'(0)\varphi\big)\big]\|_{H_+(0)} \\
+ \|R_-^{-1/2}(T)\big[P_-(T)\big(f(T) - A(T)\psi - R'(T)\psi\big)\big]\|_{H_-(T)}\Big]
\end{aligned}$$

where c depends only on a^{-1} and $\|\mathcal{A}\|$, i.e., calling c' the constant $c+1$,

$$\|u'\|_{\mathcal{W}_\mathcal{R}} \leqslant c' \Big[\|f' - \mathcal{P}'u\|_{\mathcal{V}'} + \|R_+^{-1/2}(0)\big[P_+(0)\big(f(0) - A(0)\varphi - R'(0)\varphi\big)\big]\|_{H_+(0)} + \|R_-^{-1/2}(T)\big[P_-(T)\big(f(T) - A(T)\psi - R'(T)\psi\big)\big]\|_{H_-(T)} \Big]$$

where with $\mathcal{P}'$ we denote for simplicity the operator $\mathcal{P}'u := (\mathcal{R}'u)' + \mathcal{A}'u$. By the estimate above we finally get

$$\begin{aligned} \|u\|_{\mathcal{W}_\mathcal{R}} + \|u'\|_{\mathcal{W}_\mathcal{R}} + \max_{t\in[0,T]} \|u(t)\|_{V(t)} \\ \leqslant c'' \Big[\|f\|_{\mathcal{V}'} + \|R_+^{1/2}(0)(u(0) - v(0))\|_{H_+(0)} + \|R_-^{1/2}(T)(u(T) - v(T))\|_{H_-(T)} \\ + \|f'\|_{\mathcal{V}'} + \big\|R_+^{-1/2}(0)\big[P_+(0)\big(f(0) - A(0)\varphi - R'(0)\varphi\big)\big]\big\|_{H_+(0)} \\ + \big\|R_-^{-1/2}(T)\big[P_-(T)\big(f(T) - A(T)\psi - R'(T)\psi\big)\big]\big\|_{H_-(T)} \Big] \end{aligned}$$

where c'' depends only on a^{-1}, $\|\mathcal{A}\|$, $\|\mathcal{P}'\|$. □

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