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Remarks About Morphisms on an Algebraic Curve

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Abstract. – *In a previous paper we described the collection of homological equivalence relations on a curve of genus* ≥ 2 *as the set of integral solutions of certain algebraic equations. In the present paper we improve one argument of the previous paper, and we study the equations more closely for a curve of genus* 2.

Introduction.

A classical topic in the theory of algebraic curves, originating from work of De Franchis and Severi, is concerned with the rigidity and the finiteness of morphisms from a given curve to other curves of genus ≥ 2, up to the natural equivalence, and the problem of finding a reasonable upper bound for this finite number of maps is still open. We refer to [5] for references to the classical literature, and to [3] for some update.

This collection of equivalence classes of morphisms on a given curve X is in some sense bounded above by the collection of homological equivalence relations of genus ≥ 2 on the curve, which is the locus of integral points of an algebraic subset in $H_2(X \times X, \mathbb{C})$, and is a finite set too. In our previous paper [3] we wrote explicit equations for this algebraic set, which might suggest some insight about the set of solutions. In the present paper we do two things.

First, we provide an improvement to the previous paper, replacing the argument for the equation of $H_{1,1}(X \times X)$ with a more meaningful and more general one, see § 2. Two corrections to the paper are given in Remark 1.3. Second, we want to show that the approach of [3] can be pursued some further. We work with curves of genus 2. In this case the finiteness theorem above is trivial, however there is a companion result to the effect that the equivalence classes of morphisms of a given degree n to curves of genus 1 form a finite set. This set is bounded in the same way as before, and we think that working with the equations in this special situation may be a good test. We have two main results.

We prove a characterization of the rank of the symmetric part of the Néron Severi group $NS(X \times X)$ in terms of special period matrices for the curve, see § 5. We discuss with the help of an example the claim that the problem of solving the equations may be reduced to the classical problem in Number Theory of representations of integers by positive definite quadratic forms, see § 6.

1. – Summary of previous results.

We recall the basic facts and the main results from [3]. We refer to Kani [5] for precise references to the classical literature.

1.1 – *Morphisms as homology classes in the self product.*

Let X be a curve of genus $g \geq 2$. We mean a nonsingular complex projective curve. Define $\mathcal{F}(X)$ to be the collection of equivalence classes of morphisms $f : X \to Y$ onto curves Y of genus $g' \geq 2$, up to isomorphisms $Y \to Y'$. This is a finite set, according to the theorem of De Franchis and Severi.

There is an extension, observed by Tamme. Define $\bar{\mathcal{F}}(X)$ to be the collection of equivalence classes of morphisms onto curves of genus $g' \geq 1$. Define moreover $\mathcal{F}_n(X)$ and $\bar{\mathcal{F}}_n(X)$ to be the subcollections consisting of equivalence classes of morphisms of degree n. Every $\bar{\mathcal{F}}_n(X)$ is a finite set.

Associated to a morphism $f : X \to Y$ is an effective divisor Z_f in $X \times X$, defined by $f(x) = f(y)$, that only depends on the equivalence class $[f]$ in $\mathcal{F}(X)$. The main result about this is the rigidity theorem of Kani [5] saying that, taking the homology class of the divisor, the map

$$\bar{\mathcal{F}}(X) \longrightarrow H_2(X \times X, \mathbb{Z}) \qquad [f] \mapsto [Z_f]$$

is still injective. The aim here is to describe the image of this map.

The associated homology classes inherit a number of properties, expressing the concept of an equivalence relation, and also have the property of being algebraic classes, elements of the Néron Severi group $NS(X \times X) = H_{1,1}(X \times X) \cap H_2(X \times X, \mathbb{Z})$, where $H_{1,1}$ is Poincaré dual of the Hodge space $H^{1,1}$. Define $\mathcal{F}_n(X)$ to be the collection of equivalence classes of morphisms of degree n.

DEFINITION – *Let $n \geq 1$ and $g' \geq 1$ be integers such that $n(g'-1) \leq (g-1)$. Define*

$$\mathrm{V}_{n,g'} \subset H_{1,1}(X \times X)$$

to be the subset of homology classes z which satisfy:

- *z is symmetric, under the involution $(x, y) \mapsto (y, x)$,*
- *$z \circ z = nz$, where $\circ$ is the composition of correspondences,*
- *$z \cdot \xi_1 = z \cdot \xi_2 = n$ is the degree of the correspondence,*
- *$z \cdot \Delta = n(2 - 2g')$, where Δ is the diagonal class.*

Such an homology class z may be called a homological equivalence relation, of degree n, and the integer g' may be called the virtual genus of z. The set $\mathrm{V}_{n,g'}$ will be described as an algebraic set in the affine space $H_2(X \times X, \mathbb{C})$. Define *moreover*

$$\mathrm{V}_n := \cup\, \mathrm{V}_{n,g'}$$

where the union is taken over all $g' \geq 2$ satisfying the conditions above. Then

$$\mathrm{V}_n(\mathbb{Z}) := \mathrm{V}_n \cap H_2(X \times X, \mathbb{Z})$$

will be the locus of integral points of the algebraic set.

The basic facts quoted above say that we have an inclusion

$$\mathcal{F}_n(X) \longrightarrow \mathrm{V}_n(\mathbb{Z}).$$

Then there is a classic argument implying that $\mathrm{V}_n(\mathbb{Z})$ is a finite set, see [3]. Thus we are lead to the problem of describing this possibly larger finite set.

1.2 – *Equations for special homology classes.*

We introduce coordinates in the homology of $X \times X$. Let ξ be the fundamental class in $H_2(X, \mathbb{Z})$, let p be the point in $H_0(X, \mathbb{Z})$, and let $\gamma_1, \ldots, \gamma_{2g}$ be a canonical basis in $H_1(X, \mathbb{Z})$, with respect to which the intersection product on $H_1(X, \mathbb{Z})$ is represented by the standard symplectic matrix of principal type, that we call D.

Then in $H_2(X \times X, \mathbb{Z})$ one has the basis

$$\xi_1 := \xi \times p, \quad \xi_2 := p \times \xi, \quad \gamma_i \times \gamma_j \quad i,j \in \{1, \ldots, 2g\},$$

thus an homology class $z \in H_2(X \times X, \mathbb{Z})$ is given by a pair

$$(a_1, a_2) \quad \text{and} \quad A = (a_{ij})$$

consisting of a vector in $\mathbb{Z}^2$ and a matrix in $M_{2g,2g}(\mathbb{Z})$.

The following properties hold:

- $z \cdot \xi_1 = z \cdot \xi_2 = n$ means that $a_1 = a_2 = n$,
- then z is symmetric if and only if A is antisymmetric,
- $z \circ z = nz$ if and only if

$$ADA = nA, \tag{1}$$

- $z \cdot \Delta = n(2 - 2g')$ if and only if

$$\operatorname{tr} A_{12} = -ng', \tag{2}$$

where A is viewed as a 2×2 matrix consisting of $g \times g$ blocks, and A_{12} denotes the block on the upper right corner.

The condition for homology classes of type $(1, 1)$ requires the choice of a period matrix Π for the curve. The following holds:

LEMMA 1.1. – *The condition for the* 2*-cycle z that the Poincaré dual P*(z) *is of type* $(1,1)$ *is that*

$$\Pi\, A\, {}^t\Pi = 0. \tag{3}$$

The proof given in [3], § 5, is just a computation, and only valid for symmetric homology classes. Here we replace it with a more significant and fully general one, that will be given in the next section.

We summarize the preceding results. Write A as a block matrix

$$A = \begin{pmatrix} U & W \\ -{}^tW & V \end{pmatrix}$$

with U and V antisymmetric. Then the equation $ADA = nA$ is written as

$$\begin{array}{cc} -WU - U\,{}^tW = nU & -WW + UV = nW \\ \cdots & -VW - {}^tWV = nV \end{array} \tag{1'}$$

(we omit block (2,1) by antisymmetry), in which we have the condition

$$\operatorname{tr} W = -ng' \tag{2}$$

with $g' \geq 2$ and $n(g'-1) \leq (g-1)$, and the equation $\Pi A^t \Pi = 0$ is written as

$$QW - {}^tWQ + QUQ + V = 0. \tag{3'}$$

PROPOSITION 1.2. – *The set* $\mathrm{V}_n(\mathbb{Z})$*, viewed in the space of triplets* (U, V, W)*, is the locus of integral points of the algebraic subset defined by the equations above.*

REMARK 1.3. – Corrections to the previous paper [3]. Formula (3) above corrects formula (5) of the paper, where the sign of transposition happens to be shifted to the other side. The statement of Proposition 1.2 above makes precise the statement of Proposition 6.1 of the paper, where the equations are said to describe V_n, see also Remark 2.3 later on. These corrections do not affect the rest of the paper.

2. – Homology classes of type $(1,1)$.

We introduce coordinates in De Rahm cohomology. Let θ be the dual fundamental class in $H^2(X,\mathbb{Z})$, and let $\varphi_1,\ldots,\varphi_{2g}$ in $H^1(X,\mathbb{Z})$ be the dual basis of a canonical basis $\gamma_1,\ldots,\gamma_{2g}$ in $H_1(X,\mathbb{Z})$. Then in $H^2(X\times X,\mathbb{Z})$ one has the De Rahm dual basis

$$\theta_1 := p_1^*(\theta), \quad \theta_2 := p_2^*(\theta), \quad p_1^*(\varphi_i)\wedge p_2^*(\varphi_j) \quad i,j \in \{1,\ldots,2g\}.$$

We use the rough notation $\varphi_i\varphi_j$.

Then we introduce coordinates in the Hodge groups. If $\omega_1, \dots, \omega_g$ is a basis in $H^{1,0}(X)$, then in $H^{1,1}(X \times X)$ there is the basis

$$\theta_1, \quad \theta_2, \quad p_1^*(\omega_i) \wedge p_2^*(\bar{\omega}_j), \quad p_1^*(\bar{\omega}_i) \wedge p_2^*(\omega_j), \quad i, j \in \{1, \dots, g\},$$

for which we also use the symbols $\omega_i \bar{\omega}_j$ and $\bar{\omega}_i \omega_j$.

The relation of coordinates in the De Rahm and the Hodge groups is as follows. The inclusion $H^{1,0}(X) \subset H^1(X, \mathbb{C})$ is described by

$$\omega_i = \sum_j q_{ij} \varphi_j \, .$$

Then $\bar{\omega}_i = \sum_j \bar{q}_{ij} \varphi_j$, where overline means complex conjugate. The complex matrix

$$\Pi := (q_{ij})$$

of type $(g, 2g)$ is a period matrix for X (here we adopt the convention for period matrices that is proposed in the book of Birkenhake and Lange [1], p. 210). The Riemann relations say that the basis $\omega_1, \dots, \omega_g$ may be chosen so that

$$\Pi = (Q, 1)$$

where Q is symmetric and $\mathrm{im}(Q)$ is positive definite. We say that Q is a Siegel matrix for X.

Then in $H^1(X, \mathbb{C})$ there is a second basis $\omega_1, \dots, \omega_g, \overline{\omega}_1, \dots, \overline{\omega}_g$, obtained from $\varphi_1, \dots, \varphi_{2g}$ by means of the matrix

$$\widetilde{\Pi} := \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}.$$

On the other hand we may write

$$\varphi_i = \sum_j r_{ij} \omega_j + \sum_j \overline{r}_{ij} \overline{\omega}_j$$

for some complex matrix

$$R = (r_{ij})$$

of type $(2g, g)$, so that the matrix

$$\widetilde{R} := (R, \overline{R})$$

represents the reversed change of basis.

We now describe $H^{1,1}(X \times X)$ as a subset of $H^2(X \times X, \mathbb{C})$. An arbitrary 2-form is written as

$$\Omega = b_1 \theta_1 + b_2 \theta_2 + \sum_{hk} b_{hk} \varphi_h \varphi_k$$

and is given by the coordinate vector (b_1, b_2) and the coordinate matrix

$$B := (b_{hk}).$$

For simplicity, we confine ourselves to the case of a real form, belonging to $H^2(X \times X, \mathbb{R})$.

LEMMA 2.1. – *The condition for the* 2*-form above to be of type* $(1,1)$ *is that*

$$ {}^tRBR = 0. \tag{4}$$

PROOF. – The form is of type $(1,1)$ if and only if so is the component $\sum_{hk} b_{hk}\varphi_h\varphi_k$. Using the expression of φ_h in terms of forms ω_i and $\overline{\omega}_i$, we compute

$$\varphi_h\varphi_k = \Big(\sum_i r_{hi}\omega_i + \overline{r}_{hi}\overline{\omega}_i\Big)\Big(\sum_j r_{kj}\omega_j + \overline{r}_{kj}\overline{\omega}_j\Big) = \Big(\sum_{ij} r_{hi}r_{kj}\omega_i\omega_j\Big)$$
$$+\Big(\sum_{ij} r_{hi}\overline{r}_{kj}\omega_i\overline{\omega}_j\Big) + \Big(\sum_{ij} \overline{r}_{hi}r_{kj}\overline{\omega}_i\omega_j\Big) + \Big(\sum_{ij} \overline{r}_{hi}\overline{r}_{kj}\overline{\omega}_i\overline{\omega}_j\Big)$$

and therefore the form $\sum_{hk} b_{hk}\varphi_h\varphi_k$ is written as

$$\sum_{ij}\Big(\sum_{hk} b_{hk}r_{hi}r_{kj}\Big)\omega_i\omega_j + \sum_{ij}\Big(\sum_{hk} b_{hk}r_{hi}\overline{r}_{kj}\Big)\omega_i\overline{\omega}_j$$
$$+\sum_{ij}\Big(\sum_{hk} b_{hk}\overline{r}_{hi}r_{kj}\Big)\overline{\omega}_i\omega_j + \sum_{ij}\Big(\sum_{hk} b_{hk}\overline{r}_{hi}\overline{r}_{kj}\Big)\overline{\omega}_i\overline{\omega}_j$$

and this is of type $(1,1)$ if and only if for every i, j

$$\sum_{hk} r_{hi}b_{hk}r_{kj} = 0 \quad \text{and} \quad \sum_{hk} \overline{r}_{hi}b_{hk}\overline{r}_{kj} = 0,$$

which means that ${}^tRBR = 0$ and ${}^t\overline{R}B\overline{R} = 0$. As B is a real matrix, the first condition implies the second one. □

The condition in the lemma may be expressed in terms of the period matrix. Recall that $\widetilde{R}$ is the inverse of $\widetilde{\Pi}$. If Π is of the form $(Q, 1)$ then $\widetilde{\Pi} = \begin{pmatrix} Q & 1 \\ \overline{Q} & 1 \end{pmatrix}$ and hence $(\widetilde{\Pi})^{-1} = \begin{pmatrix} 1 & -1 \\ -\overline{Q} & Q \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ where $C = \dfrac{1}{2i}\,\mathrm{im}(Q)^{-1}$. Therefore $R = \begin{pmatrix} 1 \\ -\overline{Q} \end{pmatrix} C$.

Thus ${}^tRBR = {}^tC(1, -\overline{Q})B\begin{pmatrix} 1 \\ -\overline{Q} \end{pmatrix}C$ and the equation ${}^tRBR = 0$ is satisfied if and only if $(1, -\overline{Q})B\begin{pmatrix} 1 \\ -\overline{Q} \end{pmatrix} = 0$ or equivalently

$$(1, -Q)B\begin{pmatrix} 1 \\ -Q \end{pmatrix} = 0$$

as B is a real matrix. If B is written as a block matrix (in the same way as A in the previous section) then the equation becomes

$$B_{11} - QB_{21} - B_{12}Q + QB_{22}Q = 0. \tag{4'}$$

REMARK 2.2. – It is now immediate to prove Lemma 1.1 as a corollary of Lemma 2.1. Just substitute for B in equation (4′) the expression

$$P(A) = \begin{pmatrix} -A_{22} & A_{21} \\ A_{12} & -A_{11} \end{pmatrix}$$

that has been computed in [3], § 5, for the Poincaré dual coordinate matrix.

REMARK 2.3. – It is also clear that V_n is an algebraic set in $H_2(X \times X, \mathbb{C})$. The condition for a complex 2-cycle z that the Poincaré dual $P(z)$ is of type (1,1) has to be expressed by means of two equations, arising from ${}^tRBR = 0$ and ${}^t\overline{R}B\overline{R} = 0$ in the proof above after substituting $B = P(A)$. The two equations coincide for a real 2-cycle, so one of them is sufficient in order to describe the integral locus $\mathrm{V}_n(\mathbb{Z})$, as in the statement of Proposition 1.2.

3. – Symplectic transformations.

The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the group of matrices $R \in \mathrm{M}_{2g,2g}(\mathbb{Z})$ such that ${}^tRDR = D$ where $D = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the canonical symplectic matrix of principal type. Writing $R = {}^t\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ the defining condition is equivalent to the following:

$${}^t\alpha\gamma \text{ and } {}^t\beta\delta \text{ are symmetric, and } {}^t\alpha\delta - {}^t\gamma\beta = I.$$

Similarly the group $\mathrm{Sp}_{2g}(\mathbb{Q})$ is defined. The Siegel space $\boldsymbol{H}_g$ is the space of matrices $Q \in \mathrm{M}_{g,g}(\mathbb{C})$ such that $Q = {}^tQ$ and $\mathrm{im}(Q) > 0$ is positive definite. The action of $\mathrm{Sp}_{2g}(\mathbb{Q})$ on $\boldsymbol{H}_g$ is written as

$$Q' = (\alpha Q + \beta)(\gamma Q + \delta)^{-1}.$$

We refer to [1], ch. 8, for a complete treatment of this matter.

We point out some special symplectic transformations, that will be used in the following:

(a) $Q' = Q + \beta$ with β symmetric,
(b) $Q' = -Q^{-1}$,
(c) $Q' = {}^t\alpha Q\alpha$ with α invertible.

Every symplectic transformation of the Siegel matrix Q determines a transformation of the datum (U, V, W) such that the fundamental equation $(3')$ is preserved. These induced transformations are given in the various cases by the following formulas:

(a) $W' = W - U\beta, \quad U' = U, \quad V' = -\beta W + {}^tW\beta + \beta U\beta + V,$
(b) $W' = {}^tW, \quad U' = V, \quad V' = U,$
(c) $W' = a^{-1}Wa, \quad U' = a^{-1}U\,{}^ta^{-1}, \quad V' = {}^taVa.$

Note that equation $(1')$ is not preserved by transformations of type (a), in general, while it is preserved by transformations of type (b) or (c). This is an easy verification. Moreover all these transformations preserve the trace function $\mathrm{tr}(W)$. This is obvious for types (b) and (c), and for type (a) follows from the orthogonality $\mathrm{tr}(U\beta) = 0$ of antysimmetric and symmetric matrices.

4. – Curves of genus $g = 2$.

We analyse more closely the fundamental equations for curves of genus 2. First consider equation (1). Write the coordinate matrix as

$$A = \begin{pmatrix} 0 & u & w_{11} & w_{12} \\ -u & 0 & w_{21} & w_{22} \\ -w_{11} & -w_{21} & 0 & v \\ -w_{12} & -w_{22} & -v & 0 \end{pmatrix}.$$

The product ADA is equal to

$$\begin{pmatrix} 0 & -u(w_{11}+w_{22}) & -(w_{11}^2+w_{12}w_{21})-uv & -w_{12}(w_{11}+w_{22}) \\ u(w_{11}+w_{22}) & 0 & -w_{21}(w_{11}+w_{22}) & -(w_{21}w_{12}+w_{22}^2)-uv \\ \cdot & \cdot & 0 & -v(w_{11}+w_{22}) \\ \cdot & \cdot & v(w_{11}+w_{22}) & 0 \end{pmatrix}$$

and so the equation $ADA = nA$ splits into a system of scalar equations

$$(5) \qquad \begin{array}{ll} -u(w_{11}+w_{22}) = nu & -v(w_{11}+w_{22}) = nv \\ -w_{12}(w_{11}+w_{22}) = nw_{12} & -w_{21}(w_{11}+w_{22}) = nw_{21} \\ -(w_{11}^2+w_{12}w_{21})-uv = nw_{11} & -(w_{21}w_{12}+w_{22}^2)-uv = nw_{22}. \end{array}$$

The two equations in the third row can be read as requiring that

$$w_{11}, w_{22} \text{ are roots of the polynomial } X^2 + nX + (w_{12}w_{21} + uv)$$

and for this a necessary condition is that the discriminant of the polynomial is a perfect square, equivalently that

$$4(w_{12}w_{21} + uv) + z^2 = n^2 \tag{6}$$

for some integer z.

One has $g' = 2$ if and only if $w_{11} + w_{22} = -2n$. In this case it is immediately seen that the only solution to the equations above is the matrix $-nD$. In other words the set $\mathrm{V}_{n,2}(\mathbb{Z})$ consists of the single element $n\Delta$.

One has $g' = 1$ if and only if

$$w_{11} + w_{22} = -n. \tag{7}$$

In this case in the system of equations above all equations in the first two rows are satisfied, and both equations in the third row become equal to

$$w_{11}w_{22} - w_{12}w_{21} = uv. \tag{8}$$

We point out another way of handling the equations above. The two equations in the third row of (5) in conjunction with equation (7) can now be read as requiring that

$$w_{11}, w_{22} \text{ are precisely the roots of } X^2 + nX + (w_{12}w_{21} + uv) \tag{9}$$

and therefore they are determined, up to order, provided that condition (6) is satisfied. Note that in this case

$$w_{11} - w_{22} = \pm z.$$

Next consider equation (3). Write the Siegel matrix as

$$Q = \begin{pmatrix} q_1 & q \\ q & q_2 \end{pmatrix}.$$

In the equation $QW - {}^tWQ + QUQ + V = 0$ there are three antisymmetric summands, each determined by the coefficient in place $(1,2)$. The summand $QW - {}^tWQ$ is given by $q_1w_{12} - q_2w_{21} - q(w_{11} - w_{22})$, and QUQ is given by $(q_1q_2 - q^2)u$. So the equation becomes

$$q_1w_{12} - q_2w_{21} - q(w_{11} - w_{22}) + (q_1q_2 - q^2)u + v = 0. \tag{10}$$

The following statement complements Proposition 1.2.

REMARK 4.1. – The set $\mathrm{V}_{n,1}(\mathbb{Z})$ viewed in $\mathbb{C}^6$, the space of data (u, v, W), is the locus of integral points of the algebraic set defined by equations (7), (8), (10) or equivalently by condition (9) and equation (10).

REMARK 4.2. – For every solution (u, v, W) there is a second solution (u', v', W') where

$$W' = \begin{pmatrix} w_{22} & -w_{12} \\ -w_{21} & w_{11} \end{pmatrix} \quad \text{and} \quad u' = -u,\ v' = -v.$$

This is seemingly an extension of the fact that on a curve of genus 2 morphisms to curves of genus 1 occur in pairs, see e.g. Kuhn [6].

We now give a look at the induced action of the symplectic group $\mathrm{Sp}_4(\mathbb{Z})$ on data (u, v, W), which is written as

(a) $W' = W - U\beta, \quad u' = u, \quad v' = -b_1 w_{12} + b_2 w_{21} + b(w_{11} - w_{22}) + |\beta| u + v;$
(b) $W' = {}^t W, \quad u' = v, \quad v' = u;$
(c) $W' = a^{-1} W a, \quad u' = u \det(a)^{-1}, \quad v' = v \det(a).$

REMARK 4.3. – Two little observations that will be used in the following.

(i) The expression for v' in (a) implies that: if there is a non-trivial solution with $u = 0$, and necessarily W different from wI, then after a suitable symplectic transformation of type (a), one obtains that there is a solution with $v \neq 0$. Then after another symplectic transformation of type (b) one obtains a solution with $u \neq 0$.

(ii) Transformations of type (a) and (c) preserve the condition that $u \neq 0$, and those of type (c) also preserve the condition that $v \neq 0$.

5. – Symmetric correspondences.

Let X be a curve of genus 2. We study equation (10) by means of the symplectic transformations described in § 3.

We have seen in § 1 that antisymmetric matrices A of order 4 are coordinates for the symmetric part of $H_2(X \times X, \mathbb{Z})/\langle \xi_1, \xi_2 \rangle$, which is isomorphic to $\mathbb{Z}^6$, and that equation (10) defines the symmetric part of the reduced Néron Severi group $NS(X \times X)/\ \langle \xi_1, \xi_2 \rangle$, that can be identified with the reduced Néron Severi group of the symmetric self product $NS(X \cdot X)/\langle \xi_1 + \xi_2 \rangle$, viewed as a subgroup of $\mathbb{Z}^6$. The diagonal Δ corresponds to the solution $W = -I$ with $u = v = 0$. Define

$$r := \mathrm{rk}\, NS(X \cdot X)/\langle \xi_1 + \xi_2 \rangle.$$

The rational solutions of (10) represent $NS(X \cdot X) \otimes \mathbb{Q}/\langle \xi_1 + \xi_2 \rangle$ as a subgroup of $\mathbb{Q}^6$. Note that $\mathrm{rk}_{\mathbb{Z}}\, NS(X \cdot X)/\langle \xi_1 + \xi_2 \rangle = \dim_{\mathbb{Q}} NS(X \cdot X) \otimes \mathbb{Q}/\langle \xi_1 + \xi_2 \rangle$. Note moreover that

$$r = 6 - \mathrm{rk}\langle q_1, q_2, q, |Q|, 1 \rangle,$$

where the angle brackets denote the subspace over $\mathbb{Q}$ generated in $\mathbb{C}$ by the given elements.

We define two subsets of solutions of (10), two subgroups of the reduced symmetric Néron Severi group, that will be used in the following proof:

let B be the subset of solutions with $u = v = 0$, and hence $q_1 w_{12} - q_2 w_{21} - q(w_{11} - w_{22}) = 0$, let B' be the subset of solutions with $u = 0$, and hence $q_1 w_{12} - q_2 w_{21} - q(w_{11} - w_{22}) + v = 0$.

Note that $\mathrm{rk}(B) = 4 - \mathrm{rk}\langle q_1, q_2, q\rangle$ and $\mathrm{rk}(B') = 5 - \mathrm{rk}\langle q_1, q_2, q, 1\rangle$ possibly depend on the particular choice of the Siegel matrix Q.

THEOREM 5.1. – *The possible values of the rank r are characterized in the following way.*

(1) $r \geq 2$ *if and only if the curve admits a Siegel matrix Q such that* $|Q| \in \langle q_1, q_2, q, 1\rangle$,
(2) $r \geq 3$ *if and only if the curve admits a Siegel matrix such that* $|Q|, q \in \langle q_1, q_2, 1\rangle$, *or equivalently a Siegel matrix such that* $|Q|, 1 \in \langle q_1, q_2, q\rangle$,
(3) $r \geq 4$ *if and only if the curve admits a Siegel matrix such that* $|Q|, q, 1 \in \langle q_1, q_2\rangle$, *and in this case one has precisely* $r = 4$.

PROOF. – We start with a preliminary remark.

($*$) The various conditions on the Siegel matrix which appear in the statement, such as $|Q| \in \langle q_1, q_2, q, 1\rangle$, and $1 \in \langle q_1, q_2, q\rangle$, etc., can be interpreted in terms of existence of special solutions of equation (10), with $u \neq 0$, with $u = 0, v \neq 0$, etc. Moreover, the two conditions considered above are preserved by symplectic transformations of type (a) and (c), because of Remark 4.3(*ii*).

(1) If $r \geq 2$ then, by Remark 4.3(*i*), the curve admits a Siegel matrix Q such that equation (10) has a solution with $u \neq 0$, and the converse is also true. Moreover, given Q, there is a solution with $u \neq 0$ if and only if $|Q| \in \langle q_1, q_2, q, 1\rangle$.

(2) Let $r \geq 3$. We may assume that $|Q| \in \langle q_1, q_2, q, 1\rangle$, by point (1). In this case $r = \mathrm{rk}(B') + 1$. Because $r = 6 - \mathrm{rk}\langle q_1, q_2, q, |Q|, 1\rangle = 6 - \mathrm{rk}\langle q_1, q_2, q, 1\rangle$ and $\mathrm{rk}(B') = 5 - \mathrm{rk}\langle q_1, q_2, q, 1\rangle$. Then $\mathrm{rk}(B') \geq 2$, so there is a solution of $q_1 w_{12} - q_2 w_{21} - q(w_{11} - w_{22}) + v = 0$ different from the multiples $(0, 0, wI)$.

Furthermore, we may assume that $w_{11} - w_{22} \neq 0$. In fact, it is easy to see that there is some $a \in \mathrm{SL}_2(\mathbb{Z})$ such that the conjugate matrix $a^{-1}Wa$ satisfies the required condition, and then this a determines a symplectic transformation of type (c), which preserves the initial condition $|Q| \in \langle q_1, q_2, q, 1\rangle$ by remark ($*$). Hence from the equality above one obtains $q \in \langle q_1, q_2, 1\rangle$.

Next, it is easy to see that there is some symplectic transformation of type (a), of the form $Q' = Q + \beta$ with $\beta = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, such that in the transformed equation $q'_1 w'_{12} - q'_2 w'_{21} - q'(w'_{11} - w'_{22}) + v' = 0$ one has $v' \neq 0$. Hence, for the new period matrix, one obtains $1 \in \langle q_1, q_2, q\rangle$. Note that the condition $|Q| \in \langle q_1, q_2, q, 1\rangle$ is preserved, by remark ($*$). The converse is clear.

(3) Let $r \geq 4$. We may assume that $|Q|, 1 \in \langle q_1, q_2, q\rangle$, by point (2). In this case $r = \mathrm{rk}(B) + 2$. Because $r = 6 - \mathrm{rk}\langle q_1, q_2, q, 1\rangle = 6 - \mathrm{rk}\langle q_1, q_2, q\rangle$ and $\mathrm{rk}(B) =$

$4 - \mathrm{rk}\langle q_1, q_2, q \rangle$. Then $\mathrm{rk}(B) \geq 2$, so there is a solution of $q_1 w_{12} - q_2 w_{21} - q(w_{11} - w_{22}) = 0$ different from multiples $(0, 0, wI)$.

Then, using a symplectic transformation of type (c) as in the proof of point (2), we obtain that $w_{11} - w_{22} \neq 0$. Hence from the equation above we have that $q \in \langle q_1, q_2 \rangle$. Note that the initial condition $|Q|, 1 \in \langle q_1, q_2, q \rangle$ is preserved, by remark $(*)$.

Assume therefore that $|Q|, 1, q \in \langle q_1, q_2 \rangle$. Then necessarily $\mathrm{rk}\langle q_1, q_2 \rangle = 2$, otherwise, if for instance $q_2 \in \langle q_1 \rangle$ then also $1 \in \langle q_1 \rangle$ and hence $q_1 \in \mathbb{Q}$, that is impossible. Because $\mathrm{im}(Q)$ is positive definite and $\mathrm{im}(q_1) > 0$. It follows that $r = 6 - \mathrm{rk}\langle q_1, q_2 \rangle = 4$. □

REMARK 5.2. – The exact values of r can be characterized by adding to the condition that characterizes $r \geq i$ (empty condition for $i = 1$), in which some of the elements $q_1, q_2, q, |Q|, 1$ are obtained from the others, the requirement that the remaining elements are linearly independent over $\mathbb{Q}$ (automatic for $i = 4$). This is easily seen from the proof above.

The previous result leads to the natural question: given a matrix which satisfies the Siegel conditions, is there any curve for which the given matrix is a Siegel matrix? Such a matrix represents an Abelian surface endowed with a principal polarization. It is known that every principally polarized Abelian surface is isomorphic to a Jacobian or to a product of elliptic curves (see e.g. [1], p. 341), and it is known that certain products of elliptic curves are Jacobians (see e.g. Hayashida and Nishi [4]), but a complete answer to the question which products of elliptic curves are Jacobians seems not to be available in the literature.

6. – Equivalence relations of genus $g' = 1$.

Let X be a curve of genus 2. We study the set $\mathrm{V}_{n,1}(\mathbb{Z})$ by means of equations (9), (10), see Remark 4.1. For a given curve, in order to write down the equations explicitely, the knowledge of a Siegel matrix is required (and this may be a difficult point). Given the Siegel matrix, a method for studying the equations is outlined in § 4. We apply this in the following.

EXAMPLE 6.1. – Let X be the nonsingular curve of genus 2 whose affine plane model is the curve $y^2 = x^6 - 1$. It is shown by Kuusalo and Näätänen in [7] that a Siegel matrix for this curve is

$$Q = \begin{pmatrix} \dfrac{i\sqrt{3}}{2} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{i\sqrt{3}}{2} \end{pmatrix}.$$

Using Theorem 5.1 we immediately have that $r = 4$, and this is seen again in the following computations. Equation (10) is

$$\frac{i\sqrt{3}}{2}(w_{12} - w_{21}) + \frac{1}{2}(w_{11} - w_{22}) - u + v = 0$$

and it follows that

$$w_{12} = w_{21} =: w, \qquad w_{11} - w_{22} = 2(u - v).$$

Condition (9) says that w_{11}, w_{22} are the roots of the polynomial $X^2 + nX + (w^2 + uv)$, and the necessary condition (6) requires that $4(w^2 + uv) + z^2 = n^2$ for some integer z, where necessarily $w_{11} - w_{22} = \pm z$. It follows that the ordered pair w_{11}, w_{22} is uniquely determined provided that the necessary and sufficient condition

$$4(w^2 + uv) + 4(u - v)^2 = n^2$$

is satisfied. This requires that $n = 2m$ is even, and then the equation above becomes

$$u^2 + v^2 - uv + w^2 = m^2.$$

The binary quadratic form $u^2 + v^2 - uv$ is positive definite, so the equation above may be studied as

$$u^2 + v^2 - uv = p \quad \text{with} \quad p = m^2 - w^2 \quad \text{and} \quad |w| \leq m.$$

Note that $u^2 + v^2 - uv$ is equivalent to $u^2 + v^2 + uv$, and this is a well known example in the arithmetic theory of quadratic forms. The number of integral solutions of the equation $u^2 + v^2 + uv = p$ admits the following description, see Dickson [2], p. 80, ex. 2. It is equal to

$$6E(p)$$

where $E(p)$ is the excess of the number of divisors of p of the form $3h + 1$ over the number of divisors of the form $3h + 2$. If we write $p = 2^k q$ with q odd, then $E(p) = 0$ if k is odd and $E(p) = E(q)$ if k is even. It may be interesting to understand whether the proof of this result can be made effective so to produce the list of solutions.

Now we want an upper bound for the function $E(p)$. This function is not bigger than the function which gives the total number of divisors of p, and sometimes they coincide, so we write $E(p) \leq p$.

It follows that the total number of solutions is bounded above by $6 \sum_{-m,\dots,m} (m^2 - w^2) = 6m^2(2m + 1) - 12 \sum_{1,\dots,m} w^2$. Now recall the basic formula $\sum_{1,\dots,m} w^2 = \frac{1}{6} m(m + 1)(2m + 1)$. So the final bound is $6m^2(2m + 1) -$

$2m(m+1)(2m+1) = 2m(2m+1)(2m-1)$, which may also be written as $n(n+1)(n-1)$, which is

$$O(n^3).$$

The kind of analysis in this example should also be possible in the general situation. We can prove that, in general, one is lead to the problem of finding integral solutions of Diophantine equations of the form

$$f(x_1, \ldots, x_{r-1}) = n^2$$

where f is a positive definite rational quadratic form, and r is the rank of the reduced symmetric Néron Severi group. Clearly the equation above can also be written in the form $f'(x_1, \ldots, x_{r-1}) = (dn)^2$, where f' is an integral quadratic form and d is an integer, and this is a proper Diophantine equation, of the special form in which a positive integer has to be represented by a given positive definite quadratic form. We do not develop the proof here. It may be interesting to investigate what can be obtained about this by applying the methods of Number Theory.

We end with a result in which the influence of the Néron Severi rank is made precise.

PROPOSITION 6.2. – *If the rank r is small then the set* $\mathrm{V}_{n,1}(\mathbb{Q})$ *is small:*

- *if* $r = 1$ *the set is empty,*
- *if* $r = 2$ *the cardinality of the set is* 0 *or* 2.

PROOF. – Let $r = 1$. The solutions of (10) are given by data of the form $xI, (0,0)$, and none satisfies (9).

Let $r = 2$. Suppose that there is some solution of (9),(10) independent of $(0,0,-I)$, given by (u,v,W). Then another solution is given by the datum (u',v',W') defined in Remark 4.2. They are the only solutions.

All rational solutions of (10) are given by data of the form $(yu, yv, xI + yW)$ with rational x, y. We apply the analysis in § 4. Because of (9), knowing that the polynomial $X^2 + nX + (w_{12}w_{21} + uv)$ has the roots w_{11}, w_{22}, we have to search for those values of x, y for which the coefficients $x + yw_{11}, x + yw_{22}$ are the roots of the polynomial $X^2 + nX + y^2(w_{12}w_{21} + uv)$. Because of (6) we must have that $4y^2(w_{12}w_{21} + uv) + z^2 = n^2$ for some integer z, and then the difference of roots is $y(w_{11} - w_{22}) = \pm z$, so the necessary and sufficient condition is $y^2\big(4(w_{12}w_{21} + uv) + (w_{11} - w_{22})^2\big) = n^2$. We know that $4(w_{12}w_{21} + uv) + (w_{11} - w_{22})^2 = n^2$, hence $y^2 = 1$ and the two polynomials above coincide. This implies that the pair of roots $x + yw_{11}, x + yw_{22}$ coincides with the pair w_{11}, w_{22}. It is easy to see that this is possible if and only if either $y = 1, x = 0$ or $y = -1, x = -n$. $\square$

REFERENCES

[1] C. BIRKENHAKE - H. LANGE, *Complex Abelian Varieties*, second edition. Springer-Verlag, Berlin, 2004.

[2] L. DICKSON, *Introduction to the Theory of Numbers*. Dover Publications, New York, 1957.

[3] L. GUERRA, *Morphisms on an algebraic curve and divisor classes in the self product.* Boll. Unione Mat. Ital. Sez. B (8) **10**, no. 3 (2007), 715-725.

[4] T. HAYASHIDA - M. NISHI, *Existence of curves of genus two on a product of two elliptic curves.* J. Math. Soc. Japan, **17** (1965), 1-16.

[5] E. KANI, *Bounds on the number of nonrational subfields of a function field.* Invent. Math., **85**, no. 1 (1986), 185-198.

[6] R. KUHN, *Curves of genus* 2 *with split Jacobian.* Trans. Amer. Math. Soc., **307**, no. 1 (1988), 41-49.

[7] T. KUUSALO - M. NÄÄTÄNEN, *Geometric uniformization in genus* 2. Ann. Acad. Sci. Fenn. Ser. A I Math., **20**, no. 2 (1995), 401-418.

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