Bollettino
Unione Matematica Italiana

Tuomo Kuusi, Giuseppe Mingione

Endpoint and Intermediate Potential Estimates for Nonlinear Equations

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **4** (2011), n.1, p. 149–157.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2011_9_4_1_149_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2011.

Bollettino U. M. I.
(9) **IV** (2011), 149-157

Endpoint and Intermediate Potential Estimates for Nonlinear Equations

TUOMO KUUSI - GIUSEPPE MINGIONE

Abstract. – *We describe a few results obtained in* [10], *concerning the possibility of estimating solutions of quasilinear elliptic equations via nonlinear potentials.*

1. – Introduction.

Let us consider the Poisson equation $-\triangle u = \mu$ in $\mathbb{R}^n$, and let us take here $n \geq 3$, with μ being an integrable function. The well-known representation formula via fundamental solutions

$$u(x) = \frac{1}{n(n-2)|B_1|} \int_{\mathbb{R}^n} \frac{d\mu(\xi)}{|x-\xi|^{n-2}} \,, \tag{1.1}$$

allows to establish many of the basic properties of the solution u and of its gradient Du, as for instance integrability in various functions spaces, via the analysis of related Riesz potentials. Indeed, let us define for a general Radon measure μ the Riesz potential I^{μ}_{β} as (here we omit the usual renormalizing constant)

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(\xi)}{|x-\xi|^{n-\beta}} \,, \qquad \beta \in (0, n]$$

Then the following formulae hold:

$$|u(x)| \leq c(n) I_2(|\mu|)(x) \qquad \text{and} \qquad |Du(x)| \leq c(n) I_1(|\mu|)(x) \,. \tag{1.2}$$

Although the possibility to estimate pointwise solutions by means of potentials seems to be linked to the specific structure of the Poisson equation, in recent years it has been shown that similar pointwise estimates still hold when dealing with possibly degenerate quasilinear equations. We shall consider quasilinear equations in divergence form of the type

$$-\operatorname{div} a(x, Du) = \mu \tag{1.3}$$

under the assumptions

$$\begin{cases} |a(x,z)| + |\partial a(x,z)|(|z|^2+s^2)^{1/2} \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\lambda|^2 \leq \langle \partial a(x,z)\lambda, \lambda\rangle\,, \end{cases} \tag{1.4}$$

where $p > 2 - 1/n$ and $0 < \nu \leq L < \infty$. These are supposed to hold whenever $x \in \Omega$ and $z, \lambda \in \mathbb{R}^n$; the symbol ∂a denotes the gradient of $a(\,\cdot\,)$ with respect to the gradient variable z. Finally, μ always denotes a Radon measure which we for simplicity think to be defined on the whole space $\mathbb{R}^n$. The main model here is of course given by the p-Laplacean equation with coefficients $-\mathrm{div}(\gamma(x)|Du|^{p-2}Du) = \mu$. However, the full significance of the results presented here appears in the nonlinear situation already when $p = 2$.

By now classical theorems from nonlinear potential theory allow for pointwise estimates of solutions to (1.3) in terms of the (truncated) Wolff potential $\boldsymbol{W}^{\mu}_{\beta,p}(x,R)$ defined by

$$\boldsymbol{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta > 0\,. \tag{1.5}$$

Needless to say $B(x,\varrho) \subset \mathbb{R}^n$ denotes the open ball centered at x, with radius ϱ. Wolff potentials reduce to the standard (truncated) Riesz potentials when $p = 2$

$$\boldsymbol{W}^{\mu}_{\beta/2,2}(x,R) = \boldsymbol{I}^{\mu}_{\beta}(x,R) = \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}\,, \qquad \beta > 0\,, \tag{1.6}$$

with the first equality being true for nonnegative measures. In fact, a fundamental fact due to Kilpeläinen & Malý [7] – later deduced by different approaches by Trudinger & Wang in [17, 18] – is the estimate

$$|u(x)| \leq c\boldsymbol{W}^{\mu}_{1,p}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, d\xi\,, \tag{1.7}$$

valid whenever $B(x,R) \subset \Omega$, with x being a Lebesgue point of u. Later on, this result has been upgraded to the gradient level in [4, 15], where the authors proved the estimate

$$|Du(x)| \leq c\boldsymbol{W}^{\mu}_{1/p,p}(x,R) + c \fint_{B(x,R)} (|Du| + s)\, d\xi\,. \tag{1.8}$$

Estimates (1.7) and (1.8) are the nonlinear counterparts of the linear estimates (1.2). The results we are presenting here – an excerpt from [10] – go beyond estimates (1.7)-(1.8) presenting a unified approach to the regularity of possibly degenerate quasilinear equations and, in particular, yielding oscillation estimates.

2. – Full scale and endpoint estimates.

In this note, for the sake of simplicity, we shall consider the slightly simpler case given by equations with splitting coefficients of the type

$$-\operatorname{div}(\gamma(x)a(Du)) = \mu \tag{2.1}$$

while we refer to [10] for the more general results concerning operators of the type (1.3). The vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ will satisfy assumptions (1.4) (obviously restated for the case when $a(\cdot)$ has no x-dependence) while $\gamma\colon \Omega \to \mathbb{R}$ is a bounded measurable function such that $\nu \leq \gamma(x) \leq L$ for a.e. $x \in \Omega$. We shall denote by $\omega(\cdot)$ the "integral modulus of continuity" of $\gamma(\cdot)$:

$$\omega(R) := \sup_{B_\varrho \subset \Omega, \varrho \leq R} \fint_{B_\varrho} |\gamma(x) - (\gamma)_{B_\varrho}|\, dx\,, \tag{2.2}$$

where $(\gamma)_{B_\varrho}$ denotes the integral average of $\gamma(\cdot)$ over B_ϱ (as usual B_ϱ denotes a ball with radius $\varrho > 0$). Now, to start with, consider again the Poisson equation $-\triangle u = \mu$, (we take $n \geq 3$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying $|u(x)| \leq c|x|^{2-n}$ asymptotically as $|x| \to \infty$; such a condition is satisfied for instance when μ has compact support). The formula (1.1) also gives

$$|u(x) - u(y)| \leq c[I_{2-\alpha}(|\mu|)(x) + I_{2-\alpha}(|\mu|)(y)]|x-y|^\alpha \tag{2.3}$$

whenever $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$. We can therefore read (2.3) as an intermediate relation between the two ones in (1.2), which is moreover endpoint in the sense that it allows – up to additional constants – to go back to the two estimates in (1.2) when $\alpha \to 0$ and $\alpha \to 1$. The additional feature of an estimate as (2.3) is clearly that it enables to catch *oscillation information on the solutions* by prescribing the regularity of potentials, ultimately allowing to get regularity properties of continuity in various function spaces.

The main aim of [10] is to show that something similar happens in the quasilinear case and that *estimates* (1.7) *and* (1.8) *are particular instances of more general estimates*, exactly as it happens in the linear case by (2.3), *allowing to get intermediate pointwise estimates of fractional derivatives via potentials.* There are actually several ways to express the concept of fractional differentiability. It might appear at the beginning vague to extend pointwise estimates (1.7)-(1.8) to fractional derivatives, as these are obviously non-local objects. We shall here use a *notion of fractional differentiability* introduced by DeVore & Sharpley [3] that allows to describe fractional derivatives reducing the non-locality of the definition to a minimal status, i.e. using two points only.

DEFINITION 1. – *Let $\alpha \in (0,1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function v, finite a.e. in Ω, belongs to the Calderón space*

$C^a_q(\Omega)$ *if and only if there exists a nonnegative function* $m \in L^q(\Omega)$ *such that*

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^a \tag{2.4}$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$.

Such spaces are closely related to the usual fractional Sobolev spaces $W^{a,q}$ (see [3]). Of course there could be more than one function $m(\cdot)$ working in (2.4). For this reason in their original paper DeVore & Sharpley fix $m(\cdot)$ to be the sharp fractional maximal operator of order a of v, i.e. $m = M^{\#}_a(v)$, (see [10]). Indeed, notice that it follows from the definitions that the validity of (2.4) for some $m \in L^q$ is equivalent to have $M^{\#}_a(v) \in L^q$ whenever $q > 1$. The main thing we are here interested in is the fact that (2.4) *allows to identify* $m(\cdot)$ *as "a fractional derivative of order* a*"* for v.

It is important to note here that while (1.7) holds true when the dependence on $x \mapsto a(x, \cdot)$ is just measurable, estimate (1.8) necessitates more regularity from the mapping $x \mapsto a(x, \cdot)$. Indeed, (1.8) implies the gradient boundedness for regular enough measures, for which plain continuity of coefficients is known to be insufficient, while for instance Dini continuity suffices. For this reason, when considering (2.1), intermediate moduli of continuity of $x \mapsto \gamma(x)$ will appear in the next statements according to the kind of estimates considered.

DEFINITION 2. – *With* $\omega(\cdot)$ *defined in* (2.2), *the function* $\gamma(\cdot)$ *will be called* VMO*-regular if* $\omega(r) \to 0$ *when* $r \to 0$.

In this note we shall formulate our results in the form of a priori estimates for more regular (C^0, C^1) solutions. The case of general solutions (very weak solutions) can be dealt with by well established approximation methods and integral a priori estimates. We refer for instance for [2, 4] for a description.

2.1 – *The case* $p \geq 2$.

Here, in order to strengthen the exposition, we shall confine ourselves to treat the case $p \geq 2$. The first result we present upgrades estimate (1.7) to low order fractional derivatives, and actually holds in the case $p < 2$ as well. In fact, our aim here is also to demonstrate a sharp connection between classical DeGiorgi's theory and nonlinear potential estimates. Indeed, when considering solutions to homogeneous equations as $\operatorname{div} a(x, Dw) = 0$, with measurable dependence on x, DeGiorgi's theory provides the existence of a *universal Hölder continuity exponent* $a_m \in (0, 1)$, depending only on n, p, ν, L, such that

$$w \in C^{0,a_m}_{\mathrm{loc}}(\Omega), \qquad |w(x) - w(y)| \leq c \fint_{B_R} (|w| + Rs)\, dx \cdot \left(\frac{|x - y|}{R}\right)^{a_m}, \tag{2.5}$$

where the last inequality holds whenever $x, y \in B_{R/2}$ and $B_R \subset \Omega$. The exponent α_m can be thought as the *maximal Hölder regularity exponent* associated to the vector field $a(\cdot)$, and is actually universal in that it is even independent of $a(\cdot)$ and depends only on n, p, ν, L. It then holds

THEOREM 2.1 (DeGiorgi's theory via potentials). – *Let $u \in C^0(\Omega) \cap W^{1,p}(\Omega)$ be a weak solution to the equation* (2.1) *with measurable coefficients $\gamma(\cdot)$, and let* (1.4) *hold with $p > 2 - 1/n$. Let $B_R \subset \Omega$ be such that $x, y \in B_{R/8}$, then*

$$
\begin{aligned}
|u(x) - u(y)| \leq c\Big[\boldsymbol{W}^{\mu}_{1-\alpha(p-1)/p,p}(x, R) + \boldsymbol{W}^{\mu}_{1-\alpha(p-1)/p,p}(y, R)\Big]|x - y|^{\alpha} \\
+ c\fint_{B_R}(|u| + Rs)\, d\xi \cdot \left(\frac{|x - y|}{R}\right)^{\alpha}
\end{aligned}
\tag{2.6}
$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, for every $\tilde{\alpha} < \alpha_m$, where the constant c depends only n, p, ν, L and $\tilde{\alpha}$.

In general, counterexamples show that $\alpha_m \to 0$ when $L/\nu \to \infty$, and this prevents estimate (2.6) to hold in general for the full range $\alpha \in [0, 1)$ when in presence of measurable coefficients. Let us remark that the restriction to the case $2 - 1/n < p$ is motivated by the fact that this is the range in solutions to measure data problems belong to the Sobolev space $W^{1,1}$, and we can talk about the usual gradient. In this respect the lower bound $p > 2 - 1/n$ is optimal as showed by the (so called nonlinear fundamental) solution $|x|^{\frac{p-n}{p-1}}$ (here $n \neq p$ for simplicity) to the equation $-\triangle_p u = c(n, p)\delta$, where δ is the Dirac measure charging the origin.

In order to prove estimates for higher order fractional derivatives we shall need more regularity on coefficients.

THEOREM 2.2 (Fractional nonlinear potential bound). – *Let $u \in C^1(\Omega)$ be a weak solution to* (2.1), *under the assumptions* (1.4) *with $p > 2 - 1/n$. For every $\tilde{\alpha} < 1$ there exists a positive number $\delta \equiv \delta(n, p, \nu, L, \tilde{\alpha})$ such that if*

$$
\lim_{r \to 0} \omega(r) \leq \delta, \tag{2.7}
$$

then the pointwise estimate (2.6) *holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, for a constant $c \equiv c(n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, \operatorname{diam}(\Omega))$, as soon as $x, y \in B_{R/8}$. In particular, if $\gamma(\cdot)$ is* VMO*-regular then* (2.6) *holds whenever $\alpha < 1$.*

Estimate (2.6) fails for the case $\alpha = 1$, already when considering continuous coefficients. Instead, a form of Dini continuity must be assumed as follows:

THEOREM 2.3 (Full interpolation estimate). – *Let $u \in C^1(\Omega)$ be a weak solution to* (2.1) *under the assumptions* (1.4) *with $p \geq 2$, and assume also that*

$$\int_0^r [\omega(\varrho)]^{2/p} \frac{d\varrho}{\varrho} < \infty \qquad \forall\, r < \infty\,. \tag{2.8}$$

Then (2.6) *holds uniformly $a \in [0,1]$, whenever $B_R \subset \Omega$ is a ball such that $x, y \in B_{R/8}$, where $c \equiv c(n, p, \nu, L, \omega(\,\cdot\,), \mathrm{diam}(\Omega))$.*

REMARK 2.1 (Endpoint/Interpolation nature of the estimates). – A main feature in the previous theorem is the *endpoint nature* of estimate (2.8), that holds uniformly up to including the borderline cases (1.7)-(1.8) (up to additive constants) when this is allowed by the regularity of $\gamma(\,\cdot\,)$.

We finally move towards the maximal regularity of the operator in (2.1). When considering the homogeneous equation $\operatorname{div} a(Dv) = 0$, a version of DeGiorgi's theory is again available – see [11] for a beautiful presentation – ultimately leading to the existence of a *universal maximal regularity exponent* $a_M \in (0,1)$, depending only on n, p, ν and L such that whenever $x, y \in B_{R/4}$,

$$Dv \in C^{0,a_M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)\,, \quad |Dv(x) - Dv(y)| \leq c \fint_{B_R} (|Dv| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{a_M} \tag{2.9}$$

holds *for any local solution v*. Accordingly, we have

THEOREM 2.4 (Gradient fractional bound). – *Let $u \in C^1(\Omega)$ be a weak solution to* (2.1), *under the assumptions* (1.4) *with $p \geq 2$, and assume that for $\tilde{a} < a_M$ it holds that*

$$S := \sup_r \int_0^r \frac{[\omega(\varrho)]^{2/p}}{\varrho^{\tilde{a}}} \frac{d\varrho}{\varrho} < \infty\,. \tag{2.10}$$

Then the pointwise estimate

$$\begin{aligned} |Du(x) - Du(y)| \leq{}& c\left[\boldsymbol{W}^{\mu}_{1-\frac{(1+a)(p-1)}{p},p}(x,R) + \boldsymbol{W}^{\mu}_{1-\frac{(1+a)(p-1)}{p},p}(y,R)\right]|x-y|^a \\ & + c\fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^a \end{aligned} \tag{2.11}$$

holds uniformly in $a \in [0, \tilde{a}]$, whenever $x, y \in \Omega$ and $B_R \subset \Omega$ is a ball such that $x, y \in B_{R/4}$, for a constant c depending only on $n, p, \nu, L, \omega(\,\cdot\,), \tilde{a}, S$ and $\mathrm{diam}(\Omega)$.

3. – Further main results from [10].

3.1 – *The case* $2 - 1/n < p \leq 2$.

While Theorems 2.1 and 2.2 still apply to the case $2 - 1/n < p \leq 2$, estimates leading to assertions on the gradient show up in a different form. More precisely, instead of Wolff potentials, Riesz potentials come into the play again, exactly as in the linear case. This fact has been already observed in [5], where the following estimate has been proved:

$$|Du(x)| \leq c\Big[\boldsymbol{I}_1^{|\mu|}(x,R)\Big]^{1/(p-1)} + c \fint_{B(x,R)} (|Du| + s)\, d\xi\,, \tag{3.1}$$

whenever $B(x,R) \subset \Omega$, provided $2 - 1/n < p < 2$. We refer to [10] for oscillation estimates in the subquadratic case and to [5, 6] for estimates using linear potentials.

3.2 – *Maximal estimates.*

A few preliminary lemmas useful in the proofs of some of the nonlinear potentials estimates in Section 2.1 are concerned with the pointwise estimates of a certain fractional maximal operators. For instance in [10] we prove that if the coefficient function $\gamma(\,\cdot\,)$ is VMO-regular then the estimate

$$M_{1-a,R}(Du)(x) \leq c\big[M_{p-a(p-1),R}(\mu)(x)\big]^{1/(p-1)} + cR^{1-a} \fint_{B_R} (|Du| + s)\, d\xi \tag{3.2}$$

holds whenever $B(x,R) \subset \Omega$ and $a < 1$. Here we recall that, for $\beta \in [0, n]$, the fractional maximal operator is defined via

$$\begin{aligned} M_{\beta,R}(f)(x) &:= \sup_{0<r\leq R} r^{\beta} \fint_{B(x,r)} |f|\, d\xi \\ &= \sup_{0<r\leq R} r^{\beta} \frac{|f|(B(x,r))}{|B(x,r)|} \qquad \text{in the case } f \text{ is a measure}\,. \end{aligned}$$

Estimate (3.2) can also be used to derive a non-endpoint (i.e. no stability of the constants when $a \to 0$ or $a \to 1$) and alternative form of estimate (2.6).

3.3 – *Some consequences and applications.*

The first application we present is about local regularity in fractional Sobolev spaces, and, in particular, in Hölder spaces. We recall that a function $v \in L^q(A)$

belongs to the Nikolskii space $N^{a,q}(A)$ for $a \in [0,1]$ and $q \geq 1$ iff

$$[v]^q_{a,q;A} := \sup_{|\mathrm{e}|=1} \sup_{|h|\neq 0} \int_{A_h} \frac{|v(x+h\mathrm{e}) - v(x)|^q}{|h|^{aq}}\, dx < \infty\,, \tag{3.3}$$

where $A \subset \mathbb{R}^n$ is an open subset and $A_h \subset A$ denotes the subset of A consisting of all point having distance to the boundary larger than $|h|$. Estimates in such spaces reveal to be crucial in several contexts; we mention, amongst those related to the present setting, recent applications to estimates of singular sets of vectorial problems ([8, 9, 12]) and to the differentiability properties of very weak solutions ([13, 14]). We observe that $N^{a,\infty} \equiv C^{0,a}$, so that estimates in this class of spaces imply those in Hölder spaces, and therefore *nonlinear Schauder estimates*.

By using for instance Theorem 2.3 we see that under the assumptions considered there we have, up to a standard covering argument, that the estimate

$$[u]_{a,q;B_{R/2}} \leq c\|\boldsymbol{W}_{1-a(p-1)/p,p}(\cdot,R)\|_{L^q(B_R)} + \frac{c}{R^a}\int_{B_R} (|u| + Rs)\, dx \tag{3.4}$$

holds with a constant c depending only on n, p, v, L. The previous estimate tells us that in order to look for fractional differentiability one can confine himself to require the needed integrability properties of the potential. In turn, via (3.6) below, this immediately yields the necessary integrability assumptions on μ. Indeed, let us recall that the Wolff potential is dominated by the so called Havin-Mazya potential, that is the composition of standard Riesz potentials appearing on the right hand side of the next inequality

$$\boldsymbol{W}_{\beta,p}(x,R) \leq I_\beta\left[I_\beta(|\mu|)^{1/(p-1)}\right](x)\,, \qquad \beta p < n\,, \qquad R > 0\,. \tag{3.5}$$

In turn, the last inequality implies for instance bounds in Lebesgue spaces:

$$\|\boldsymbol{W}^\mu_{\beta,p}\|_{L^{\frac{n\gamma(p-1)}{n-\beta\gamma p}}(\Omega)} \leq c\|\mu\|_{L^\gamma(\Omega)}\,, \qquad \beta\gamma p < n\,, \tag{3.6}$$

in any open subset $\Omega \subset \mathbb{R}^n$; similar bounds are actually available in several other rearrangement invariant functions spaces.

For further applications we again refer the reader to [10] while for recent developments and an overview on the problems treated here we quote the recent survey [16].

Acknowledgments. The authors are supported by the ERC grant 207573 “Vectorial Problems” and by the Academy of Finland project “Potential estimates and applications for nonlinear parabolic partial differential equations”.

REFERENCES

[1] R. A. Adams - J. J. F. Fournier, *Sobolev Spaces. Second edition*, Pure and Appl. Math., **140**, Elsevier/Academic Press, Amsterdam, 2003.
[2] L. Boccardo - T. Gallouët, *Nonlinear elliptic equations with right-hand side measures*, Comm. PDE, **17** (1992), 641-655.
[3] R. A. De Vore - R. C. Sharpley, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., **47**, no. 293 (1984).
[4] F. Duzaar - G. Mingione, *Gradient estimates via nonlinear potentials*, Amer. J. Math., to appear.
[5] F. Duzaar - G. Mingione, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal., **259** (2010), 2961-2998.
[6] F. Duzaar - G. Mingione, *Local Lipschitz regularity for degenerate elliptic systems*, Ann. Inst. H. Poincaré Anal. Non Linèaire, **27** (2010), 1361-1396.
[7] T. Kilpeläinen - J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137-161.
[8] J. Kristensen - G. Mingione, *The singular set of minima of integral functionals*, Arch. Ration. Mech. Anal., **180** (2006), 331-398.
[9] J. Kristensen - G. Mingione, *Boundary regularity in variational problems*, Arch. Rational Mech. Anal., **198** (2010), 369-455.
[10] T. Kuusi - G. Mingione, *Universal potential estimates*, Preprint 2010.
[11] J. J. Manfredi, *Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations*, Ph. D. Thesis, University of Washington, St. Louis.
[12] G. Mingione, *Bounds for the singular set of solutions to non linear elliptic systems*, Calc. Var. PDE, **18** (2003), 373-400.
[13] G. Mingione, *The Calderón-Zygmund theory for elliptic problems with measure data*, Ann. Scu. Norm. Sup. Pisa Cl. Sci. (V), **6** (2007), 195-261.
[14] G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann., **346** (2010), 571-627.
[15] G. Mingione, *Gradient potential estimates*, J. European Math. Soc., **13** (2011), 459-486.
[16] G. Mingione, *Nonlinear aspects of Calderón-Zygmund theory*, Jahresbericht der DMV, **112** (2010), 159-191.
[17] N. S. Trudinger - X. J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, Amer. J. Math., **124** (2002), 369-410.
[18] N. S. Trudinger - X. J. Wang, *Quasilinear elliptic equations with signed measure data*, Disc. Cont. Dyn. Systems, **124** (2002), 369-410.

Tuomo Kuusi, Aalto University Institute of Mathematics
P.O. Box 11100 FI-00076 Aalto, Finland
E-mail: tuomo.kuusi@tkk.fi

Giuseppe Mingione, Dipartimento di Matematica, Università di Parma
Parco Area delle Scienze 53/a, Campus, 43100 Parma, Italy
E-mail: giuseppe.mingione@unipr.it.

Received October 3, 2010 and in revised form October 12, 2010