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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.1, p. 47–68.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_1_47_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 47-68

On the Computation of the Spectrum of the Linearized Boltzmann Collision Operator for Maxwellian Molecules

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Dedicated to the memory of Carlo Cercignani

Abstract. – *In this article we provide a complete and self-contained treatment of the spectrum of the linearized Boltzmann collision operator for Maxwellian molecules.*

1. – Introduction and motivations.

This paper deals with some mathematical aspects concerning the so-called *linearized Boltzmann collision operator for Maxwellian molecules*, namely

$$L_b[h](\boldsymbol{v}) := \int_{\mathbb{R}^3}\int_{S^2} M(\boldsymbol{w})\ [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})] \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{w}\,. \tag{1}$$

The original idea to derive this linear operator from the *nonlinear Boltzmann collision operator* goes back to Hilbert [9], who laid the foundations of the theory of the linearized Boltzmann equation. Successively, Grad [8] elaborated and adapted these pioneering insights to the specific case of the *spatially homogeneous Boltzmann equation for Maxwellian molecules*, with the aim at investigating the asymptotic behavior of its solution. See [4, 15] for basic and complete information about the Boltzmann equation. Actually, Grad succeeded in establishing a remarkable connection between certain spectral properties of L_b and the rapidity of convergence to equilibrium of the solution of the original nonlinear Boltzmann equation. Even though the Grad estimation is valid only in a small neighborhood of the equilibrium itself – which is far from being a satisfactory conclusion – it is worth stressing that the importance of that estimation lies in its quantitative nature, which can be made explicit just in the case of Maxwellian molecules. Indeed, as shown in [1, 3, 16], the spectrum of L_b *can be given in closed form*, after specifying a suitable domain for this operator, and the dependence of the eigenvalues on the *angular collision kernel* b is made explicit.

It is just this feature – which is unexpected in the case of other interaction potentials – to make Maxwellian molecules of paramount importance, at least from a mathematical point of view. Recently, Mouhot [10, 11] has established an analog of the Grad estimation in the case of the so-called *hard potential* by introducing new strategies for the linearization of the Boltzmann collision operator. Anyway, in the hard potential case, no explicit formula has been given until now for the eigenvalues of the linearized operator in terms of the angular collision kernel.

This paper aims at reviewing some specific spectral properties of the operator L_b, in the case of Maxwellian molecules, when its domain is the Hilbert space

$$\mathcal{H} := \left\{ z : \mathbb{R}^3 \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^3} M(\boldsymbol{v}) z^2(\boldsymbol{v}) \mathrm{d}\boldsymbol{v} < +\infty \right\}$$

endowed with the scalar product $(z_1, z_2)_* := \int_{\mathbb{R}^3} M(\boldsymbol{v}) z_1(\boldsymbol{v}) z_2(\boldsymbol{v}) \mathrm{d}\boldsymbol{v}$. Here, M stands for the *standard Maxwellian probability density function*, i.e.

$$M(\boldsymbol{v}) := \left(\frac{1}{2\pi}\right)^{3/2} e^{-|\boldsymbol{v}|^2/2} .$$

The choice of $\mathcal{H}$ agrees with the original Hilbert's investigations and is suitable for the study of L_b, which turns out to be a *self-adjoint* and *negative* operator on this domain. Moreover, $\mathcal{H}$ admits an explicit Fourier basis made of eigenfunctions of L_b, introduced for the first time in [6]. On the other hand, the choice of $\mathcal{H}$ is unnatural with respect to the nonlinear problem, for it is tantamount to asserting that the solution of the nonlinear Boltzmann equation belongs to $L^2(\mathbb{R}^3, M^{-1}(\boldsymbol{x})\mathrm{d}\boldsymbol{x})$, a very strong condition not yet entirely understood. Indeed, even if this condition is imposed on the initial datum, it is not propagated in time in any useful way and there is no straightforward strategy to control the differences between the true nonlinear evolution and the linearized one, except when the initial data are extremely close to equilibrium and possess rapidly decaying Gaussian tails. See [5]. In any case, since the nonlinear problem is not dealt with here, the spectral analysis is carried out by viewing L_b as an operator on $\mathcal{H}$, with the aim at unifying well-known results and different methods which, in part, are still scattered in various sources. See Section 6 of [1], Section 5 of Chapter III of [3] and [16]. This way, a uniform and self-contained exposition of the subject is reached in a coherent notation, and this is achieved by simplifying some of the existing arguments in Section 3. The proofs of some technical statements are deferred to the Appendix, while some preliminary facts, concerning the linearization of the Boltzmann collision operator, are recalled in Section 2.

This section concludes with a detailed description of the symbols introduced in (1) and not yet explained. First, u_{S^2} stands for the uniform probability measure on the sphere S^2, which is thought of as embedded in $\mathbb{R}^3$. The symbols $\boldsymbol{v}_*$ and $\boldsymbol{w}_*$ are abbreviations for the expressions

$$\begin{aligned} \boldsymbol{v}_* &= \boldsymbol{v} + [(\boldsymbol{w} - \boldsymbol{v}) \cdot \omega]\, \omega \\ \boldsymbol{w}_* &= \boldsymbol{w} - [(\boldsymbol{w} - \boldsymbol{v}) \cdot \omega]\, \omega \end{aligned}$$

which are obtained in such a way that each binary collision preserves momentum and kinetic energy, i.e.

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{v}_* + \boldsymbol{w}_* \quad \text{and} \quad |\boldsymbol{v}|^2 + |\boldsymbol{w}|^2 = |\boldsymbol{v}_*|^2 + |\boldsymbol{w}_*|^2 \ . \tag{2}$$

Lastly, the measurable function $b : (-1, 1) \to [0, +\infty)$ is the so-called Maxwellian collision kernel, which describes the microscopical interaction between the particles. This function constitutes the main variable of interest and the aim of the paper is to show the dependence of the spectrum of L_b on b. For the physical meaning of b and ensuing mathematical properties, see Section 5.II of [4] and Subsections 3.3-6 in Chapter 2A of [15]. The following two hypotheses are assumed henceforth: The former is a *symmetry condition*, encapsulated in the equation

$$b(x) = b(\sqrt{1-x^2})\, \frac{|x|}{\sqrt{1-x^2}} \tag{3}$$

which is valid for all x in $(-1, 1)$. The latter is the so-called *cutoff hypothesis*, given in the form of

$$\int_0^1 b(x) \mathrm{d}x = 1 \ . \tag{4}$$

It is worth recalling that assumption (3) is not restrictive: Indeed, if b does not satisfy (3), then it can be replaced by the function

$$b^*(x) := \frac{1}{2} \left[b(|x|) + b(\sqrt{1-x^2})\, \frac{|x|}{\sqrt{1-x^2}} \right] \ ,$$

which meets (3), without changing the numerical value of L_b, i.e.

$$L_b[h] = L_{b^*}[h] \ .$$

On the contrary, the cutoff hypothesis is restrictive and can be mainly justified on the basis of arguments of a mathematical nature. Throughout the paper, (4) will be always in force since, otherwise, the integral in (1) could be meaningless for a general h in $\mathcal{H}$. A non-integrable kernel would require a domain strictly smaller than $\mathcal{H}$, formed of smoother functions. Finally, the constant 1 in (4) derives from a conventional choice.

2. – Derivation of L_b and basic properties.

The aforementioned nonlinear Boltzmann collision kernel for Maxwellian molecules reads

$$C_b[f;f](\boldsymbol{v}) := \int_{\mathbb{R}^3}\int_{S^2} [f(\boldsymbol{v}_*)f(\boldsymbol{w}_*) - f(\boldsymbol{v})f(\boldsymbol{w})] b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{w}$$

and it turns out to be well-defined, under the hypotheses (3)-(4), for every f in $L^1(\mathbb{R}^3)$. With a view to the study of the nonlinear evolution equation

$$\frac{\partial}{\partial t} f(\boldsymbol{v},t) = C_b[f(\cdot,t);f(\cdot,t)](\boldsymbol{v}) \ , \tag{5}$$

which corresponds to the *spatially homogeneous Boltzmann equation for Maxwellian molecules*, the domain of C_b is restricted to the set

$$\mathcal{P}_2 := \left\{ f : \mathbb{R}^3 \to [0,+\infty) \ \Big| \ \int_{\mathbb{R}^3} f(\boldsymbol{v})\mathrm{d}\boldsymbol{v} = 1, \ \int_{\mathbb{R}^3} |\boldsymbol{v}|^2 f(\boldsymbol{v})\mathrm{d}\boldsymbol{v} < +\infty \right\}$$

whose elements are *probability density functions with finite second moments*. The most important feature of the operator C_b is that it *conserves mass, momentum* and *kinetic energy*, namely

$$\int_{\mathbb{R}^3} \varphi(\boldsymbol{v}) C_b[f;f](\boldsymbol{v})\mathrm{d}\boldsymbol{v} = 0 \tag{6}$$

when $\varphi(\boldsymbol{v}) = 1, \boldsymbol{v}, |\boldsymbol{v}|^2$ respectively, for every f in $\mathcal{P}_2$. This property is a transposition in macroscopic terms of the identities (2) which pertain to the relative microscopic dynamics. See Section 6 of Chapter II of [4]. It is also well-known that the only solutions in $\mathcal{P}_2$ to the equation $C_b[f;f](\boldsymbol{v}) = 0$ are the *Maxwellian probability density functions*

$$M_{\boldsymbol{u},\sigma^2}(\boldsymbol{v}) := \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} e^{-|\boldsymbol{v}-\boldsymbol{u}|^2/2\sigma^2}$$

with *mean* $\boldsymbol{u}$ in $\mathbb{R}^3$ and *variance* σ^2 in $(0,+\infty)$. A complete proof of this fact is given in [14]. It is a cornerstone in kinetic theory that, for any given initial datum $f_0(\,\cdot\,)$ in $\mathcal{P}_2$, the relative solution $f(\cdot,t)$ converges strongly in $L^1(\mathbb{R}^3)$ to the specific Maxwellian $M_{\boldsymbol{u},\sigma^2}(\,\cdot\,)$ with $\boldsymbol{u} = \int_{\mathbb{R}^3} \boldsymbol{v} f_0(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$ and $\sigma^2 = \int_{\mathbb{R}^3} |\boldsymbol{v}-\boldsymbol{u}|^2 f_0(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$, a choice which agrees with (6). This is proved in [2].

In view of (6), it is nonrestrictive to assume $\int_{\mathbb{R}^3} \boldsymbol{v} f_0(\boldsymbol{v})\mathrm{d}\boldsymbol{v} = \boldsymbol{0}$ and $\int_{\mathbb{R}^3} |\boldsymbol{v}|^2 f_0(\boldsymbol{v})\mathrm{d}\boldsymbol{v} = 3$, so that *the only equilibrium point for the nonlinear dynamics is represented by* M. These remarks leads to consider

$$f(\boldsymbol{v},t) = M(\boldsymbol{v})(1 + h(\boldsymbol{v},t))$$

with $h(\cdot, t)$ becoming smaller and smaller as t increases. Putting this in (5) gives

$$\frac{\partial}{\partial t} h(\boldsymbol{v}, t) = L_b[h(\cdot, t)](\boldsymbol{v}) + R_b[h(\cdot, t)](\boldsymbol{v})$$

with

$$R_b[h(\cdot, t)](\boldsymbol{v}) := \frac{1}{M(\boldsymbol{v})} C_b[M(\,\cdot\,)h(\cdot, t); M(\,\cdot\,)h(\cdot, t)](\boldsymbol{v}) \ .$$

Since h is supposed to be small and R_b is quadratic in h, while L_b is linear, it is tempting to assume that R_b is negligible with respect to L_b. On the one hand, this need not be true but, on the other hand, it represents a hint about how to study the operator L_b. See [5].

An important property of L_b is that

$$\begin{aligned}(L_b[h], g)_* = -\frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} & M(\boldsymbol{v}) M(\boldsymbol{w}) \; [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})] \\ & \times [g(\boldsymbol{v}_*) + g(\boldsymbol{w}_*) - g(\boldsymbol{v}) - g(\boldsymbol{w})] \\ & \times b\left(\frac{\boldsymbol{w} - \boldsymbol{v}}{|\boldsymbol{w} - \boldsymbol{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \mathrm{d}\boldsymbol{v} \mathrm{d}\boldsymbol{w}\end{aligned} \tag{7}$$

holds true for every h and g in $\mathcal{H}$. See Appendix A.1 for a proof. A straightforward consequence of (7) is the following

PROPOSITION 2.1. – *The operator* $L_b : \mathcal{H} \to \mathcal{H}$ *is* self-adjoint *and* negative.

Another remarkable property of L_b is connected with a natural decomposition which will constitute the starting point for the computations contained in the next section. Since (4) entails $\int_{S^2} b(\boldsymbol{u} \cdot \boldsymbol{\omega}) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 1$ for every $\boldsymbol{u}$ in S^2, equality

$$L_b[h](\boldsymbol{v}) = K_b[h](\boldsymbol{v}) - \int_{\mathbb{R}^3} M(\boldsymbol{w}) h(\boldsymbol{w}) \mathrm{d}\boldsymbol{w} - h(\boldsymbol{v}) \tag{8}$$

holds true with

$$K_b[h](\boldsymbol{v}) := \int_{\mathbb{R}^3} \int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*)] M(\boldsymbol{w}) b\left(\frac{\boldsymbol{w} - \boldsymbol{v}}{|\boldsymbol{w} - \boldsymbol{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \mathrm{d}\boldsymbol{w} \ . \tag{9}$$

It is evident that the only term which, in the above decomposition, deserves attention is $K_b[h]$.

To simplify the ensuing computations, (3) can be exploited to prove that

$$K_b[h](\boldsymbol{v}) = 2 \int_{\mathbb{R}^3} \int_{S^2} h(\boldsymbol{v}_*) M(\boldsymbol{w}) b\left(\frac{\boldsymbol{w} - \boldsymbol{v}}{|\boldsymbol{w} - \boldsymbol{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \mathrm{d}\boldsymbol{w} \ . \tag{10}$$

See Appendix A.2. Finally, observe that

$$\| K_b[h] - K_{b'}[h] \|_* \leq 2\sqrt{2}\, \| h \|_* \| b - b' \|_{L^1(0,1)}^{1/2} \tag{11}$$

for every h in $\mathcal{H}$, and for any pair (b, b') of collision kernels satisfying (3)-(4), $\| \cdot \|_*$ being the norm in $\mathcal{H}$. The proof of (11), which is an immediate consequence of the Jensen inequality, is deferred to Appendix A.3.

3. – Computation of the spectrum.

The main result of the paper is encapsulated in the following

THEOREM 3.1. – *Let the collision kernel b be any measurable function satisfying* (3)-(4). *Then, the spectrum of $L_b : \mathcal{H} \to \mathcal{H}$ is discrete and the eigenvalues are given by*

$$\lambda_{n,l} = 2a(n,l) - \delta_{0,n}\delta_{0,l} - 1 \tag{12}$$

with n, l in $\mathbb{N}_0 := \{0, 1, \ldots\}$ and

$$a(n,l) := \int_0^{\pi/2} P_l(\sin\theta) \sin^{l+2n+1}\theta \; b(\cos\theta)\mathrm{d}\theta$$

*P_l and $\delta_{i,j}$ standing for the l-*th *Legendre polynomial and the standard Kronecker symbol, respectively. Moreover, each $\lambda_{n,l}$ has multiplicity $2l+1$ and $\lambda_{0,0} = \lambda_{1,0} = \lambda_{0,1} = 0$, the remaining eigenvalues being strictly negative. The value Λ_b of the least negative eigenvalues gives the* spectral gap *of L_b and can be obtained for any (n,l) such that $l + 2n = 4$, i.e.*

$$\Lambda_b = -2\int_0^1 x^2(1-x^2)b(x)\mathrm{d}x. \tag{13}$$ □

The rest of the section is devoted to a proof, based on the study of the operator K_b, of this theorem. Thanks to (11) and the standard approximation technique for functions in L^p, the proof will be carried out first for a bounded b and then completed by resorting to a suitable approximation process.

To start with, change the variables in (10) according to $\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{v}$ to obtain

$$\begin{aligned} K_b[h](\boldsymbol{v}) &= 2\int_{\mathbb{R}^3}\int_{S^2} h(\boldsymbol{v} + (\boldsymbol{z}\cdot\boldsymbol{\omega})\boldsymbol{\omega})M(\boldsymbol{v}+\boldsymbol{z})b\left(\frac{\boldsymbol{z}}{|\boldsymbol{z}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{z} \\ &= 4\int_{\mathbb{R}^3}\int_{S^2_+} h(\boldsymbol{v} + (\boldsymbol{z}\cdot\boldsymbol{\omega})\boldsymbol{\omega})M(\boldsymbol{v}+\boldsymbol{z})b\left(\frac{\boldsymbol{z}}{|\boldsymbol{z}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{z} \end{aligned}$$

where $S^2_+ := \{\boldsymbol{x} = (x_1, x_2, x_3) \in S^2 \mid x_3 > 0\}$. After putting

$$\begin{aligned}
\mathbb{R}^3_+ &:= \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\} \\
\mathbb{R}^3_- &:= \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < 0\} \\
\Omega_{1,+} &:= \{(\boldsymbol{z}, \boldsymbol{\omega}) \in \mathbb{R}^3 \times S^2_+ \mid \boldsymbol{z} \cdot \boldsymbol{\omega} > 0\} \\
\Omega_{1,-} &:= \{(\boldsymbol{z}, \boldsymbol{\omega}) \in \mathbb{R}^3 \times S^2_+ \mid \boldsymbol{z} \cdot \boldsymbol{\omega} < 0\} \\
\Omega_{2,+} &:= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \mid \boldsymbol{x} \cdot \boldsymbol{y} = 0\} \\
\Omega_{2,-} &:= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^3_- \times \mathbb{R}^3 \mid \boldsymbol{x} \cdot \boldsymbol{y} = 0\} \ ,
\end{aligned}$$

consider the diffeomorphism $\boldsymbol{T} : \Omega_{1,+} \cup \Omega_{1,-} \to \Omega_{2,+} \cup \Omega_{2,-}$ given by

$$(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{T}(\boldsymbol{z}, \boldsymbol{\omega}) = ((\boldsymbol{z} \cdot \boldsymbol{\omega})\boldsymbol{\omega}, \boldsymbol{z} - (\boldsymbol{z} \cdot \boldsymbol{\omega})\boldsymbol{\omega}) \ .$$

If $\mathscr{L}_d$ denotes the d-dimensional Lebesgue measure (1), then $\boldsymbol{T}$ transforms the measure $\mathscr{L}_3 \otimes u_{S^2}$ on $\Omega_{1,+} \cup \Omega_{1,-}$ into the measure $\dfrac{1}{4\pi}|\boldsymbol{x}|^{-2} m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathscr{L}_3(\mathrm{d}\boldsymbol{x})$ on $\Omega_{2,+} \cup \Omega_{2,-}$, with $m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})$ defined as follows. First, set $\Pi(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^3 \mid \boldsymbol{x} \cdot \boldsymbol{y} = 0\}$ and let $p_3 : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $p_3 : (x_1, x_2, x_3) \mapsto (x_1, x_2)$. Second, for every $\boldsymbol{x}$ in $\mathbb{R}^3_0 := \mathbb{R}^3_+ \cup \mathbb{R}^3_-$, let $Q^{(\boldsymbol{x})}$ be any element of the group $\mathrm{SO}(3)$ such that $Q^{(\boldsymbol{x})}\boldsymbol{x} = |\boldsymbol{x}|\boldsymbol{e}_3$. Finally, for every A in $\mathscr{B}(\mathbb{R}^3)$, put

$$m_{\boldsymbol{x}}(A) := \mathscr{L}_2\big(p_3\big[Q^{(\boldsymbol{x})}(A \cap \Pi(\boldsymbol{x}))\big]\big)$$

which turns out to be well-defined, independently of the specific choice of $Q^{(\boldsymbol{x})}$. All these facts are analyzed and proved in Appendix A.4. Then,

$$\begin{aligned}
&4 \int_{\mathbb{R}^3} \int_{S^2_+} h(\boldsymbol{v} + (\boldsymbol{z} \cdot \boldsymbol{\omega})\boldsymbol{\omega}) M(\boldsymbol{v} + \boldsymbol{z}) b\left(\frac{\boldsymbol{z}}{|\boldsymbol{z}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{z} \\
&= 4 \int_{\Omega_{1,+} \cup \Omega_{1,-}} h(\boldsymbol{v} + \boldsymbol{x}(\boldsymbol{z}, \boldsymbol{\omega})) M(\boldsymbol{v} + \boldsymbol{x}(\boldsymbol{z}, \boldsymbol{\omega}) + \boldsymbol{y}(\boldsymbol{z}, \boldsymbol{\omega})) \\
&\times b\left(\frac{|\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{\omega})|}{\sqrt{|\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{\omega})|^2 + |\boldsymbol{y}(\boldsymbol{z}, \boldsymbol{\omega})|^2}}\right) \mathrm{d}\boldsymbol{z}\, u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \\
&= \frac{1}{\pi} \int_{\Omega_{2,+} \cup \Omega_{2,-}} h(\boldsymbol{v} + \boldsymbol{x}) M(\boldsymbol{v} + \boldsymbol{x} + \boldsymbol{y}) b\left(\frac{|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2 + |\boldsymbol{y}|^2}}\right) |\boldsymbol{x}|^{-2} m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{\pi} \int_{\mathbb{R}^3_0} h(\boldsymbol{v} + \boldsymbol{x}) \left[\int_{\Pi(\boldsymbol{x})} M(\boldsymbol{v} + \boldsymbol{x} + \boldsymbol{y}) b\left(\frac{|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2 + |\boldsymbol{y}|^2}}\right) m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y}) \right) \frac{\mathrm{d}\boldsymbol{x}}{|\boldsymbol{x}|^2} \ .
\end{aligned}$$

(1) when no ambiguity will arise, $\mathscr{L}_d(\mathrm{d}\boldsymbol{x})$ will be shortened to $\mathrm{d}\boldsymbol{x}$.

Setting $q(t) := b\left(\frac{1}{\sqrt{1+t^2}}\right)$, one gets $b\left(\frac{|\boldsymbol{x}|}{\sqrt{|\boldsymbol{x}|^2+|\boldsymbol{y}|^2}}\right) = q(|\boldsymbol{y}|/|\boldsymbol{x}|)$ and

$$K_b[h](\boldsymbol{v}) = \frac{1}{\pi}\int_{\mathbb{R}^3_0} h(\boldsymbol{v}+\boldsymbol{x})\left[\int_{\Pi(\boldsymbol{x})} M(\boldsymbol{v}+\boldsymbol{x}+\boldsymbol{y})q(|\boldsymbol{y}|/|\boldsymbol{x}|)m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\right]\frac{\mathrm{d}\boldsymbol{x}}{|\boldsymbol{x}|^2}$$
$$= \frac{1}{\pi}\int_{\mathbb{R}^3_{\boldsymbol{v}}} h(\boldsymbol{r})\left[\int_{\Pi(\boldsymbol{r}-\boldsymbol{v})} M(\boldsymbol{r}+\boldsymbol{y})q(|\boldsymbol{y}|/|\boldsymbol{r}-\boldsymbol{v}|)m_{\boldsymbol{r}-\boldsymbol{v}}(\mathrm{d}\boldsymbol{y})\right]\frac{\mathrm{d}\boldsymbol{r}}{|\boldsymbol{r}-\boldsymbol{v}|^2}$$

where $\mathbb{R}^3_{\boldsymbol{v}} := \boldsymbol{v} + \mathbb{R}^3_0$. The last formula shows that $K_b[h]$ is an *integral operator* of the type of

$$K_b[h](\boldsymbol{v}) = \int_{\mathbb{R}^3} \mathrm{K}(\boldsymbol{v};\boldsymbol{r})h(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$$

with

$$\mathrm{K}(\boldsymbol{v};\boldsymbol{r}) := \begin{cases} \dfrac{1}{\pi|\boldsymbol{r}-\boldsymbol{v}|^2}\displaystyle\int_{\Pi(\boldsymbol{r}-\boldsymbol{v})} M(\boldsymbol{r}+\boldsymbol{y})q\left(\frac{|\boldsymbol{y}|}{|\boldsymbol{r}-\boldsymbol{v}|}\right)m_{\boldsymbol{r}-\boldsymbol{v}}(\mathrm{d}\boldsymbol{y}) & \text{if } \boldsymbol{r}\neq\boldsymbol{v} \\ 0 & \text{if } \boldsymbol{r}=\boldsymbol{v}\ . \end{cases} \tag{14}$$

It is now useful to rewrite the integral in (14) as follows. Fix $\boldsymbol{r}$ and $\boldsymbol{v}$ in such a way that $\boldsymbol{r}\neq\boldsymbol{v}$ and use the fact that $\boldsymbol{y}\cdot(\boldsymbol{r}-\boldsymbol{v})=0$ to write

$$M(\boldsymbol{r}+\boldsymbol{y}) = \left(\frac{1}{2\pi}\right)^{3/2}\exp\left\{-\frac{1}{2}(|\boldsymbol{r}|^2+|\boldsymbol{y}|^2+2\boldsymbol{r}\cdot\boldsymbol{y})\right\}$$
$$= \left(\frac{1}{2\pi}\right)^{3/2}\exp\left\{-\frac{1}{2}(|\boldsymbol{r}|^2+|\boldsymbol{y}|^2+(\boldsymbol{r}+\boldsymbol{v})\cdot\boldsymbol{y})\right\}\ .$$

Then, decompose the vector $\frac{1}{2}(\boldsymbol{r}+\boldsymbol{v})$ as

$$\frac{1}{2}(\boldsymbol{r}+\boldsymbol{v}) = \boldsymbol{\zeta} + \alpha(\boldsymbol{r}-\boldsymbol{v}) \tag{15}$$

where $\boldsymbol{\zeta}$ is orthogonal to $(\boldsymbol{r}-\boldsymbol{v})$ and α is a constant. To find the value of such a constant, take the scalar product of both members of (15) by $(\boldsymbol{r}-\boldsymbol{v})$ so that

$$\alpha = \frac{1}{2}\frac{(\boldsymbol{r}+\boldsymbol{v})\cdot(\boldsymbol{r}-\boldsymbol{v})}{|\boldsymbol{r}-\boldsymbol{v}|^2}\ .$$

Then, the triple-product identities show that

$$\boldsymbol{\zeta} = \frac{1}{2}\left[(\boldsymbol{r}+\boldsymbol{v}) - \frac{(\boldsymbol{r}+\boldsymbol{v})\cdot(\boldsymbol{r}-\boldsymbol{v})}{|\boldsymbol{r}-\boldsymbol{v}|^2}(\boldsymbol{r}-\boldsymbol{v})\right] = \frac{(\boldsymbol{v}\wedge\boldsymbol{r})\wedge(\boldsymbol{v}-\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{v}|^2} \tag{16}$$

where $\wedge$ stands for the usual wedge product between vectors in $\mathbb{R}^3$. Immediate consequences of (16) are that $\boldsymbol{\zeta}$ belongs to $\Pi(\boldsymbol{r}-\boldsymbol{v})$ and

$$|\zeta| = \frac{|\boldsymbol{v} \wedge \boldsymbol{r}|}{|\boldsymbol{r}-\boldsymbol{v}|} \ .$$

Thanks to (15), the integral in (14) becomes

$$\frac{1}{\pi|\boldsymbol{r}-\boldsymbol{v}|^2} \int\limits_{\Pi(\boldsymbol{r}-\boldsymbol{v})} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}(|\boldsymbol{r}|^2 + |\boldsymbol{y}|^2 + 2\boldsymbol{\zeta}\cdot\boldsymbol{y})\right\} q\left(\frac{|\boldsymbol{y}|}{|\boldsymbol{r}-\boldsymbol{v}|}\right) m_{(\boldsymbol{r}-\boldsymbol{v})}(\mathrm{d}\boldsymbol{y})$$

whose expression can be simplified after the following change of variables (from Cartesian to polar coordinates)

$$\boldsymbol{y} = \rho\cos\phi\frac{\boldsymbol{\zeta}}{|\zeta|} + \rho\sin\phi\frac{\boldsymbol{\zeta}^{\perp}}{|\zeta^{\perp}|} \ ,$$

$\boldsymbol{\zeta}^{\perp}$ being any fixed (non null) vector in $\Pi(\boldsymbol{r}-\boldsymbol{v})$ orthogonal to $\boldsymbol{\zeta}$ and ϕ varying in $[0, 2\pi)$. So, the integral at issue assumes the form

$$\frac{1}{\pi|\boldsymbol{r}-\boldsymbol{v}|^2} \int\limits_0^{+\infty}\int\limits_0^{2\pi} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}(|\boldsymbol{r}|^2 + \rho^2 + 2|\zeta|\rho\cos\phi)\right\} q\left(\frac{\rho}{|\boldsymbol{r}-\boldsymbol{v}|}\right)\rho\mathrm{d}\rho\mathrm{d}\phi \ .$$

Changing again the coordinates in the above ρ-integral, according to $\dfrac{\rho}{|\boldsymbol{r}-\boldsymbol{v}|} = \tan\theta$ with θ in $(0, \pi/2)$, leads to

$$\frac{1}{\pi|\boldsymbol{r}-\boldsymbol{v}|^2} \int\limits_0^{\pi/2}\int\limits_0^{2\pi} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}\left(|\boldsymbol{r}|^2 + |\boldsymbol{r}-\boldsymbol{v}|^2\tan^2\theta\right.\right.$$

$$\left.\left. + 2\frac{|\boldsymbol{v}\wedge\boldsymbol{r}|}{|\boldsymbol{r}-\boldsymbol{v}|}|\boldsymbol{r}-\boldsymbol{v}|\tan\theta\cos\phi\right)\right\} q(\tan\theta)|\boldsymbol{r}-\boldsymbol{v}|\tan\theta\ |\boldsymbol{r}-\boldsymbol{v}|\cos^{-2}\theta\mathrm{d}\theta\mathrm{d}\phi$$

$$= \frac{1}{\pi}\int\limits_0^{\pi/2}\int\limits_0^{2\pi} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}\left(|\boldsymbol{r}|^2 + |\boldsymbol{r}-\boldsymbol{v}|^2\tan^2\theta\right.\right.$$

$$\left.\left. + 2|\boldsymbol{v}\wedge\boldsymbol{r}|\tan\theta\cos\phi\right)\right\} q(\tan\theta)\tan\theta\cos^{-2}\theta\mathrm{d}\theta\mathrm{d}\phi \ .$$

At this stage, observe that the definition of q entails $q(\tan\theta) = b(\cos\theta)$ and recall both the definition of the *modified Bessel function of order* v, I_v, and the integral representation $I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\phi}\,\mathrm{d}\phi$. See, for example, (2) in Subsection 3.7 and (9) in Subsection 3.71 of [17], respectively. Thus, the integral in (14) can be written as

$$
(17)\qquad \begin{aligned} &2\int_0^{\pi/2}\left(\frac{1}{2\pi}\right)^{3/2}\exp\left\{-\frac{1}{2}\left(|\boldsymbol{r}|^2+|\boldsymbol{r}-\boldsymbol{v}|^2\tan^2\theta\right)\right\}\\ &\times I_0(|\boldsymbol{v}\wedge\boldsymbol{r}|\tan\theta)\ b(\cos\theta)\ \frac{\sin\theta}{\cos^3\theta}\,\mathrm{d}\theta. \end{aligned}
$$

Denote by ψ the angle in $[0,\pi]$ between the vectors $\boldsymbol{v}$ and $\boldsymbol{r}$ to obtain $\boldsymbol{r}\cdot\boldsymbol{v}=|\boldsymbol{r}|\,|\boldsymbol{v}|\cos\psi$ and $|\boldsymbol{v}\wedge\boldsymbol{r}|=|\boldsymbol{r}|\,|\boldsymbol{v}|\sin\psi$, so that (17) becomes

$$
(18)\qquad \begin{aligned} &2M(\boldsymbol{r})\int_0^{\pi/2}\exp\left\{-\frac{1}{2}\tan^2\theta(|\boldsymbol{v}|^2+|\boldsymbol{r}|^2)\right\}\\ &\times\exp\left\{|\boldsymbol{r}|\,|\boldsymbol{v}|\cos\psi\tan^2\theta\right\}\cdot I_0(|\boldsymbol{r}|\,|\boldsymbol{v}|\sin\psi\tan\theta)b(\cos\theta)\frac{\sin\theta}{\cos^3\theta}\mathrm{d}\theta\ . \end{aligned}
$$

With a view to further developments, recall the definition of the *l^{th}-Legendre polynomial*, P_l, contained, for example, in Subsection 10.10 of [7], along with the identity

$$
(19)\qquad \begin{aligned} &\exp\{z\cos\gamma\cos\delta\}I_0(z\sin\gamma\sin\delta)\\ &=\left(\frac{\pi}{2z}\right)^{1/2}\sum_{l=0}^{\infty}(2l+1)I_{l+1/2}(z)P_l(\cos\gamma)P_l(\cos\delta) \end{aligned}
$$

which holds true for every real γ, δ and z. When $z=0$, put $z^{-1/2}I_{l+1/2}(z):=[\Gamma(3/2)]^{-1}\delta_{l,0}$ for every l in $\mathbb{N}_0$. The validity of (19) is stated in Appendix A.5. After putting $z=|\boldsymbol{r}|\,|\boldsymbol{v}|\sin\theta\cos^{-2}\theta$, $\gamma=\psi$ and $\delta=\pi/2-\theta$, (18) is rendered into

$$
(20)\qquad \begin{aligned} &2M(\boldsymbol{r})\int_0^{\pi/2}\exp\left\{-\frac{1}{2}\tan^2\theta(|\boldsymbol{v}|^2+|\boldsymbol{r}|^2)\right\}\cdot\left[\frac{\pi\cos^2\theta}{2|\boldsymbol{r}|\,|\boldsymbol{v}|\sin\theta}\right]^{1/2}\\ &\times\left[\sum_{l=0}^{\infty}(2l+1)I_{l+1/2}\left(|\boldsymbol{r}|\,|\boldsymbol{v}|\frac{\sin\theta}{\cos^2\theta}\right)P_l(\cos\psi)P_l(\sin\theta)\right]b(\cos\theta)\frac{\sin\theta}{\cos^3\theta}\,\mathrm{d}\theta. \end{aligned}
$$

Now, it can be proved that

$$
(21)\qquad \sum_{l=0}^{\infty}(2l+1)\int_0^{\pi/2}\exp\left\{-\frac{1}{2}\tan^2\theta(|\boldsymbol{v}|^2+|\boldsymbol{r}|^2)\right\}I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{\cos^2\theta}\right)\frac{\mathrm{d}\theta}{\cos^2\theta}<+\infty
$$

which, after noticing that $|P_l(x)|\le 1$ for every x in $[-1,1]$ and $b(x)\le B$ for every x in $(-1,1)$, is sufficient to justify the exchange of the integral with the series in (20), by a straightforward monotone convergence argument. Hence, the expression (20) is equal to

$$\sqrt{\frac{2\pi}{|\boldsymbol{r}|\,|\boldsymbol{v}|}}\,M(\boldsymbol{r})\sum_{l=0}^{\infty}(2l+1)P_l(\cos\psi)\int_0^{\pi/2}P_l(\sin\theta)I_{l+1/2}\left(|\boldsymbol{r}|\,|\boldsymbol{v}|\frac{\sin\theta}{\cos^2\theta}\right)$$
$$\times\exp\left\{-\frac{1}{2}\tan^2\theta(|\boldsymbol{v}|^2+|\boldsymbol{r}|^2)\right\}\frac{\sqrt{\sin\theta}}{\cos^2\theta}b(\cos\theta)\mathrm{d}\theta\ . \tag{22}$$

At this stage, it is worth rewriting (22) according to the following procedure. Recall the definition of the *Laguerre polynomial of order* α and degree n, L_n^α, given, for example, in Subsection 10.12 of [7], along with the so-called *Hille-Hardy formula*, i.e.

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\alpha+1)}L_n^\alpha(x)L_n^\alpha(y)t^n$$
$$=\frac{(xyt)^{-\alpha/2}}{1-t}\exp\left\{\frac{-t(x+y)}{1-t}\right\}I_\alpha\left(\frac{2(xyt)^{1/2}}{1-t}\right) \tag{23}$$

which is valid for $|t|<1$ and $\alpha>-1$. See (20) in Section 10.12 of [7]. Put $\alpha=l+1/2$, $t=\sin^2\theta$, $x=|\boldsymbol{v}|^2/2$ and $y=|\boldsymbol{r}|^2/2$ in (23) to obtain

$$\sqrt{\frac{2\pi}{|\boldsymbol{r}|\,|\boldsymbol{v}|}}\,M(\boldsymbol{r})\sum_{l=0}^{\infty}(2l+1)P_l(\cos\psi)$$
$$\times\int_0^{\pi/2}P_l(\sin\theta)\sqrt{\sin\theta}\ b(\cos\theta)\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|\sin\theta}{2}\right)^{l+1/2} \tag{24}$$
$$\times\left[\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+l+3/2)}L_n^{l+1/2}(|\boldsymbol{v}|^2/2)L_n^{l+1/2}(|\boldsymbol{r}|^2/2)\sin^{2n}\theta\right]\mathrm{d}\theta$$

which is the desired rewrite of (22). To proceed, restrict the analysis to the case in which both $\boldsymbol{v}$ and $\boldsymbol{r}$ are different from the null vector and take account of the asymptotic relation

$$L_n^{l+1/2}(x)=O\left[\frac{e^{x/2}}{\sqrt{x}}\left(\frac{n}{x}\right)^{l/2}\right]\qquad(n\to\infty)\ ,$$

which holds uniformly for all x in every bounded interval $[a,b]$ with $0<a<b$, to conclude that the series

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+l+3/2)}|L_n^{l+1/2}(|\boldsymbol{v}|^2/2)|\cdot|L_n^{l+1/2}(|\boldsymbol{r}|^2/2)|\int_0^{\pi/2}\sin^{2n}\theta\mathrm{d}\theta$$
$$=\frac{\sqrt{\pi}}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\Gamma(n+l+3/2)}|L_n^{l+1/2}(|\boldsymbol{v}|^2/2)|\cdot|L_n^{l+1/2}(|\boldsymbol{r}|^2/2)| \tag{25}$$

is convergent. For the above asymptotic expression for $L_n^{l+1/2}$ see Section 6 of Chapter III of [13]. By resorting again to a monotone convergence argument, the convergence of the series (25) entails the possibility of exchanging the integral with the series, since b is bounded. Thus, (24) can be written as

$$\begin{aligned}(26)\qquad &\sqrt{\pi}\, M(\boldsymbol{r}) \sum_{l=0}^{\infty} (2l+1) \left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{2}\right)^{l} P_l(\cos\psi) \\ &\times \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+l+3/2)} L_n^{l+1/2}(|\boldsymbol{v}|^2/2) L_n^{l+1/2}(|\boldsymbol{r}|^2/2) a(n,l)\ .\end{aligned}$$

For any unit vector $\boldsymbol{\omega} = (\sin\alpha\cos\beta,\, \sin\alpha\sin\beta,\, \cos\alpha)$ with α in $[0,\pi]$ and β in $[0,2\pi)$, define the *real spherical harmonics*, $Y_l^m(\boldsymbol{\omega})$, with l in $\mathbb{N}_0$ and m in $\{-l,\dots,l\}$, as $Y_0^0(\boldsymbol{\omega}) := (4\pi)^{-1/2}$ and, for every l in $\mathbb{N}$,

$$Y_l^m(\boldsymbol{\omega}) := \begin{cases} \sqrt{\dfrac{2l+1}{4\pi}} P_l(\cos\alpha) & \text{for } m=0 \\ \sqrt{\dfrac{2l+1}{2\pi} \cdot \dfrac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\alpha)\cos m\beta & \text{for } m=1,\dots,l \\ \sqrt{\dfrac{2l+1}{2\pi} \cdot \dfrac{(l+m)!}{(l-m)!}}\, P_l^{-m}(\cos\alpha)\sin m\beta & \text{for } m=-l,\dots,-1 \end{cases}$$

where P_l^m is the *associated Legendre function of the first kind* od order l and degree m, defined, for example, in Section 17 of Chapter III of [13]. From (4) in the aforementioned section of [13], it follows that the set of functions $\{Y_l^m\}$ constitutes an orthonormal basis for $L^2(S^2, \mathrm{d}\boldsymbol{\omega})$, $\mathrm{d}\boldsymbol{\omega}$ being the surface measure. Consequently, in view of (16_1)-(16_2) in Section 1 of Chapter IV of [13], the set of functions

$$e_{n,l,m}(\boldsymbol{x}) := \pi^{3/4} \left[\frac{2n!}{\Gamma(n+l+3/2)}\right]^{1/2} \left(\frac{|\boldsymbol{x}|}{\sqrt{2}}\right)^{l} L_n^{l+1/2}(|\boldsymbol{x}|^2/2) Y_l^m(\boldsymbol{x}/|\boldsymbol{x}|)$$

with n and l varying in $\mathbb{N}_0$ and m in $\{-l,\dots,l\}$ forms an orthonormal basis for $\mathcal{H}$. In this notation, the well-known *addition theorem for the spherical harmonics* reads

$$(27)\qquad P_l(\cos\psi) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\boldsymbol{v}/|\boldsymbol{v}|) Y_l^m(\boldsymbol{r}/|\boldsymbol{r}|)$$

leading to the equality

$$\mathrm{K}(\boldsymbol{v};\boldsymbol{r}) = 2M(\boldsymbol{r}) \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e_{n,l,m}(\boldsymbol{v}) e_{n,l,m}(\boldsymbol{r}) a(n,l)$$

which is valid when $\boldsymbol{v} \neq \boldsymbol{r}$ and both $\boldsymbol{v}$ and $\boldsymbol{r}$ are different from the null vector. Since every h in $\mathcal{H}$ admits a Fourier expansion of the type of

$$h(\boldsymbol{r}) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{n,l,m} e_{n,l,m}(\boldsymbol{r})$$

with $c_{n,l,m} = (h, e_{n,l,m})_*$, one gets

$$K_b[h](\boldsymbol{v}) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 2c_{n,l,m} a(n,l) e_{n,l,m}(\boldsymbol{v}) \ ,$$

which proves (12) under the extra hypothesis that b is bounded.

The removing of this additional condition is straightforward. Consider any collision kernel b, satisfying only (3)-(4), and a sequence $(b_j)_{j\geq 1}$ of bounded collision kernels satisfying (3)-(4) and approximating b in $L^1(-1,1)$. Then,

$$L_{b_j}[e_{n,l,m}] = \lambda_{n,l}^{(j)} e_{n,l,m}$$

for each n, l in $\mathbb{N}_0$, m in $\{-l, \dots, l\}$ and j in $\mathbb{N}$, where

$$\lambda_{n,l}^{(j)} = 2a_j(n,l) - \delta_{0,n}\delta_{0,l} - 1$$

and

$$a_j(n,l) := \int_0^{\pi/2} P_l(\sin\theta) \sin^{l+2n+1}\theta \ b_j(\cos\theta) \mathrm{d}\theta \ .$$

Now, $\lim_{j\to+\infty} \lambda_{n,l}^{(j)} = \lambda_{n,l}$ and, since the eigenfunctions $e_{n,l,m}$ do not depend on b, (11) entails $\lim_{j\to+\infty} L_{b_j}[e_{n,l,m}] = L_b[e_{n,l,m}]$, so that each $\lambda_{n,l}$ is an eigenvalue of L_b. Next, to prove that $S := \{\lambda_{n,l} \mid n, l \in \mathbb{N}_0\}$ *coincides* with the spectrum of L_b, it is enough to prove that any eigenvalue λ^* of L_b must belong to S. Indeed, there is an eigenfunction h^* for which

$$L_b[h^*] = \lambda^* h^* \tag{28}$$

and, since $h^* = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c^*_{n,l,m} e_{n,l,m}$ for some $c^*_{n,l,m}$, (28) implies

$$c^*_{n,l,m} \lambda_{n,l} = c^*_{n,l,m} \lambda^*$$

for every n, l in $\mathbb{N}_0$ and m in $\{-l, \dots, l\}$. To verify the claim, suffice it to observe that the Fourier coefficient $c^*_{n,l,m}$ cannot be simultaneously equal to zero.

Finally, it must be proved that Λ_b is the value of the spectral gap. To this end, notice that the inequality $|P_l(x)| \leq 1$, valid for every x in $[-1,1]$, implies that

$$\Lambda_b \geq \lambda_{n,l} \tag{29}$$

holds when either n belongs to $\mathbb{N}$ or $n = 0$ and $l \geq 4$. The cases $(n, l) = (0,0), (0,1), (1,0)$, which correspond to the five null eigenvalues, do not count. It remains to consider the cases $(n, l) = (0,2), (0,3)$, in which (29) can be derived from the inequalities $P_2(x) \leq x^2$ and $P_3(x) \leq x$, which are valid for every x in $[0,1]$.

A. – Appendix.

As mentioned in Section 1, proofs of some of the statements scattered in previous sections are gathered in this appendix.

A.1 – *Proof of* (7)

The identity at issue follows immediately after proving

i)

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\, g(\boldsymbol{v}) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\, g(\boldsymbol{w}) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$

ii)

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\, g(\boldsymbol{v}_*) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\, g(\boldsymbol{w}_*) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$

iii)

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\; g(\boldsymbol{v}) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$
$$= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\, g(\boldsymbol{v}_*) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}.$$

Now, i) and ii) hold true in view of the change of variable $(\boldsymbol{v},\boldsymbol{w}) \mapsto (\boldsymbol{w},\boldsymbol{v})$. As far as iii) is concerned, exchange the order of integration in the left-hand side, passing from $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}$ to $\int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}$. After noting that $|\boldsymbol{v}-\boldsymbol{w}| = |\boldsymbol{v}_*-\boldsymbol{w}_*|$ and $(\boldsymbol{v}-\boldsymbol{w})\cdot\boldsymbol{\omega} = -(\boldsymbol{v}_*-\boldsymbol{w}_*)\cdot\boldsymbol{\omega}$, take account of (2) and (3) to deduce

$$\begin{aligned}&\int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\; g(\boldsymbol{v}) \\ &\times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w} \\ &= \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\; g(\boldsymbol{v}) \\ &\times b\left(\frac{\boldsymbol{w}_*-\boldsymbol{v}_*}{|\boldsymbol{w}_*-\boldsymbol{v}_*|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v}_*)M(\boldsymbol{w}_*) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}.\end{aligned} \tag{30}$$

Then, consider the linear transformation $\boldsymbol{L}_{\boldsymbol{\omega}} : (\boldsymbol{v},\boldsymbol{w}) \mapsto (\boldsymbol{v}_*,\boldsymbol{w}_*)$ from $\mathbb{R}^6$ into itself: A straightforward computation shows that $\boldsymbol{L}_{\boldsymbol{\omega}}^2 \equiv \mathrm{Id}$, from which $det(Jac[\boldsymbol{L}_{\boldsymbol{\omega}}]) = 1$. Hence, choosing $\boldsymbol{v}_*$ and $\boldsymbol{w}_*$ as new variables of integration in the integral on the right-hand side of (30) leads to

$$\int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [h(\boldsymbol{v}_*) + h(\boldsymbol{w}_*) - h(\boldsymbol{v}) - h(\boldsymbol{w})]\; g(\boldsymbol{v}) \times b\left(\frac{\boldsymbol{w}_*-\boldsymbol{v}_*}{|\boldsymbol{w}_*-\boldsymbol{v}_*|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v}_*)M(\boldsymbol{w}_*) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$
$$= \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [h(\boldsymbol{v}) + h(\boldsymbol{w}) - h(\boldsymbol{v}_*) - h(\boldsymbol{w}_*)]\; g(\boldsymbol{v}_*) \times b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) M(\boldsymbol{v})M(\boldsymbol{w}) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$

and the desired conclusion follows after exchanging the order of integration from $\int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}$ to $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}$.

A.2 – *Proof of* (10)

It is enough to prove that

$$\int_{S^2} h(\boldsymbol{v}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = \int_{S^2} h(\boldsymbol{w}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \tag{31}$$

holds for every bounded and continuous $h : \mathbb{R}^3 \to \mathbb{R}$ and for every fixed $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^3$ with $\boldsymbol{v} \neq \boldsymbol{w}$. Since the change $\boldsymbol{\omega} \to -\boldsymbol{\omega}$ leaves $\boldsymbol{v}_*$ unchanged, putting $\boldsymbol{r} := \frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}$ and $S^2_{\boldsymbol{r}} := \{\boldsymbol{\omega} \in S^2 \setminus \{\boldsymbol{r}\} \mid \boldsymbol{\omega}\cdot\boldsymbol{r} > 0\}$ yields

$$\int_{S^2} h(\boldsymbol{v}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 2\int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{v}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$

$$\int_{S^2} h(\boldsymbol{w}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 2\int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{w}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$

which lead to the equivalence between (31) and

$$\int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{v}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = \int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{w}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\ . \tag{32}$$

At this stage, the map

$$\boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\omega}) := \frac{\boldsymbol{r} - (\boldsymbol{r}\cdot\boldsymbol{\omega})\boldsymbol{\omega}}{\sqrt{1-(\boldsymbol{r}\cdot\boldsymbol{\omega})^2}}$$

turns out to be a diffeomorphism from $S^2_{\boldsymbol{r}}$ into itself, such that

$$\begin{aligned} \boldsymbol{v} \ &+\ [(\boldsymbol{w}-\boldsymbol{v})\cdot\boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\omega})]\boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\omega}) \ = \ \boldsymbol{w}_* \\ \boldsymbol{w} \ &-\ [(\boldsymbol{w}-\boldsymbol{v})\cdot\boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\omega})]\boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\omega}) \ = \ \boldsymbol{v}_* \end{aligned}$$

for every $\boldsymbol{\omega}$ in $S^2_{\boldsymbol{r}}$. The change of variables $\boldsymbol{\omega} = \boldsymbol{G}_{\boldsymbol{r}}(\boldsymbol{\tau})$ entails

$$\int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{v}_*) b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = \int_{S^2_{\boldsymbol{r}}} h(\boldsymbol{w}_*(\boldsymbol{\tau})) b(\sqrt{1-(\boldsymbol{r}\cdot\boldsymbol{\tau})^2}) u_{S^2}\circ\boldsymbol{G}_{\boldsymbol{r}}(\mathrm{d}\boldsymbol{\tau})$$

where $\boldsymbol{w}_*(\boldsymbol{\tau}) = \boldsymbol{w} - [(\boldsymbol{w}-\boldsymbol{v})\cdot\boldsymbol{\tau}]\,\boldsymbol{\tau}$. A direct computation shows that

$$u_{S^2}\circ \boldsymbol{G}_{\boldsymbol{r}}(\mathrm{d}\boldsymbol{\tau}) = \frac{\boldsymbol{r}\cdot\boldsymbol{\tau}}{\sqrt{1-(\boldsymbol{r}\cdot\boldsymbol{\tau})^2}}u_{S^2}(\mathrm{d}\boldsymbol{\tau})$$

which, combined with the property (3), yields (32).

A.3 – *Proof of* (11)

Fix h in $\mathcal{H}$ and a pair (b, b') of collision kernels satisfying (3)-(4). Now, (10) entails

$$\| K_b[h] - K_{b'}[h] \|_*^2 = 4\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\int_{S^2} h(\boldsymbol{v}_*)M(\boldsymbol{w})$$
$$\times\left[b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)\right]u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{w}\Big|^2 M(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$$

and, after an application of the Jensen inequality,

$$\| K_b[h] - K_{b'}[h] \|_*^2 \le 4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big|\int_{S^2} h(\boldsymbol{v}_*)$$
$$\times\left[b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)\right]u_{S^2}(\mathrm{d}\boldsymbol{\omega})\Big|^2 M(\boldsymbol{v})M(\boldsymbol{w})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}\ .$$

If $b \equiv b'$, (11) holds trivially. Otherwise,

$$\| b-b' \|_{L^1(0,1)}^2\left\{\int_{S^2} h(\boldsymbol{v}_*)\ \frac{b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)}{\| b-b' \|_{L^1(0,1)}}u_{S^2}(\mathrm{d}\boldsymbol{\omega})\right\}^2$$
$$\le \| b-b' \|_{L^1(0,1)}^2\left\{\int_{S^2} h(\boldsymbol{v}_*)\frac{\left|b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)\right|}{\| b-b' \|_{L^1(0,1)}}u_{S^2}(\mathrm{d}\boldsymbol{\omega})\right\}^2$$
$$\le \| b-b' \|_{L^1(0,1)}\int_{S^2} h^2(\boldsymbol{v}_*)\ \left|b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)\right|\ u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$

by the Jensen inequality again. A combination of the Tonelli theorem with the properties of the linear transformation $\boldsymbol{L}_{\boldsymbol{\omega}} : (\boldsymbol{v}, \boldsymbol{w}) \mapsto (\boldsymbol{v}_*, \boldsymbol{w}_*)$ explained in A.1 gives

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} h^2(\boldsymbol{v}_*)\ |b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)|$$
$$\times M(\boldsymbol{v})M(\boldsymbol{w})u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}$$
$$= \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h^2(\boldsymbol{v}_*)\ |b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)|$$
$$\times M(\boldsymbol{v}_*)M(\boldsymbol{w}_*)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} h^2(\boldsymbol{v})\ |b\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right) - b'\left(\frac{\boldsymbol{w}-\boldsymbol{v}}{|\boldsymbol{w}-\boldsymbol{v}|}\cdot\boldsymbol{\omega}\right)|$$
$$\times M(\boldsymbol{v})M(\boldsymbol{w})\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{w}u_{S^2}(\mathrm{d}\boldsymbol{\omega})$$
$$\leq 2\ \|h\|_*^2$$

which yields the conclusion.

A.4 – *Study of* $\boldsymbol{T}$

First, it is an elementary verification that $\boldsymbol{T}_{|\ \Omega_{1,+}} : \Omega_{1,+} \to \Omega_{2,+}$ and $\boldsymbol{T}_{|\ \Omega_{1,-}} : \Omega_{1,-} \to \Omega_{2,-}$ are diffeomorphisms. Then, proving that $\boldsymbol{T}$ transforms $\mathscr{L}_3 \otimes u_{S^2}$ into $\frac{1}{4\pi}|\boldsymbol{x}|^{-2}m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathscr{L}_3(\mathrm{d}\boldsymbol{x})$ is the same as verifying the identities

$$\int_{\Omega_{1,+}} f_+(\boldsymbol{T}(\boldsymbol{z},\boldsymbol{\omega}))\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{\omega} = \int_{\Omega_{2,+}} f_+(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}|^{-2}m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathrm{d}\boldsymbol{x} \tag{33}$$

$$\int_{\Omega_{1,-}} f_-(\boldsymbol{T}(\boldsymbol{z},\boldsymbol{\omega}))\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{\omega} = \int_{\Omega_{2,-}} f_-(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}|^{-2}m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathrm{d}\boldsymbol{x} \tag{34}$$

for every $f_+ : \Omega_{2,+} \to \mathbb{R}$ and $f_- : \Omega_{2,-} \to \mathbb{R}$ continuous with compact support. Since (33) implies (34) after an obvious change of sign, it is enough to deal with the former. The definition of $m_{\boldsymbol{x}}(\,\cdot\,)$ and the Fubini theorem yield the equality

$$\int_{\Omega_{2,+}} f_+(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}|^{-2}m_{\boldsymbol{x}}(\mathrm{d}\boldsymbol{y})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3_+}\int_{\mathbb{R}^2} f_+(\boldsymbol{x},{}^tQ^{(\boldsymbol{x})}\tilde{\boldsymbol{u}})|\boldsymbol{x}|^{-2}\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{x}$$

with $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ and $\tilde{\boldsymbol{u}} := (u_1, u_2, 0) \in \mathbb{R}^3$. Next, use polar coordinates to rewrite the variable $\boldsymbol{x}$ belonging to $\mathbb{R}^3_+$ according to $\boldsymbol{x} = \rho\boldsymbol{\omega}$, with ρ in $(0, +\infty)$ and $\boldsymbol{\omega}$ in S^2_+, and note that $\mathrm{d}\boldsymbol{x}$ changes into $\rho^2\mathrm{d}\rho\mathrm{d}\boldsymbol{\omega}$ and $Q^{(\boldsymbol{x})} = Q^{(\boldsymbol{\omega})}$. Whence,

$$\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^2} f_+(\boldsymbol{x},{}^tQ^{(\boldsymbol{x})}\tilde{\boldsymbol{u}})|\boldsymbol{x}|^{-2}\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{x} = \int_0^{+\infty}\int_{S^2_+}\int_{\mathbb{R}^2} f_+(\rho\boldsymbol{\omega},{}^tQ^{(\boldsymbol{\omega})}\tilde{\boldsymbol{u}})\mathrm{d}\boldsymbol{u}\mathrm{d}\boldsymbol{\omega}\mathrm{d}\rho\ .$$

To conclude, for any fixed $\boldsymbol{\omega}$ in S^2_+, consider the map $\boldsymbol{Z}_\omega : \mathbb{R}^3_+ \to \{\boldsymbol{z} \in \mathbb{R}^3 \mid \boldsymbol{z}\cdot\boldsymbol{\omega} > 0\}$ given by

$$\boldsymbol{Z}_\omega : \begin{pmatrix} u_1 \\ u_2 \\ \rho \end{pmatrix} \mapsto \rho\boldsymbol{\omega} + {}^tQ^{(\omega)}\tilde{\boldsymbol{u}} = {}^tQ^{(\omega)}\begin{pmatrix} u_1 \\ u_2 \\ \rho \end{pmatrix}$$

which is linear and $det(Jac[\boldsymbol{Z}_\omega]) = 1$, in view of the orthogonality of $Q^{(\omega)}$. Putting $\boldsymbol{z} = \boldsymbol{Z}_\omega({}^t(u_1, u_2, \rho))$ yields $\rho = \boldsymbol{z}\cdot\boldsymbol{\omega}$ and

$$\int_0^{+\infty}\int_{\mathbb{R}^2} f_+(\rho\boldsymbol{\omega}, {}^tQ^{(\omega)}\tilde{\boldsymbol{u}})\mathrm{d}\boldsymbol{u}\mathrm{d}\rho = \int_{\{\boldsymbol{z}\in\mathbb{R}^3 \mid \boldsymbol{z}\cdot\boldsymbol{\omega}>0\}} f_+((\boldsymbol{z}\cdot\boldsymbol{\omega})\boldsymbol{\omega}, \boldsymbol{z} - (\boldsymbol{z}\cdot\boldsymbol{\omega})\boldsymbol{\omega})\mathrm{d}\boldsymbol{z}$$

for every $\boldsymbol{\omega}$ in S^2_+. By resorting to the definitions of $\boldsymbol{T}$ and $\Omega_{1,+}$, this last equality leads immediately to (33).

A.5 – *Proof of* (19)

An application of the integral representation $I_0(z) = \dfrac{1}{\pi}\int_0^\pi e^{z\cos\phi}\mathrm{d}\phi$ yields

$$e^{z\cos\gamma\cos\delta}I_0(z\sin\gamma\sin\delta) = \frac{1}{\pi}\int_0^\pi \exp\{z(\cos\gamma\cos\delta + \sin\gamma\sin\delta\sin\phi)\}\mathrm{d}\phi\ .$$

Then, in view of well-known equality

$$e^{zx} = \sqrt{\frac{\pi}{2z}}\sum_{l=0}^\infty (2l+1)I_{l+1/2}(z)P_l(x)\ , \tag{35}$$

valid for every real z and x with $|x| \le 1$, one gets

$$\begin{aligned} e^{z\cos\gamma\cos\delta}I_0(z\sin\gamma\sin\delta) = &\frac{1}{\pi}\sqrt{\frac{\pi}{2z}}\int_0^\pi \Bigg[\sum_{l=0}^\infty (2l+1)I_{l+1/2}(z) \\ &\times P_l(\cos\gamma\cos\delta + \sin\gamma\sin\delta\sin\phi)\Bigg]\mathrm{d}\phi\ . \end{aligned} \tag{36}$$

See Section 11.5 of [17] for details about (35). Now, since $|P_l(x)| \le 1$ for every x in $[-1, 1]$ and the series $\sum\limits_{l=0}^\infty (2l+1)I_{l+1/2}(z)$ is convergent for every real z, it is possible to exchange the integral with the series in the right-hand side of (36).

Whence,

$$
\begin{aligned}
(37)\qquad e^{z\cos\gamma\cos\delta} I_0(z\sin\gamma\sin\delta) =& \frac{1}{\pi}\sqrt{\frac{\pi}{2z}}\sum_{l=0}^{\infty}(2l+1)I_{l+1/2}(z) \\
&\times \int_0^{\pi} P_l(\cos\gamma\cos\delta + \sin\gamma\sin\delta\sin\phi)\mathrm{d}\phi
\end{aligned}
$$

and the conclusion follows by resorting to formula (VII') in Section 19 of Chapter III of [13] – the so-called *addition theorem for Legendre polynomials* – which shows that

$$
\frac{1}{\pi}\int_0^{\pi} P_l(\cos\gamma\cos\delta + \sin\gamma\sin\delta\sin\phi)\mathrm{d}\phi = P_l(\cos\gamma)P_l(\cos\delta)
$$

for every l in $\mathbb{N}$.

A.6 – *Proof of* (21).

Through the change $\cos^2\theta = t$ the integral term in (21) becomes

$$
(38)\qquad \int_0^1 \exp\Big\{-\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t}\Big\} I_{l+1/2}\Big(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\Big)\frac{1}{2t\sqrt{t(1-t)}}\mathrm{d}t
$$

and, after fixing ε in $(0,1)$ in such a way that $(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)(1-\varepsilon) > 2|\boldsymbol{r}|\,|\boldsymbol{v}|$, the integral in (38) can be split into two integrals as

$$
(39)\qquad \Big(\int_0^{\varepsilon} + \int_{\varepsilon}^{1}\Big)\exp\Big\{-\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t}\Big\} I_{l+1/2}\Big(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\Big)\frac{1}{2t\sqrt{t(1-t)}}\mathrm{d}t\ .
$$

Now, to avoid trivialities, assume that both $|\boldsymbol{r}|$ and $|\boldsymbol{v}|$ are different from zero and start by studying the former integral to obtain

$$
\begin{aligned}
&\int_0^{\varepsilon} \exp\Big\{-\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t}\Big\} I_{l+1/2}\Big(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\Big)\frac{1}{2t\sqrt{t(1-t)}}\mathrm{d}t \\
(40)\qquad &\leq \frac{1}{2\sqrt{1-\varepsilon}}\int_0^{+\infty} \exp\Big\{-\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-\varepsilon}{t}\Big\} I_{l+1/2}\Big(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\Big) t^{-3/2}\mathrm{d}t \\
&= \frac{1}{2\sqrt{1-\varepsilon}}\int_0^{+\infty} \exp\Big\{-\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)(1-\varepsilon)\tau\Big\} I_{l+1/2}(|\boldsymbol{r}|\,|\boldsymbol{v}|\tau)\tau^{-1/2}\mathrm{d}\tau\ .
\end{aligned}
$$

Since the last member in (40) can be viewed as a Laplace transform, formula (7) in Subsection 3.15.1 of [12] can be applied to conclude that

$$
\begin{aligned}
&\int_0^{\varepsilon} \exp\left\{ -\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t} \right\} I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\right) \frac{1}{2t\sqrt{t(1-t)}}\mathrm{d}t \\
&\leq \frac{1}{2}\sqrt{\frac{2}{\pi(1-\varepsilon)|\boldsymbol{r}|\,|\boldsymbol{v}|}}\, Q_l(\xi)
\end{aligned}
\tag{41}
$$

where $\xi := \dfrac{(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)(1-\varepsilon)}{2|\boldsymbol{r}|\,|\boldsymbol{v}|} > 1$ and Q_l stands for the *Legendre function of the second kind of order l*, defined, for example, in Section 10.10 of [7]. At this stage, use (31) in Section 10.10 of [7] to deduce that

$$
|Q_l(\xi)| \leq \frac{(\xi - \sqrt{\xi^2 - 1})^l}{\xi - 1}
$$

and, since $0 < \xi - \sqrt{\xi^2 - 1} < 1$, it is immediate to conclude that

$$
\sum_{l=0}^{\infty} (2l+1) \int_0^{\varepsilon} \exp\left\{ -\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t} \right\} I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\right) \frac{1}{2t\sqrt{t(1-t)}}\,\mathrm{d}t < +\infty\ .
$$

Finally, the monotonicity of the restriction of each $I_{l+1/2}$ to $[0, +\infty)$ entails

$$
\begin{aligned}
&\int_{\varepsilon}^{1} \exp\left\{ -\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t} \right\} I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\right) \frac{1}{2t\sqrt{t(1-t)}}\,\mathrm{d}t \\
&\leq \frac{1}{2}\varepsilon^{-3/2} I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{\varepsilon}\right) \int_0^1 \frac{1}{\sqrt{1-t}}\mathrm{d}t
\end{aligned}
$$

which, in view of (35), leads to

$$
\sum_{l=0}^{\infty} (2l+1) \int_{\varepsilon}^{1} \exp\left\{ -\frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{r}|^2)\frac{1-t}{t} \right\} I_{l+1/2}\left(\frac{|\boldsymbol{r}|\,|\boldsymbol{v}|}{t}\right) \frac{1}{2t\sqrt{t(1-t)}}\,\mathrm{d}t < +\infty\,.
$$

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Received November 30, 2010 and in revised form December 16, 2010