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Abstract. – *For $\Omega \subset \mathbb{R}^n, n \geq 2$, and $N \geq 1$ we consider vector valued minimizers $u \in W^{m,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of a uniformly convex integral functional of the type*

$$\mathcal{F}[u, \Omega] := \int_\Omega f(x, D^m u)\, dx,$$

where f is a Carathéorody function satisfying $p(x)$ growth conditions with respect to the second variable. We show that if the dimension n is small enough, dependent on the structure conditions of the functional, there holds

$$D^k u \in C^{0,\beta}_{loc}(\Omega) \text{ for } k \in \{0, \dots, m-1\},$$

for some β, also depending on the structural data, provided that the nonlinearity exponent p is uniformly continuous with modulus of continuity ω satisfying

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) = 0.$$

1. – Introduction.

The manuscript on hand is concerned with regularity results for vector valued minimizers $u \in W^{m,p(\cdot)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $N \geq 1$ of convex integral functionals of order $m \geq 1$. More precisely we consider integral functionals of the type

$$\mathcal{F}(u) := \int_\Omega f(x, D^m u)\, dx\,, \tag{1.1}$$

on a bounded open set $\Omega \subset \mathbb{R}^n, n \geq 2$. In this context, f denotes a convex integrand function, satisfying non standard $p(x)$ growth conditions (see 2.4). The aim of this paper is to show, that under an optimal condition on the modulus of continuity of the exponent function p, provided that the dimension n is small enough, the derivative $D^k u$ is locally Hölder continuous everywhere on Ω, with exponent that depends on the global lower bound of the exponent function p.

Problems with $p(x)$ growth became of increasing interest within the end of the nineties, basically for two reasons: On one hand, they are interesting from the mathematical point of view, since they represent a borderline case between standard p growth conditions with constant p and so-called $p-q$ growth conditions, originally introduced by Marcellini [21]. On the other hand, a number of applications in mathematical physics, as for example the modeling of electro-rheological fluids introduced by Růžička [23] or image-processing models introduced by Chen, Levine and Rao [6], involve equations and energy-functionals with structures of $p(x)$ growth type. In a first step, convexity of the integrand, as $(2.4)_2$ provides the existence of a unique local minimizer (to a given boundary data) in the generalized Sobolev space $W^{m,p(\cdot)}_{\text{loc}}(\Omega, \mathbb{R}^N)$, the space of all measurable maps $u : \Omega \to \mathbb{R}^N$, whose distributional derivatives $D^a u$ for any mulitindex a of order $|a| \le m$ belong to the generalized Lebesgue space

$$L^{p(\cdot)}_{\text{loc}}(\Omega, \mathbb{R}^N) := \left\{ g : \Omega \to \mathbb{R}^N : \quad \int_K |g|^{p(x)}\,dx < +\infty, \text{ for all } K \Subset \Omega \right\}.$$

Properties of the generalized spaces $L^{p(\cdot)}(\Omega)$, $W^{m,p(\cdot)}(\Omega)$ and their local versions are interesting by themselves and have been intensively studied by a variety of authors. Just to mention some of them at this point, we refer the reader for example to [20, 22, 11, 12, 23, 9].

Investigations in regularity theory for functionals and equations with $p(x)$ growth started with the paper of Zhikov [24], showing that in the case of the Dirichlet $p(x)$ energy functional

$$\mathcal{F}(u) = \int_\Omega |Du|^{p(x)}\,dx,$$

higher integrability of the local minimizer can be achieved, provided that the exponent function $p : \Omega \to (1, +\infty)$ is "logarithmic Hölder continuous", i.e. that there holds

$$|p(x) - p(y)| \le \frac{c}{-\log|x-y|}, \qquad \text{for all } x, y \in \Omega, \text{ with } |x-y| \le \frac{1}{2}. \tag{1.2}$$

Furthermore, Zhikov showed that the failure of (1.2) causes the loss of hardly any type of higher regularity. Starting from the point of higher integrability, localization and freezing techniques allow to prove higher regularity, such as $C^{0,a}$ or $C^{1,a}$ regularity in the scalar case ($N = 1$), provided that the modulus of continuity of the exponent function satisfies stronger assumptions than (1.2), so for example (2.6) to gain Hölder continuity of the minimizer u. This was proved in 2001 by Acerbi & Mingione [1]. To achive analogue results for the derivative Du, counter examples show that one has to assume Hölder continuity of p itself (see [1] for the regularity proof). There have been a large amount of generalizations of the results in [1] to the situation of more general integrands (see for example [13]) or functionals involving derivatives of higher order, as done in [18].

In 2001, Acerbi & Mingione [1] proved $C^{0,a}$ regularity for scalar functionals with $p(x)$ growth, under the continuity condition (2.6) which is slightly stronger than (1.2). Condition (2.6) appears in a large amount of regularity proofs and turns out to be sufficient to prove that local minimizers u are in fact locally Hölder continuous to any Hölder exponent $a \in (0,1)$ (see also [14] for Hölder continuity in case of one sided obstacle problems). On the other hand, the existence of one (small) Hölder continuity exponent $a \in (0,1)$ can be shown under the weaker condition (1.2), which was done in [16]. As Zhikov [24] showed, (1.2) is optimal in the sense that it suffices to prove higher integrability of local minimizers, whereas hardly any type of regularity – and especially higher integrability in the sense of Lemma 5.1 – fails if (1.2) is violated. In the case that the exponent function p itself is Hölder continuous, even the gradient Du of minimizers can be shown to be Hölder continuous (see [7] for the Dirichlet $p(x)$ integral and [1] for more general functionals). More recently, gradient estimates of Calderón-Zygmund type for solutions of nonlinear elliptic systems with $p(x)$ growth structure were proved under condition (2.6) (see [4] for general equations and the $p(x)$ Laplacean system; [19] for more general elliptic systems with $p(x)$ growth).

However, in the case of vector valued minimization problems ($N > 1$) or systems of PDEs, classical counter examples show that (local) $C^{0,a}$ or $C^{1,a}$ regularity on whole Ω in general cannot hold. Instead, one may attain only partial regularity, i.e. regularity on an open subset $\Omega_0 \subset \Omega$ of full Lebesgue measure. For functionals with $p(x)$ growth structure, partial regularity was first proved in [2]. For the generalization to higher order functionals we refer the reader to [18].

Nevertheless, in the situation that the dimension n is small enough with respect to the structure data, it is possible to prove full regularity even in the vectorial case, a fact which was first shown by Campanato [5] for nonlinear elliptic systems with constant p growth. The result proved in this paper is the analogue to Campanato's result – in the context of minimizers of integral functionals instead of solutions of systems – for functionals with nonstandard $p(x)$ growth structure.

2. – Notations and Setting.

We consider local minimizers $u \in W^{m,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of the higher order functional

$$\mathcal{F}[u, \Omega] \equiv \int_\Omega f(x, D^m u)\, dx, \tag{2.3}$$

where $f : \Omega \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ with $\mathcal{N} := \binom{n+m-1}{m}$ denotes a Carathéodory function, C^2 with respect to the second variable. Denoting by $p : \Omega \to (1, +\infty)$ a continuous exponent function, we impose the following nonstandard growth, convexity and continuity conditions on the integrand f:

$$
(2.4)\qquad \begin{cases} L^{-1}\Big(1+|z|^2\Big)^{\frac{p(x)}{2}} \le f(x,z) \le L\Big(1+|z|^2\Big)^{\frac{p(x)}{2}}, \\ L^{-1}\Big(1+|z|^2\Big)^{\frac{p(x)-2}{2}}|\lambda|^2 \le \langle f_{zz}(x,z)\lambda,\lambda\rangle \le L\Big(1+|z|^2\Big)^{\frac{p(x)-2}{2}}|\lambda|^2, \\ |f(x,z)-f(x_0,z)| \le L\omega(|x-x_0|)\Big[\Big(1+|z|^2\Big)^{\frac{p(x)}{2}}+\Big(1+|z|^2\Big)^{\frac{p(x_0)}{2}}\Big] \\ \qquad\qquad\qquad \cdot\Big[1+\log\Big(1+|z|^2\Big)\Big], \end{cases}
$$

for all $x, x_0 \in \Omega$, $z, \lambda \in \mathbb{R}^{\mathcal{N}}$, with a constant $L \ge 1$ and with a function $\omega : [0,\infty) \to [0,\infty)$ that is continuous, non decreasing, and satisfies $\omega(0) = 0$. Condition $(2.4)_1$ describes a non-degenerate nonstandard growth condition, whereas condition $(2.4)_2$ on the second derivative of f expresses the convexity of the integrand, adopted with respect to the growth condition $(2.4)_1$. $(2.4)_3$ represents uniform continuity of the integrand with respect to the space variable x, which is expressed in terms of the modulus of continuity ω. Certainly we impose the same continuity assumption also on the exponent function p, i.e.

$$
|p(x)-p(y)| \le \omega(|x-y|),
$$

for all $x, y \in \Omega$. Without loss of generality we assume ω to be concave with $\omega(\rho) \le 1$ for all $\rho > 0$. Note here that since ω is concave and hence sublinear we shall very often use

$$
\omega(tr) \le t\,\omega(r), \quad \omega(r+s) \le \omega(r)+\omega(s),
$$

for all $r, s \ge 0$ and $t \ge 1$. Additionally, since our results are local in nature, we will assume for the whole paper that

$$
(2.5)\qquad 1<\gamma_1 \le p(x) \le \gamma_2 < +\infty,
$$

for all $x \in \Omega$. We adapt the following notion of a local minimizer of the functional (2.3):

DEFINITION 2.1. – *$u \in W^{m,1}_{\mathrm{loc}}(\Omega)$ is called a local minimizer of the functional $\mathcal{F}$, if $|D^m u|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$
\mathcal{F}[u, \operatorname{spt}\phi] \le \mathcal{F}[u+\phi, \operatorname{spt}\phi],
$$

for any $\phi \in W^{m,1}_0(\Omega, \mathbb{R}^N)$ with $|D^m\phi|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and $\operatorname{spt}\phi \Subset \Omega$.

Our aim is to prove the following

THEOREM 2.1. – *Let $u \in W^{m,1}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, satisfying the structure conditions* (2.4)*, let $p : \Omega \to [1,\infty)$ be an exponent* *function which satisfies* (2.5) *and whose modulus of continuity fulfills*

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) = 0. \tag{2.6}$$

Then there exists $\varepsilon > 0$ *such that if*

$$n < (m-k)\gamma_1 + 2 + \varepsilon, \tag{2.7}$$

then for the derivative of u *of order* $k \in \{0, \dots, m-1\}$ *there holds*

$$D^k u \in C^{0,\beta}_{loc}(\Omega) \text{ with } \beta = m - k - \frac{n-(2+\varepsilon)}{\gamma_1}, \quad \textit{if } n > (m-k-1)\gamma_1 + 2 + \varepsilon,$$
$$D^k u \in C^{0,\beta}_{loc}(\Omega) \text{ for any } \beta \in (0,1), \qquad \textit{if } n \le (m-k-1)\gamma_1 + 2 + \varepsilon.$$

REMARK 2.2. – Note at this point, that in dimension $n = 2$, (2.7) is satisfied for any $m \ge 1$ and $0 \le k \le m-1$, furthermore for all choices of m and k we end up in the second case, which means that $D^k u \in C^{0,\beta}_{\text{loc}}(\Omega)$ for any $\beta \in (0,1)$.

3. – Preliminaries.

3.1 – *Function Spaces.*

By means of functional analysis it is easy to prove that the non standard growth structure of the integral functionals considered in this paper imply the existence of weak local minimizers in generalized Sobolev spaces. Let us give the definitions of these spaces in the sequel.

DEFINITION 3.1 (Generalized Lebesgue and Sobolev spaces). – *Let* $p : \Omega \to (1, +\infty)$ *be a measurable function. The generalized Lebesgue space* $L^{p(\cdot)}(\Omega)$ *is defined as the set*

$$L^{p(\cdot)}(\Omega) := \left\{ f \in L^1(\Omega) : \int_\Omega |f(x)|^{p(x)}\, dx < +\infty \right\}. \tag{3.8}$$

Endowed with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} \equiv \|f\|_{p(\cdot)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}, \tag{3.9}$$

the space $\left(L^{p(\cdot)}(\Omega); \|\cdot\|_{p(\cdot)}\right)$ *becomes a Banach space, which is separable, if and only if the exponent function p is bounded. For* $m \in \mathbb{N}$, *the generalized Sobolev space* $W^{m,p(\cdot)}(\Omega)$ *is defined as*

$$W^{m,p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega) : \; D^a f \in L^{p(\cdot)}(\Omega, \mathbb{R}) \textit{ for all } a \textit{ with } |a| \le m \right\}. \tag{3.10}$$

Defining the norm

$$\|f\|_{W^{m,p(\cdot)}(\Omega)} \equiv \|f\|_{m,p(\cdot)} := \sum_{|a|\leq m} \|D^a f\|_{p(\cdot)}, \tag{3.11}$$

the space $\left(W^{m,p(\cdot)}(\Omega); \|\cdot\|_{m,p(\cdot)}\right)$ *also becomes a Banach space.*

The desired Hölder continuity is shown by proving a quantitative control of the oscillations of the minimizer u respectively its derivatives. The spaces of functions whose oscillation on a ball B_ρ measured in an appropriate L^p sense is controlled by a power of the radius ρ are the well known Morrey and Campanato spaces. We recall the definitions of these spaces and state a well established theorem by Campanato, which links the control of oscillations to the classical Hölder continuity.

DEFINITION 3.2. – *(Morrey and Campanato spaces) Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* $p > 1$ *be constant. For* $\mu \in [0, n]$ *the Morrey space* $L^{p,\mu}(\Omega, \mathbb{R}^N)$ *is defined as*

$$L^{p,\mu}(\Omega, \mathbb{R}^N) \equiv \left\{u \in L^p(\Omega, \mathbb{R}^N) : \quad \|u\|^p_{L^{p,\mu}(\Omega)} < \infty\right\},$$

where

$$\|u\|_{L^{p,\mu}(\Omega)} \equiv \left[\sup_{\rho>0} \rho^{-\mu} \int_{B(x_0,\rho)\cap\Omega} |u(x)|^p \, dx\right]^{1/p}.$$

The local Morrey space $L^{p,\mu}_{loc}(\Omega, \mathbb{R}^N)$ *is the space of all functions* u *with* $u \in L^{p,\mu}(\Omega', \mathbb{R}^N)$ *for all* $\Omega' \Subset \Omega$. *For* $\mu \in (n, n+p]$, *the Campanato space* $\mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^N)$ *is defined as*

$$\mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^N) \equiv \left\{u \in L^p(\Omega, \mathbb{R}^N) : \quad [u]^p_{\mathcal{L}^{p,\mu}(\Omega)} < \infty\right\},$$

with the Campanato norm

$$[u]_{\mathcal{L}^{p,\mu}(\Omega)} \equiv \left[\sup_{\rho>0} \rho^{-\mu} \int_{B(x_0,\rho)\cap\Omega} |u(x) - (u)_{B(x_0,\rho)\cap\Omega}|^p \, dx\right]^{1/p},$$

where $(u)_{B(x_0,\rho)\cap\Omega} := \fint_{B(x_0,\rho)\cap\Omega} u \, dx := \frac{1}{|B(x_0,\rho)\cap\Omega|} \int_{B(x_0,\rho)\cap\Omega} u \, dx$ *denotes the mean value of* u *over the set* $B(x_0, \rho) \cap \Omega$

Analoguously to the definition of local Morrey spaces we also have the local variant of Campanato spaces $\mathcal{L}^{p,\mu}_{loc}(\Omega, \mathbb{R}^N)$.

THEOREM 3.1. – *Let Ω be a bounded open Lipschitz domain, and let $n<\lambda<n+p$. Then the space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to $C^{0,a}$ with $a=\dfrac{\lambda-n}{p}$. Furthermore, if $\lambda\geq n+p$, the space $\mathcal{L}^{p,\lambda}(\Omega)$ imbeds into $C^{0,a}$ for any $a\in(0,1)$.*

3.2 – *Technical Lemma.*

Within the paper we shall widely use the so-called *V-function* whose definition and elementary properties will be given now. For given $k\in\mathbb{N}$ and constant $p>1$ we define the function $V_p:\mathbb{R}^k\to\mathbb{R}^k$ as

$$V_p(z):=\Big(1+|z|^2\Big)^{\frac{p-2}{4}}z. \tag{3.12}$$

The following properties of V_p are elementary and will be used at many stages of the proofs.

LEMMA 3.3 (Properties of V_p). – *Let $p>1$ and let $V\equiv V_p:\mathbb{R}^k\to\mathbb{R}^k$ be as in* (3.12). *Then for any $z,\eta\in\mathbb{R}^k$ there holds*

(1) $|V(tz)|\leq\max\{t,t^{p/2}\}|V(z)|$, *for any* $t>0$;

(2) $|V(z+\eta)|\leq c\Big(|V(z)|+|V(\eta)|\Big)$;

(3) $c^{-1}|z-\eta|\leq\dfrac{|V(z)-V(\eta)|}{(1+|z|^2+|\eta|^2)^{(p-2)/4}}\leq c|z-\eta|$.

Moreover for any $z\in\mathbb{R}^k$ we have

$$(4)\quad \begin{array}{ll} \textit{if } p\in(1,2): & \dfrac{1}{\sqrt{2}}\min\{|z|,|z|^{p/2}\}\leq|V(z)|\leq\min\{|z|,|z|^{p/2}\}, \\ \textit{if } p\geq 2: & \max\{|z|,|z|^{p/2}\}\leq|V(z)|\leq\sqrt{2}\max\{|z|,|z|^{p/2}\}, \end{array}$$

$$(5)\quad \begin{array}{lll} \textit{if } p\in(1,2): & |V(z)-V(\eta)|\leq c|V(z-\eta)|, & \textit{for any } \eta\in\mathbb{R}^k \\ \textit{if } p\geq 2: & |V(z)-V(\eta)|\leq c(M)|V(z-\eta)|, & \textit{for } |\eta|\leq M \end{array}$$

$$(6)\quad \begin{array}{lll} \textit{if } p\in(1,2): & |V(z-\eta)|\leq c(M)|V(z)-V(\eta)|, & \textit{for } |\eta|\leq M \\ \textit{if } p\geq 2: & |V(z-\eta)|\leq c|V(z)-V(\eta)|, & \textit{for any } \eta\in\mathbb{R}^k \end{array}$$

with $c(M),c\equiv c(k,p)>0$ and $c(M)\to\infty$ if $M\to\infty$. If $1<\gamma_1\leq p\leq\gamma_2$ all the constants $c(k,p)$ may be replaced by a single constant $c\equiv c(k,\gamma_1,\gamma_2)$.

Finally we state a well known iteration lemma, which can for example be found in [17]. It will be useful for our purposes to deduce the final excess decay estimate.

LEMMA 3.4. – *Let $\Phi : [0,\infty) \to [0,\infty)$ be a non-decreasing, positive function and assume that there exists a number $\tau \in (0,1)$ and a radius $\rho_0 > 0$ such that for every $\rho < \rho_0$*

$$\Phi(\tau\rho) \le \tau^{\delta}\Phi(\rho) + B\rho^{\beta},$$

with $0 < \beta < \delta$. Then for every $\sigma < \rho \le \rho_0$ there holds

$$\Phi(\sigma) \le c\left[\left(\frac{\sigma}{\rho}\right)^{\beta}\Phi(\rho) + B\sigma^{\beta}\right], \tag{3.13}$$

where c is a constant depending only on τ, δ and β. Moreover, in the case that $\beta \ge \delta$, inequality (3.13) *holds with $\delta - \varepsilon$ instead of β for an arbitrary small $\varepsilon > 0$.*

4. – Reference estimate.

The proof of Theorem 2.1 will be done by comparison of the original minimizer u to the minimizer of a suitable "frozen" problem with constant growth exponent. We will take use of the following well known regularity result by Campanato (see [5], Theorem 1.VI.) about regularity of solutions of elliptic systems with constant p growth in low dimensions.

THEOREM 4.1. – *Let $w \in W^{m,p}(\Omega, \mathbb{R}^N), p > 1$, be a solution of the system*

$$\int_{\Omega} \langle A(D^m w), D^m \phi\rangle\, dx = 0 \qquad \forall \phi \in C_c^{\infty}(\Omega, \mathbb{R}^N), \tag{4.14}$$

where A satisfies the structure conditions

$$\begin{cases} |A(z)| \le L\left(1 + |z|^2\right)^{\frac{p-1}{2}}, \\ \langle D_z A\,\lambda, \lambda\rangle \ge \nu\left(1 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \end{cases} \tag{4.15}$$

for any $z, \lambda \in \mathbb{R}^{\mathcal{N}}$, with $0 < \nu \le 1 \le L$. Additionally assume in the case $1 < p < 2$ that

$$2 \le n < \frac{4}{2-p}.$$

Then there exists $r > 1$ such that for all $B_R \Subset \Omega$ and for all $0 < \rho \le R$ there holds

$$\int_{B_\rho} \left(1 + |D^m w|^2\right)^{\frac{p-2}{2}} |D^m w|^2\, dx \le c\left(\frac{\rho}{R}\right)^{\mu_0} \int_{B_R} \left(1 + |D^m w|^2\right)^{\frac{p-2}{2}} |D^m w|^2\, dx, \tag{4.16}$$

with $c \equiv c(n, N, m, p, L/\nu)$ and $\mu_0 = 2 + n(1 - 1/r)$.

REMARK 4.1. – A closer look to the proof of this Theorem in [5] shows that in the case of $p \in [\gamma_1, \gamma_2]$ the appearing constants may be replaced by constants that depend only on γ_1, γ_2 instead of p.

5. – Proof of Theorem 2.1.

First let us remark that the fact that the integrand $f(x, z)$ is of class C^2 with respect to the second variable enables us to use the corresponding Euler-Lagrange equation for our purposes. The strategy is to establish an appropriate comparison estimate to the minimizer of a "frozen" problem, which is due to the C^2 structure of the integrand the solution to the corresponding Euler-Lagrange system. Exploiting Campanato's estimates for elliptic systems in low dimensions provides a suitable control of the oszillations of this solution on shrinking balls. Finally, by our comparison estimate, this control carries over to the minimizer of the original problem.

5.1 – *General assumptions.*

Let us first remark that assumption (2.6) certainly allows us to assume without loss of generality that for all $\rho \in (0, 1]$ there holds

$$\omega(\rho) \log \frac{1}{\rho} \leq L, \tag{5.17}$$

where L is the constant in the growth condition $(2.4)_1$.

5.2 – *Higher integrability.*

A first consequence of the minimizing property, combined with the structure conditions is the following higher integrability result, whose proof in the situation of first order functionals goes back to Zhikov [24] and for the higher order case can be found in [18].

LEMMA 5.1. – *Let $O \subset \Omega$ be open and $u \in W^{m,1}(O, \mathbb{R}^N)$ a local minimizer of the functional*

$$w \mapsto \int_O f(x, D^m w)\, dx,$$

where $f : O \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ satisfies the growth and continuity assumptions (2.4). *Furthermore assume that*

$$\int_O |D^m u|^{p(x)}\, dx \leq M < \infty, \tag{5.18}$$

and the modulus of continuity ω *satisfies* (5.17). *Then there exist* $\delta, c \equiv \delta, c(n, \gamma_1, \gamma_2, L, M, m) > 0$ *and a radius* $\rho_0 \equiv \rho_0(n, \gamma_1, \omega(\,\cdot\,))$, *such that for every ball* $B_\rho \Subset O$ *with* $\rho \le \rho_0$ *there holds*

$$\left[\fint_{B_{\rho/2}} |D^m u|^{p(x)(1+\delta)}\, dx \right]^{\frac{1}{1+\delta}} \le c \left[\fint_{B_\rho} |D^m u|^{p(x)}\, dx + 1 \right]. \tag{5.19}$$

A second a priori higher integrability result needed for the proof of the regularity theorem is an up-to-the-boundary result for the minimizer of a "frozen" functional. For a proof in the first order case we refer the reader to [8]. The higher order case – also in a more general setting case – is proved in [18].

LEMMA 5.2. – *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* p *constant with* $1 < \gamma_1 \le p \le \gamma_2 < \infty$, $B_R \subset \Omega$ *and* $g : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ *be a continuous function, satisfying the growth condition*

$$L^{-1}(1+|z|^2)^{\frac{p}{2}} \le |g(z)| \le L(1+|z|^2)^{\frac{p}{2}},$$

for all $z \in \mathbb{R}^{\mathcal{N}}$, *with* $L > 1$. *For given* $h \in W^{m,q}(B_R)$ *with* $q > p$ *let* v *be the unique solution of the minimization problem*

$$\min\left\{ \int_{B_R} g(D^m w)\, dx, \quad w \in h + W_0^{m,p}(B_R) \right\}.$$

Then there exists $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(m, \gamma_1, \gamma_2, L, m) \in (0, \tilde{m})$, *with* $\tilde{m} = \dfrac{q}{p} - 1$, *and constants* c_1, c_2, *depending only on* $n, \gamma_1, \gamma_2, L, m$, *such that there holds*

$$\left[\fint_{B_R} |D^m v|^{p(1+\tilde{\varepsilon})}\, dx \right]^{\frac{1}{1+\tilde{\varepsilon}}} \le c_1 \fint_{B_R} |D^m v|^p\, dx + c_2 \left[\fint_{B_R} |D^m h|^{p(1+\tilde{m})}\, dx \right]^{\frac{1}{1+\tilde{m}}}.$$

5.3 – *Localizing I.*

Since our results are local, having in mind the a priori higher integrability (5.19) from Lemma 5.1 we can assume that

$$\int_\Omega |D^m u|^{p(x)(1+\delta)}\, dx < \infty, \tag{5.20}$$

where δ is the higher integrability exponent from Lemma 5.1. Without loss of generality we will assume that the exponent δ is small enough to satisfy

$$0 < \delta \le 4(\gamma_1 - 1), \tag{5.21}$$

where $\gamma_1 > 1$ denotes the global bound from (2.5). Let $\rho_0 > 0$ be a radius such that

$$\omega(8\rho_0) \leq \frac{\delta}{4},$$

and let $O \Subset \Omega$ be an open set whose diameter does not exceed ρ_0 from Lemma 5.1. We set

$$p_m := \max\{p(x) : x \in \overline{B}_{\rho_0}\}. \tag{5.22}$$

Now we consider balls $B(x_c, 4\rho) \equiv B_{4\rho} \Subset B_{\rho_0/4}$ and define

$$p_2 := \max_{\overline{B}_{4\rho}} p(x), \qquad p_1 := \min_{\overline{B}_{4\rho}} p(x). \tag{5.23}$$

Since $p_2 - p_1 \leq \omega(8\rho) \leq 8\omega(\rho)$ and by (5.21) there holds

$$p_2(1 + \delta/4) \leq p(x)(1 + \delta/4 + \omega(8\rho)) \leq p(x)(1 + \delta) \qquad \text{in } B_{4\rho} \tag{5.24}$$

$$p_m(1 + \delta/4) \leq p(x)(1 + \delta) \qquad \text{in } B_{\rho_0}. \tag{5.25}$$

Note that by the property (5.17) of the modulus of continuity ω there holds for any $0 < \rho < 8\rho_0 < 1$

$$\rho^{-n\omega(\rho)} \leq \exp(nL) = c(n, L), \qquad \rho^{-\frac{n\omega(\rho)}{1+\omega(\rho)}} \leq c(n, L). \tag{5.26}$$

A direct consequence of higher integrability results in Lemma 5.1 and Lemma 5.2 and the localizing is the following p_2-energy bound:

$$\int_{B_\rho} |D^m u|^{p_2}\, dx \leq c(M), \tag{5.27}$$

where $c \equiv c(M)$ is a constant depending on the structure parameters of the functional and on the bound M for the $p(x)$ energy on the set O (see (5.18)). The proof of the p_2-energy bound for u is simple, combining higher integrability (5.19), localization (5.24) and (5.26):

$$\begin{aligned}
\int_{B_\rho} |D^m u|^{p_2}\, dx &\leq \rho^n \fint_{B_\rho} \left(1 + |D^m u|^{p(x)(1+\omega(8\rho))}\right) dx \\
&\leq \rho^n \left[\fint_{B_{2\rho}} (1 + |D^m u|^{p(x)})\, dx \right]^{1+\omega(8\rho)} \\
&\leq \rho^{-n\omega(8\rho)} \left[\int_{B_{2\rho}} (1 + |D^m u|^{p(x)})\, dx \right]^{1+\omega(8\rho)} \\
&\leq c(n, L)(\rho^n + M)^{1+\omega(8\rho)} \leq c(n, L, M).
\end{aligned}$$

5.4 – *Freezing.*

Let $B(x_c, 4\rho) \equiv B_{4\rho}$ be a ball as described above. We consider the function $g(z) := f(x_0, z)$ and the Dirichlet problem

$$\min\left\{ \int_{B_\rho} g(D^m w)\, dx : w \in u + W_0^{m,p_2}(B_\rho, \mathbb{R}^N) \right\}, \tag{5.28}$$

where $x_0 \in \overline{B}_{4\rho}$ is a point with $p(x_0) = p_2 = \max_{x \in \overline{B}_{4\rho}} p(x)$, and let $v \in u + W_0^{m,p_2}(B_\rho, \mathbb{R}^N)$ be the unique solution. Note first, that due to the freezing in x_0, the frozen functional has p_2 growth structure, i.e. satisfies the conditions

$$\begin{cases} L^{-1}\left(1+|z|^2\right)^{\frac{p_2}{2}} \leq g(z) \leq L\left(1+|z|^2\right)^{\frac{p_2}{2}}, \\ L^{-1}\left(1+|z|^2\right)^{\frac{p_2-2}{2}}|\lambda|^2 \leq \langle D^2 g(z)\lambda, \lambda\rangle \leq L\left(1+|z|^2\right)^{\frac{p_2-2}{2}}|\lambda|^2, \end{cases} \tag{5.29}$$

for any $z, \lambda \in \mathbb{R}^{\mathcal{N}}$, with $L \geq 1$ out of (2.4). Furthermore we note that the growth conditions $(5.29)_1$ of the frozen functional together with the minimizing property of v immediately imply the following p_2-energy bound for v:

$$\int_{B_\rho} |D^m v|^{p_2}\, dx \leq L^2 \int_{B_\rho} (1 + |D^m u|^{p_2})\, dx \leq c(M), \tag{5.30}$$

where the constant c also depends on the $p(x)$-energy bound M for the original minimizer u.

STEP 1 (Reference system, energy estimate). – Since the integrand g is of class C^2, defining $A := D_z g(\,\cdot\,) = D_z f(x_0, \cdot)$, the function v is a solution of the corresponding Euler system

$$\int_{B_\rho} \langle A(D^m v), D^m \phi\rangle\, dx = 0, \qquad \text{for all } \phi \in W_0^{m,p_2}(B_\rho, \mathbb{R}^N), \tag{5.31}$$

which satisfies the following structure conditions:

$$|A(z)| \leq L(1+|z|^2)^{\frac{p_2-1}{2}}, \tag{5.32}$$

$$\langle DA(z)\lambda, \lambda\rangle \geq L^{-1}(1+|z|^2)^{\frac{p_2-2}{2}}|\lambda|^2, \tag{5.33}$$

for all $z, \lambda \in \mathbb{R}^{\mathcal{N}}$. Therefore Theorem 4.1 provides $\bar{\varepsilon} > 0$ (note that we set $\bar{\varepsilon} \equiv \frac{n}{2}\left(1 - \frac{1}{r}\right)$ with r of Theorem 4.1), such that if

$$\begin{cases} 2 \le n < \dfrac{4}{2-p_2} & \text{in the case } 1 < p_2 < 2, \\ n \ge 2 \text{ arbitrary} & \text{in the case } p_2 \ge 2, \end{cases}$$

there holds

$$\int_{B_\sigma} \left|V_{p_2}(D^m v)\right|^2 dx \le c\left(\frac{\sigma}{\rho}\right)^{\mu_0} \int_{B_\rho} \left|V_{p_2}(D^m v)\right|^2 dx, \tag{5.34}$$

for any $0 < \sigma \le \rho$, with $\mu_0 \equiv 2 + 2\bar{\varepsilon}$, and with a constant $c \equiv c(n, N, \gamma_1, \gamma_2, L, m)$.

STEP 2 (Comparison). – Between the solution v of the frozen problem and the solution u of the original one, we have the following comparison estimate:

$$\begin{aligned} I := \int_{B_\rho} (1 + |D^m u|^2 + |D^m v|^2)^{\frac{p_2-2}{2}} |D^m u - D^m v|^2 \, dx \\ \le c\omega(\rho) \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}} (1 + |D^m u|^{p_2}) \, dx + c\omega(\rho)\rho^n, \end{aligned} \tag{5.35}$$

with a constant $c \equiv c(n, \gamma_1, \gamma_2, L, M, m)$.

REMARK 5.3. – By Lemma 3.3 the above mentioned comparison estimate directly gives

$$\int_{B\rho} \left|V_{p_2}(D^m u) - V_{p_2}(D^m v)\right|^2 dx \le c\omega(\rho) \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}} (1 + |D^m u|^{p_2}) \, dx + c\omega(\rho)\rho^n, \tag{5.36}$$

which in the case $p_2 \ge 2$ by Lemma 3.3 $(6)_2$ and $(4)_2$ immediately provides

$$\int_{B_\rho} |D^m u - D^m v|^{p_2} \, dx \le c\omega(\rho) \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}} (1 + |D^m u|^{p_2}) \, dx + c\omega(\rho)\rho^n, \tag{5.37}$$

with constants c depending on $n, L, \gamma_1, \gamma_2, M, m$.

PROOF OF STEP 2. – First we show that

$$I \le \int_{B_\rho} [f(x_0, D^m u) - f(x_0, D^m v)] \, dx.$$

Recalling $g(z) \equiv f(x_0, z)$, the convexity condition (5.29) and differentiability of g provide

$$\begin{aligned} &\int_{B_\rho} [g(D^m u) - g(D^m v)] \, dx \\ &\quad = \int_{B_\rho} \langle Dg(D^m v), D^m u - D^m v \rangle dx \qquad [\, = 0] \end{aligned}$$

$$+\int_{B_\rho}\int_0^1 (1-t)\langle D^2 g(tD^m u+(1-t)D^m v)(D^m u-D^m v), D^m u-D^m v\rangle\,dt\,dx$$

$$\geq L^{-1}\int_{B_\rho}\int_0^1 (1-t)(1+|tD^m u+(1-t)D^m v|^2)^{\frac{p_2-2}{2}}|D^m u-D^m v|^2\,dt\,dx$$

$$\geq c^{-1}\int_{B_\rho}(1+|D^m u|^2+|D^m v|^2)^{\frac{p_2-2}{2}}|D^m u-D^m v|^2\,dx.$$

Therefore we have

$$\begin{aligned}
I &\leq c\int_{B_\rho}[f(x_0,D^m u)-f(x_0,D^m v)]\,dx\\
&\leq c\int_{B_\rho}[f(x_0,D^m u)-f(x,D^m u)]\,dx + c\int_{B_\rho}[f(x,D^m u)-f(x,D^m v)]\,dx\\
&\quad + c\int_{B_\rho}[f(x,D^m v)-f(x_0,D^m v)]\,dx\\
&= I_1+I_2+I_3,
\end{aligned}$$

with the obvious labeling of I_1 to I_3. The minimizing property of u implies $I_2 \leq 0$. I_1 is estimated, using the continuity condition $(2.4)_3$, finally arguing in an analog way to [4] (see the comparison estimate and the $L\log L$-estimate there):

$$\begin{aligned}
I_1 &\leq c\int_{B_\rho}\omega(|x-x_0|)((1+|D^m u|^2)^{\frac{p(x)}{2}}+(1+|D^m u|^2)^{\frac{p_2}{2}})(1+\log(1+|D^m u|^2))\,dx\\
&\leq c\omega(8\rho)\int_{B_\rho}(1+|D^m u|^2)^{\frac{p_2}{2}}(1+\log(1+|D^m u|^2))\,dx\\
&\leq c\omega(\rho)\int_{B_\rho\cap\{|D^m u|\geq e\}}|D^m u|^{p_2}\log|D^m u|^{p_2}\,dx + c\omega(\rho)\rho^n\\
&\leq c\omega(\rho)\rho^n \fint_{B_\rho}|D^m u|^{p_2}\log\left(e+\||D^m u|^{p_2}\|_{L^1(B_\rho)}\right)dx\\
&\quad + c\omega(\rho)\int_{B_\rho}|D^m u|^{p_2}\log\left[e+\frac{|D^m u|^{p_2}}{\||D^m u|^{p_2}\|_{L^1(B_\rho)}}\right]dx + c\omega(\rho)\rho^n\\
&= I_{11}+I_{12}+I_{13}.
\end{aligned}$$

Higher integrability (5.19) in combination with the localization (5.24) and (5.26) and finally the bound (5.18) for the $p(x)$ energy allows to estimate I_{12} as follows:

$$
\begin{aligned}
I_{12} &\le c\omega(\rho)\rho^n \left[\fint_{B_\rho} |D^m u|^{p_2(1+\frac{\delta}{4})}\, dx \right]^{\frac{1}{1+\frac{\delta}{4}}} \\
&\le c\omega(\rho)\rho^n + c\omega(\rho)\rho^n \left[\fint_{B_\rho} |D^m u|^{p(x)(1+\frac{\delta}{4}+\omega(\rho))}\, dx \right]^{\frac{1}{1+\frac{\delta}{4}}} \\
&\le c\omega(\rho)\rho^n + c\omega(\rho)\rho^n \left[\fint_{B_{2\rho}} |D^m u|^{p(x)}\, dx + 1 \right]^{\frac{1+\frac{\delta}{4}+\omega(\rho)}{1+\frac{\delta}{4}}} \\
&\le c\omega(\rho)\rho^n + c\omega(\rho)\rho^n \rho^{-n\frac{\omega(\rho)}{1+\frac{\delta}{4}}} \left[\fint_{B_{2\rho}} |D^m u|^{p(x)}\, dx \right] \left[\int_{B_{2\rho}} |D^m u|^{p(x)}\, dx \right]^{\frac{\omega(\rho)}{1+\frac{\delta}{4}}} \\
&\le c\omega(\rho)\rho^n + c\omega(\rho)\rho^n \left[\fint_{B_{2\rho}} 1 + |D^m u|^{p_2}\, dx \right] \left[\int_{B_{2\rho}} |D^m u|^{p(x)}\, dx \right]^{\frac{\omega(\rho)}{1+\frac{\delta}{4}}} \\
&\le c\omega(\rho)\rho^n + c\omega(\rho) \cdot M \cdot \int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx,
\end{aligned}
$$

with $c \equiv c(n, L, \gamma_1, \gamma_2, L, M, m, \delta)$.

We treat I_{11}, using estimates for the $L \log L$-norm of $|D^m u|^{p_2}$, which can for example be found in [4]:

$$
\begin{aligned}
I_{11} &\le c\omega(\rho) \log \left[\rho^{-n} e + \rho^{-n} \int_{B_\rho} |D^m u|^{p_2}\, dx \right] \int_{B_\rho} |D^m u|^{p_2}\, dx \\
&\le c\omega(\rho) \int_{B_\rho} |D^m u|^{p_2}\, dx \cdot \log \left[e + \int_{B_\rho} |D^m u|^{p_2}\, dx \right] \\
&\quad + c\omega(\rho) \log \left(\frac{1}{\rho} \right) \int_{B_\rho} |D^m u|^{p_2}\, dx \\
&\le c(\delta)\omega(\rho) \left[1 + \int_{B_\rho} |D^m u|^{p_2}\, dx \right]^{\frac{\delta}{4}} \int_{B_\rho} |D^m u|^{p_2}\, dx \\
&\quad + c\omega(\rho) \log \left(\frac{1}{\rho} \right) \int_{B_\rho} |D^m u|^{p_2}\, dx \\
&\le c(M, n, \delta) \left[\omega(\rho) + \omega(\rho) \log \left(\frac{1}{\rho} \right) \right] \int_{B_\rho} (1 + |D^m u|^{p_2})\, dx.
\end{aligned}
$$

Taking all these estimates together and additionally exploiting that $\log \frac{1}{\rho} \geq 1$, we end up with

$$I_1 \leq c\omega(\rho) \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n,$$

where, recalling the dependencies of δ in Lemma (5.1), the constant c depends on $n, L, \gamma_1, \gamma_2, m$ and M.

To estimate I_3 we proceed in exactly the same way as for the estimate of I_1. Doing the same splitting into terms I_{31} to I_{33} as we did with I_{11} to I_{13}, we use higher integrability up to the boundary for v (Lemma 5.2 with $q = p\left(1 + \frac{\delta}{4}\right), h = u \in W^{m,p(1+\frac{\delta}{4})}$ to estimate the term I_{32} (note that $\tilde{\varepsilon} \in (0, \delta)$ is the higher integrability exponent given by Lemma 5.2):

$$\begin{aligned}
I_{32} &\leq c\omega(\rho)\rho^n \left[\fint_{B_\rho} |D^m v|^{p_2(1+\frac{\tilde{\varepsilon}}{4})}\, dx \right]^{\frac{1}{1+\frac{\tilde{\varepsilon}}{4}}} \\
&\leq c\omega(\rho)\rho^n \fint_{B_\rho} |D^m v|^{p_2}\, dx + c\omega(\rho)\rho^n \left[\fint_{B_\rho} |D^m u|^{p_2(1+\frac{\delta}{4})}\, dx \right]^{\frac{1}{1+\frac{\delta}{4}}} \\
&\leq c\omega(\rho)\rho^n \int_{B_\rho} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n \left[\fint_{B_\rho} |D^m u|^{p_2(1+\frac{\delta}{4})}\, dx \right]^{\frac{1}{1+\frac{\delta}{4}}} \\
&\leq c\omega(\rho)\rho^n + c(M)\omega(\rho) \int_{B_\rho} (1 + |D^m u|^{p_2})\, dx,
\end{aligned}$$

with $c \equiv c(n, L, \gamma_1, \gamma_2, m, M)$. Note that from the second to the third line we also made use of the p_2 energy estimate (5.30).

I_{31} is estimated in an analog way to I_{11}, additionally using (5.30) for passing over from the p_2 energy of v to the energy of u. Alltogether, again remarking that $\log \frac{1}{\rho} \geq 1$, we end up with

$$I_3 \leq c\omega(\rho) \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n,$$

with c depending on $n, L, \gamma_1, \gamma_2, M, m$.

Combining the estimates for I_1 to I_3, we end up with the desired comparison estimate (5.35).

5.5 – *Excess decay estimate.*

We distinguish the cases $1<p_2<2$ and $p_2 \geq 2$. In the case $p_2 \geq 2$ we estimate by Lemma 3.3 (4)

$$\begin{aligned}\int_{B_\sigma} |D^m u|^{p_2}\, dx &\leq \int_{B_\sigma} \big|V_{p_2}(D^m u)\big|^2\, dx\\ &\leq 2\int_{B_\sigma} \big|V_{p_2}(D^m v)\big|^2\, dx + 2\int_{B_\sigma} \big|V_{p_2}(D^m u) - V_{p_2}(D^m v)\big|^2\, dx\\ &= [R_1] + [C_1].\end{aligned}$$

The comparison estimate (5.36) provides

$$[C_1] \leq c\omega(\rho)\log\left(\frac{1}{\rho}\right)\int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n,$$

with $c \equiv c(n, L, \gamma_1, \gamma_2, M, m)$. On the other hand the reference estimate (5.34) gives

$$[R_1] \leq c\left(\frac{\sigma}{\rho}\right)^{\mu_0}\int_{B_\rho} \big|V_{p_2}(D^m v)\big|^2\, dx,$$

with μ_0 defined at the end of Theorem 4.1 and with $c \equiv c(n, N, \gamma_1, \gamma_2, L, m)$. This, in combination with (5.36) and Lemma 3.3 (4), leads to

$$\begin{aligned}[R_1] &\leq c\left(\frac{\sigma}{\rho}\right)^{\mu_0}\left[\int_{B_\rho} \big|V_{p_2}(D^m u)\big|^2\, dx + \int_{B_\rho} \big|V_{p_2}(D^m u) - V_{p_2}(D^m v)\big|^2\, dx\right]\\ &\leq c\left(\frac{\sigma}{\rho}\right)^{\mu_0}\int_{B_\rho} \big|V_{p_2}(D^m u)\big|^2\, dx\\ &\quad + c\omega(\rho)\log\left(\frac{1}{\rho}\right)\int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n\\ &\leq c\left[\left(\frac{\sigma}{\rho}\right)^{\mu_0} + \omega(\rho)\log\left(\frac{1}{\rho}\right)\right]\int_{B_{2\rho}} (1 + |D^m u|^{p_2})\, dx + c\omega(\rho)\rho^n,\end{aligned}$$

with $c \equiv c(n, N, L, \gamma_1, \gamma_2, M, m)$. In the case that $1<p_2<2$ we estimate similary, observing in a first step that by Lemma 3.3 (4) we have that

$$\int_{B_\sigma} |D^m u|^{p_2}\, dx \leq \sigma^n + \int_{B_\sigma} \big|V_{p_2}(D^m u)\big|^2\, dx.$$

From this point on we proceed the estimate exactly as in the case $p_2 \geq 2$, noting that again by Lemma 3.3 (4), we may finally pass over from the V_{p_2} function to the L^{p_2} norm. Additionally exploiting that $\omega(\rho) \leq 1$, we therefore end up for any $p_2 > 1$ with the estimate

$$\int_{B_\sigma} (1+|D^m u|^{p_2})\,dx \leq c\left[\left(\frac{\sigma}{\rho}\right)^{\mu_0} + \omega(\rho)\log\left(\frac{1}{\rho}\right)\right]\int_{B_{2\rho}} (1+|D^m u|^{p_2})\,dx + c\sigma^n, \tag{5.38}$$

with a constant that depends on $n, N, L, \gamma_1, \gamma_2, m, M$.

We define the excess functional

$$\Phi(r) = \Phi(x_0, r) := r^{-\mu_1}\int_{B_{x_0,r}} (1+|D^m u|^{p_2})\,dx + r^{n-\mu_1},$$

with $\mu_1 := 2+\bar{\varepsilon}$ and where $\bar{\varepsilon}$ denotes the exponent of the reference estimate (5.34). For the convenience of the reader we recall also the definition of $\mu_0 = 2 + 2\bar{\varepsilon}$. and note that by their definitions the quantities satisfy

$$\bar{\varepsilon} \in (0, n/2), \quad \mu_0 \in (2, 2+n), \quad \mu_1 \in (2, 2+n/2).$$

We remark that $\bar{\varepsilon}$ provided by Theorem 4.1 is typically very small. Therefore it is no restriction to assume that $\bar{\varepsilon} \leq 1/2$. Exploiting (5.38) we deduce

$$\begin{aligned}
\Phi(\sigma) &\leq \sigma^{-\mu_1}\left[\sigma^n + c\left(\frac{\sigma}{\rho}\right)^{\mu_0} + c\omega(\rho)\log\left(\frac{1}{\rho}\right)\right]\int_{B_{2\rho}} (1+|D^m u|^{p_2})\,dx + \sigma^{n-\mu_1} \\
&\leq 2\left(\frac{\sigma}{\mu}\right)^{n-\mu_1}\rho^{n-\mu_1} \\
&\quad + c\left[\left(\frac{\sigma}{\rho}\right)^{\mu_0-\mu_1} + \left(\frac{\sigma}{\rho}\right)^{-\mu_1}\omega(\rho)\log\left(\frac{1}{\rho}\right)\right]\rho^{-\mu_1}\int_{B_{2\rho}} (1+|D^m u|^{p_2})\,dx.
\end{aligned}$$

Noting that $n-\mu_1 \geq \mu_2 - \mu_1$ and $\mu_2 - \mu_1 = \bar{\varepsilon}$ we conclude

$$\Phi(\sigma) \leq \tilde{c}\left[\left(\frac{\sigma}{\rho}\right)^{\bar{\varepsilon}} + \left(\frac{\sigma}{\rho}\right)^{-2-\bar{\varepsilon}}\omega(\rho)\log\left(\frac{1}{\rho}\right)\right]\Phi(2\rho),$$

with $\tilde{c} \equiv \tilde{c}(n, N, L, \gamma_1, \gamma_2, M, m)$.

For given $a \in (0, \bar{\varepsilon}/2)$ we define $\tau \equiv \tau(n, N, L, \gamma_1, \gamma_2, m, M)$ so small that

$$\tau^{\bar{\varepsilon}} \leq \frac{1}{3\tilde{c}}\tau^{2a}.$$

Such a choice is possible since $\bar{\varepsilon} - 2a > 0$. Now fixing $\rho_0 \equiv \rho_0(n, N, L, \gamma_1, \gamma_2, M, m, \omega(\,\cdot\,)) > 0$ so small that

$$\omega(\rho)\log\left(\frac{1}{\rho}\right) \leq \frac{2}{3\tilde{c}}\tau^{2+\bar{\varepsilon}+2a}, \qquad \text{for all} \quad \rho \leq \rho_0,$$

which is possible due to (2.6), we end up with

$$\Phi(\tau\rho) \leq \tau^{2a}\Phi(2\rho),$$

for all $\rho \leq \rho_0$. At this stage, Lemma 3.4 with the choices $\delta \equiv 2a$, $B \equiv 0$ provides

$$\Phi(\tilde{\rho}) \leq \left(\frac{\tilde{\rho}}{\rho}\right)^{2a-\tilde{\varepsilon}} \Phi(\rho), \tag{5.39}$$

for any $\tilde{\rho} < \rho \leq \rho_0$ and with $\tilde{\varepsilon} > 0$.

5.6 – *Finishing the proof.*

Now fix $\rho_0 \equiv \rho_0(n, N, L, \gamma_1, \gamma_2, m, M, \omega(\,\cdot\,))$ so small that all the smallness conditions imposed are satisfied and let $0 < \rho < \rho_0$ be arbitrary. Then, taking $\tilde{\varepsilon} > 0$ small enough and recalling the definitions of the excess functional Φ and the quantity μ_1, the decay estimate (5.39) provides

$$\int_{B_\rho} (1 + |D^m u|^{p_2})\, dx \leq c\rho^{\mu_1}\rho^{2a-\tilde{\varepsilon}} \leq c\rho^{2+2\varepsilon},$$

with $\varepsilon \equiv \varepsilon(\bar{\varepsilon}, a) > 0$ and a constant $c \equiv c(n, N, L, \gamma_1, \gamma_2, m, M)$.

Now let $k \in \{0, \dots, m-1\}$ and P_ρ be a polynomial on B_ρ of order $m-1$ whose coefficients are chosen in such a way that there holds

$$\fint_{B_\rho} D^\ell(u - P_\rho)\, dx = 0, \qquad \text{for } \ell = 0, \dots, m-1.$$

We refer the reader to [10] for existence, uniqueness, further properties and explicit representation formulas of such polynomials.

By Poincaré's inequality we obtain for any $k \in \{0, \dots, m-1\}$:

$$\int_{B_\rho} |D^k u - D^k P_\rho|^{p_2}\, dx \leq c\rho^{p_2(m-k)} \int_{B_\rho} |D^m u|^{p_2}\, dx \leq c\rho^{p_2(m-k)+2+2\varepsilon}.$$

From [10] we also infer that $(D^k P_\rho)_{B_\rho} = (D^k u)_{B_\rho}$, which allows to conclude that

$$D^k u \in \mathcal{L}^{p_2, (m-k)p_2+2+2\varepsilon}(B_\rho).$$

Now recalling Lemma 3.1, we have:

$$\begin{array}{lll} D^k u \in C^{0,\beta}(B_\rho) \text{ with } \beta = m - k - \frac{n-(2+2\varepsilon)}{p_2}, & \text{if} & \beta = m - k - \frac{n-(2+2\varepsilon)}{p_2}, \\ D^k u \in C^{0,\beta}(B_\rho) \text{ for all } \beta \in (0,1), & \text{if} & (m-k)p_2 + 2 + 2\varepsilon \geq n + p_2. \end{array}$$

Therefore we conclude the desired statement by a standard covering argument, noting that

$$\min_{\gamma_1 \le p_2 \le \gamma_2} \left(m - k - \frac{n-(2+2\varepsilon)}{p_2} \right) = m - k - \frac{n-(2+2\varepsilon)}{\gamma_1}. \qquad \square$$

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