BOLLETTINO
UNIONE MATEMATICA ITALIANA

LOREDANA BIACINO

Density and Tangential Properties of the Graph of Hölder Functions

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **3** (2010), n.3, p. 493–503.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2010_9_3_3_493_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2010.

Bollettino U. M. I.
(9) **III** (2010), 493-503

Density and Tangential Properties of the Graph of Hölder Functions

Loredana Biacino

Abstract. – *In this paper the circular densities (with respect to the Hausdorff or packing measure) of graphs of Hölder continuous functions are studied. They are related to the local behaviour of the functions making use of some geometric properties.*

1. – Introduction, prerequisites, notations.

The Hausdorff dimension of a bounded set E is defined in terms of the *k-dimensional measure* of E, denoted by $H^k(E)$ and given by:

$$H^k(E) = lim_{\delta \to 0}\, inf\, \{\Sigma_i |E_i|^k, E \subseteq E_i, |E_i| < \delta\}$$

where $|E_i|$ denotes the diameter of E_i and the infimum is over all (countable) $\delta-$covers $\{E_i\}$ of E (see Falconer [10], [11]). It is given by:

$$H - dim\, E = inf\, \{k > 0 : H^k(E) = 0\}. \tag{1}$$

A subset $E \subseteq R^n$ is said an *s-set*, $s > 0$, if $0 < H^s(E) < +\infty$. There are other classes of covers that define measures leading to Hausdorff dimension: for example the class of spherical balls or the class of dyadic cubes. Moreover other definitions are in widespread use. Among them it is worth mentioning the upper and lower box dimensions, defined in the following way. Let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which cover E. Then the following numbers:

$$\underline{dim_B}\, E = \underline{lim}_{\delta \to 0} \frac{log\, N_\delta(E)}{-log\, \delta} \tag{2}$$

$$\overline{dim_B}\, E = \overline{lim}_{\delta \to 0} \frac{log\, N_\delta(E)}{-log\, \delta} \tag{3}$$

are called respectively the *lower and upper box dimensions* of E and, if they agree, their common value is the *box dimension* of E.

In order to find the box dimensions (2) or (3) it is possible to take $N_\delta(E)$ to be the number of mesh cubes of side δ meeting E, a δ-mesh cube being a cube of the form: $[m_1\delta,(m_1+1)\delta]\times\ldots\times[m_n\delta,(m_n+1)\delta]$, where $m_1,\ldots,m_n$ are integers. By definitions (1), (2) and (3) it follows: $H-dim\,E\le \underline{dim_B}\,E\le \overline{dim_B}E$ for every $E\subseteq R^n$.

To avoid the difficulties arising from the fact that the box dimension is not a measure, consider: $P^s_\delta(E)=\sup\{\varSigma|B_i|^s:\{B_i\}$ collection of disjoint balls of radii at most δ with centers in E$\}$; it is not difficult to see that $lim_{\delta\to 0}P^s_\delta(E)=P^s_0(E)$ is not a measure, but:

$$P^s(E)=\inf\{\varSigma P^s_0(E_i):E\subseteq\bigcup E_i\},$$

defines a measure on R^n, known as the *s-dimensional packing measure*. Then the packing dimension is defined as usual by:

$$dim_p\,E=\sup\{s>0:P^s(E)=\infty\}=\inf\{s>0:P^s(E)=0\}.$$

C. Tricot in [19] proved that dim_pE coincides with the *modified upper box dimension* of E, that is with the number:

$$\overline{dim_{MB}}\,E=\inf\{\sup_i\overline{dim_B}\,E_i:E\subseteq E_i\}.$$

Let $E\subseteq R^n$ be an $s-$set, $P\in R^n$ and let $B_r(P)$ denote the closed ball of centre P and radius r, so that $|B_r(P)|=2r$. The *circular upper and lower densities* of the set E at P are defined respectively in the following way:

$$\overline{D^s}(E,P)=\overline{lim}_{r\to 0}\,(2r)^{-s}H^s(E\bigcap B_r(P))$$

$$\underline{D^s}(E,P)=\underline{lim}_{r\to 0}\,(2r)^{-s}H^s(E\bigcap B_r(P))$$

If they are equal then the common value is called *density* of E at P and is denoted by $D^s(E,P)$. Of course it is possible to define the *upper and lower densities correspondingly to every given measure* μ on R^n in the following way:

$$\varTheta^{*s}(\mu,P)=\overline{lim}_{r\to 0}\,(2r)^{-s}\mu(B_r(P));\,\varTheta^s_*(\mu,P)=\underline{lim}_{r\to 0}\,(2r)^{-s}\mu(B_r(P)).$$

If they agree, their common value, denoted by $\varTheta^s(\mu,P)$, is the *density of E at P with respect to* μ. In particular in the sequel the packing measure will be considered, restricted to a given set E, $P^s_{\lfloor E}(F)=P^s(E\bigcap F)$ for every $F\subseteq R^n$; the densities corresponding to it will be denoted by the symbols $\varTheta^{*s}(P^s_{\lfloor E},P)$ and $\varTheta^s_*(P^s_{\lfloor E},P)$. More about these topic can be found also in [10], [11], [13], [15], [16], [17]. In this paper sets of R^2 that are graphs of a-Hölder continuous functions are investigated and their local density behaviour with respect to the Hausdorff or packing measure, making use of some properties of their geometric configurations, is studied.

In order to enunciate the main result observe that in this paper the following notation for a continuous function $f:[a,b]\to R$, $a>0$ and $c\in[a,b]$ will be used:

$$ord_{x\to c}[f(x)-f(c)]=a \Leftrightarrow 0<\underline{lim}_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}\leq \overline{lim}_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}<+\infty;$$

$$ord_{x\to c}[f(x)-f(c)]>a \ \ (resp.<a)$$

$$\Leftrightarrow$$

$$lim_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}=0 \ \ (resp.\ \ lim_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}=+\infty);$$

$$ord_{x\to c}[f(x)-f(c)]\geq a \Leftrightarrow 0=\underline{lim}_{x\to c}\frac{|f(x)-f(c)}{|x-c|^a}\leq \overline{lim}_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}<+\infty;$$

$$ord_{x\to c}[f(x)-f(c)]\leq a \Leftrightarrow 0<\underline{lim}_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}\leq \overline{lim}_{x\to c}\frac{|f(x)-f(c)|}{|x-c|^a}s=+\infty.$$

Obviously $\leq$ is not an order relation.

Let $f:[a,b]\to R$ be an a-Hölder continuous function, let $P=(x_o,f(x_o))$, $x_o\in\ ]a,b[$, and let L be the Hölder coefficient of f. Some results of this paper involve the rectangular neighborhoods $T_r(P)$ of P defined by:

$$T_r(P)=\{(x,y):x_o-r\leq x\leq x_o+r, |y-f(x_o)|\leq Lr^a\},$$

with $r>0$ such that $[x_o-r,x_o+r]\subseteq[a,b]$. For a rather similar approach see [18]. The following theorem holds (Theorem 3.1):

If $ord_{x\to x_o}|f(x)-f(x_o)|\leq\dfrac{1}{2-a}$ *then the upper Hausdorff density of the graph G of f at the point* $P=(x_0,f(x_0))$ *is finite and if* $ord_{x\to x_0}|f(x)-f(x_0)|<\dfrac{1}{2-a}$ *the Hausdorff density exists and is zero. If f is uniformly essentially a-Holder continuous in a sense that will be specified in the sequel then if* $ord_{x\to x_o}|f(x)-f(x_o)|\geq\dfrac{1}{2-a}$ *the lower packing density of G at P is positive and if* $ord_{x\to x_o}|f(x)-f(x_o)|>\dfrac{1}{2-a}$ *the packing density at P is infinite.*

As a consequence, the subset of G constituted by the points $(x_o,f(x_o))$ such that $ord_{x\to x_o}|f(x)-f(x_o)|<\dfrac{1}{2-a}$ has zero H^{2-a}-measure.

2. – Rectangular density at a point of the graph of an a-Hölder continuous function.

In the sequel it will be useful to have lower and upper estimates or relations for the Hausdorff or packing measure of general subsets of R^n. These are furnished by the following theorem whose proof is an immediate consequence of well known facts (see for example [11]):

THEOREM 2.1. – *If E is a subset of R^n, then, for every $s > 0$, it is:*

$$H^s(E) \leq \underline{lim}_{\delta \to 0} N_\delta(E)(\delta\sqrt{n})^s, \tag{4}$$

where $N_\delta(E)$ is the number of δ-mesh cubes that intersect E; moreover:

$$\overline{lim}_{\delta \to 0} N_\delta(E)\delta^s \leq 2^s P_0^s(E); inf\{\Sigma_{i \in N} \overline{lim}_{\delta \to 0} N_\delta(E_i)\delta^s, E \subseteq \cup E_i\} \leq 2^s P^s(E) \tag{5}$$

where the inf is taken over all covers $(E_i)_{i \in N}$ of E.

PROOF. – The proof of (4) is obvious. In order to prove the first of (5), let $N'_\delta(E)$ be the largest number of disjoint balls of radius δ centred in E and let $B_1, \ldots, B_{N'_\delta}$ be disjoint balls of radii δ centred in E. If $x \in E$ then x must be within distance δ of one of the B_i, otherwise the ball of centre x and radius δ could be added to the previous system forming a larger collection of disjoint balls. Then the $N'_\delta(E)$ balls, concentric to the previous ones and with radius 2δ constitute a cover of E. This implies $N_{4\delta}(E) \leq N'_\delta(E)$. Obviously $N'_\delta(E)(2\delta)^s \leq P^s_\delta(E)$, whence:

$$N_{4\delta}(E)(4\delta)^s \leq 4^s N'_\delta(E)\delta^s \leq 2^s P^s_\delta(E).$$

Passing to the limit for $\delta \to 0$, we obtain the first of (5). Now let $(E_i)_{i \in N}$ be a cover of E and let the E_i be covered by δ-meshes: since the first of (5) holds for every E_i, passing to the sum it gives:

$$\Sigma_i \overline{lim}_{\delta \to 0} N_\delta(E_i)\delta^s \leq 2^s \Sigma_i P^s_0(E_i).$$

This inequality holds for every cover $(E_i)_{i \in N}$ of E, therefore, passing to the g.l.b., we obtain the second inequality in (5). □

By (4) it follows that if G is the graph of an a-Hölder continuous function $f : [a, b] \to R$, then $H^{2-a}(G) \leq L(\sqrt{2})^{2-a}(b - a)$, where L is the Hölder coefficient of f. In order to prove an analogous inequality with respect to the packing measure, a definition is needed.

An a-Hölder continuous function $f : [a, b] \to R$ is called *uniformly essentially a-Hölder continuous in* $[a, b]$ if there exist $\lambda > 0$, a decreasing infinitesimal sequence of positive numbers $(\delta_n)_{n \in N}$ and an increasing sequence of sets E_n, where $E_n = \{x^n_0, \ldots, x^n_{i_n}\}$, $x^n_0 = a, x^n_{i_n} = b$, $\delta_n = x^n_j - x^n_{j-1}, j = 1, \ldots, i_n$, such that for every $x \in E_n$ it is:

$$\omega(\delta_n, f, x) \geq \lambda\delta^a_n$$

where $\omega(\delta, f, x) = sup\{|f(x') - f(x'')|, |x - x'| < \delta, |x - x''| < \delta\}$. A similar condition can be found in [12], (see for example Theorem 3.1). Let $f : [a, b] \to R$ be continuous; then, if $G = G[a, b] = \{(x, y) : x \in [a, b], y = f(x)\}$ it is $H - dim\, G \geq 1$ (see [10], Lemma 3.2); in general the Hausdorff dimension of an a-Hölder continuous function is less or equal to $2 - a$ (see [11], Corollary 11.2), but if f is uniformly essentially a-Hölder continuous in $[a, b]$, or in a subinterval of $[a, b]$, the box dimension of the graph of f is $2 - a$ (see [9], Theorem 2.1). However the previous condition is not necessary: this fact and other relations between the dimension of the graph of a continuous function f and the local or global behaviour of f are established in [8].

THEOREM 2.2. – *Let G be the graph of a uniformly essentially a-Hölder continuous function then there exists a constant $C > 0$ such that*

$$C(b-a) \leq P^{2-a}(G).$$

PROOF. – Since f is uniformly essentially a-Hölder continuous, there exist $\lambda > 0$, a decreasing infinitesimal sequence of positive numbers $(\delta_n)_{n\in N}$ and an increasing sequence of sets $(E_n)_{n\in N}$, where $E_n = \{x_0^n, \ldots, x_{i_n}^n\}$, $x_0^n = a, x_{i_n}^n = b$, $\delta_n = x_j^n - x_{j-1}^n, j = 1, \ldots, i_n$, such that for every $x \in E_n$ it is:

$$\omega(\delta_n, f, x) \geq \lambda\delta_n^a.$$

Let $F \subseteq [a,b]$, F closed and consider a cover of G made of $3\delta_n$ -meshes. Correspondingly F is covered by intervals of length $3\delta_n$. Since in the third middle part of each of these intervals there is a point belonging to E_n, the oscillation of f in each interval is not less than $\lambda\delta_n^a$ and therefore it is: $N_{3\delta_n}(G_F)(3\delta_n)^{2-a} \geq \lambda\dfrac{m(F)}{3^a}$, where $m(F)$ denotes the Lebesgue measure of F. Let $G_F = \{(x,y) \in R^2 : x \in F, y = f(x)\}$; it follows:

$$\overline{lim}_{\delta\to 0} N_\delta(G_F)\delta^{2-a} \geq \frac{\lambda m(F)}{3^a}.$$

Let H be a closed subset of R^2 and let $F \subseteq R$ be the projection of $H \cap G$ onto the x-axis: then F is a closed set, $H \cap G = G_F$ and therefore, by the previous inequality, it is:

$$\overline{lim}_{\delta\to 0} N_\delta(H \cap G)\delta^{2-a} \geq \frac{\lambda m(F)}{3^a}.$$

Now let $(E_i)i \in I$ be a cover of G made of closed subsets of R^2. Then, by the previous inequality and by the first of (5), there exists $c > 0$ such that:

$$P_o^{2-a}(E_i \cap G) \geq c\ \overline{lim}_{\delta\to 0} N_\delta(E_i \cap G)\delta^{2-a} \geq c\lambda\frac{m(F_i)}{3^a}$$

where F_i is the projection of $E_i \cap G$ on the x-axis, as before. Since $P^{2-a}(G) = inf\{\Sigma P_0^{2-a}(E_i \cap G) : G \subseteq \cup E_i, E_i\ closed\}$ (see [16], 5.10) and $\Sigma m(F_i) \geq b - a$, considering first the sum with respect to i in the previous inequality and then considering the g.l.b. with respect to all possible closed covers, the theorem is proven. $\square$

In order to establish some results about the circular density of the graph G of an a-Hölder continuous function $f : [a,b] \to R$ at a point $P = (x_o, f(x_o))$,

($x_o \in \,]a,b[$), consider the set:

$$T_r(P) = \{(x,y) : x_o - r \leq x \leq x_o + r, |y - f(x_o)| \leq Lr^a\}$$

where $r > 0$ is such that $[x_o - r, x_o + r] \subseteq [a,b]$ and L is the Hölder coefficient of f. By Theorem 2.1 and Theorem 2.2 the proof of the following theorem is immediate:

THEOREM 2.3. – *Let $f : [a,b] \to R$ be an a-Hölder continuous function, $0 < a < 1$, $x_o \in \,]a,b[$, $P = (x_o, f(x_o))$ and let G be the graph of f. Then,*

i) *for every $r \in R^+$ such that $[x_o - r, x_o + r] \subseteq [a,b]$, it is:*

$$H^{2-a}(G \cap T_r(P)) \leq cr \tag{6}$$

where c > 0 is a constant, depending only on a and f.

ii) *if f is uniformly essentially a-Hölder continuous in $[x_o - \rho, x_o + \rho] \subseteq [a,b]$, then there exists a positive constant $C > 0$ such that for every $r \in \,]0,\rho[$:*

$$P^{2-a}(G \cap T_r(P)) \geq Cr. \tag{7}$$

REMARK. – It is noteworthy that Theorem 2.3 ii) holds in particular for the functions introduced by Besicovitch and Ursell in [5] in the following way:

$$f(x) = \Sigma_{n \in N} a_n \varphi(b_n x)$$

where we assume, for sake of simplicity, that $b_n = d^n$, with $d \in N$, $d \geq 2$, $a_n = d^{-n\delta}$ $0 < \delta < 1$ and φ is defined setting $\varphi(x) = 2x$ if $0 \leq x \leq 1/2$; $\varphi(x) = \varphi(-x) = \varphi(x+1)$ elsewhere. Besicovitch and Ursell in [5] prove that the so defined f is δ-Hölder continuous. Besides it is possible to prove (see [9]) that there exist $\lambda > 0$ and $t \in N$ such that, if $x = \dfrac{k}{d^n}$, $k = 0, 1, \ldots, d^{n-1}$ and $h = \dfrac{1}{2d^m}$ then, for every $m \geq n + t$ it is:

$$|f(x+h) - f(x)| \geq \lambda h^a;$$

therefore f is uniformly essentially δ-Hölder continuous.

We conclude this section with the following theorem that is another consequence of Theorem 2.1:

THEOREM 2.4. – *Let $f : [a,b] \to R$ be a uniformly essentially a-Hölder continuous function, $0 < a < 1$, let B be a subset of $[a,b]$, $A = \{(x, f(x)) : x \in B, y = f(x)\}$ and assume that $P^{2-a}(A) = 0$. Then it is $m(B) = 0$, with m Lebesgue measure.*

PROOF. – Since f is uniformly essentially a-Hölder continuous, there exist $\lambda > 0$, a decreasing infinitesimal sequence of positive numbers $(\delta_n)_{n \in N}$ and an increasing sequence of sets E_n, where $E_n = \{x_0^n, , x_{i_n}^n\}$, $x_0^n = a, x_{i_n} = b$, $\delta_n = x_j^n - x_{j-1}^n, j = 1, \dots, i_n$, such that, for every $x \in E_n$ it is: $\omega(\delta_n, f, x) \geq \lambda\delta_n^a$. Since $P^{2-a}(A) = 0$ for every $\varepsilon > 0$ there exists a sequence of Borel sets $(A_j)_{j\in N}$ such that $A \subseteq \cup A_j$ and $\Sigma_j P_0^{2-a}(A_j) < \varepsilon$. Let $B_j = \{x : (x, f(x)) \in A_j\}$: then for every cover (I_i^J) of B_j constituted by $3\delta_n$-meshes, consider in the third middle part of I_i^J a point $x \in E_n$. It is possible to obtain a cover of A_j constituted by $3\delta_n$-meshes. To calculate the number $N_{3\delta_n}$ consider that, for every I_i^j, at least $\frac{\lambda|I_i^j|^a}{3^a|I_i^j|}$ squares in vertical whose edge is $3\delta_n$ are needed. Then A_j is covered at least by $\Sigma_i \frac{\lambda|I_i^j|^a}{3^a|I_i^j|}$ squares with edge $3\delta_n$ and therefore, by the first inequality in (5) of Theorem 2.1:

$$P_0^{2-a}(A_j) \geq 2^{a-2}\overline{lim}_{n\to\infty}\Sigma_i \frac{\lambda|I_i^j|^a}{3^a|I_i^j|} |I_i^j|^{2-a} \geq 3^{-a}2^{a-2}\lambda m(B_j).$$

Since $\cup B_j \supseteq B$, summing the terms on the left and on the right of this inequality, it follows that $3^{-a}2^{a-2}\lambda m(B) \leq 3^{-a}2^{a-2}\lambda\Sigma m(Bj) < \varepsilon$, and, since ε is arbitrary, $m(B) = 0$. $\square$

3. – Circular density at a point of the graph of a uniformly essentially a-Hölder continuous function.

It is well known (see for example Lemma 3.5 of [10]) that every rectifiable curve is a *regular 1-set*, that is at almost every point of it the circular Hausdorff density exists and equals 1. On the contrary every s-set in R^2 with $1 < s < 2$ is *irregular* (see for example Corollary 4.10 of [10]), that is at almost H^s-every point of it the circular density either fails to exist or is different from 1. In this Section it will be proved that if an s-set in R^2 with $1 < s < 2$ is the graph of an a-Hölder continuous function then the behaviour of the lower and upper circular densities at a point $P = (x_o, f(x_o))$ are strictly related to the behaviour of

$$\underline{lim}_{x\to x_0} \frac{|f(x) - f(x_o)|}{|x - x_o)|^{\frac{1}{2-a}}} \quad \text{and} \quad \overline{lim}_{x\to x_0} \frac{|f(x) - f(x_o)|}{|x - x_o|^{\frac{1}{2-a}}}$$

This will be clear in the following theorem; before recall that the lower and upper square densities with respect to the packing measure are defined in a similar fashion to the circular ones, referring to the system of the squares $Q_r(P)$ with centre P and edges of width $2r$ parallel to the coordinate axes instead that to the system of the balls $B_r(P)$. As it is easily seen, for our purpose it is equivalent to consider the circular densities or the square ones. In a similar way square densities will be considered with respect to the Hausdorff measure.

THEOREM 3.1. – *Let* $f:[a,b]\to R$ *be an a-Hölder continuous function,* $0<a<1$, $x_o\in\,]a,b[$, $P=(x_o,f(x_o))$, *and let:*

$$ord_{x\to x_o}|f(x)-f(x_o)|<\frac{1}{2-a}; \tag{8}$$

then

$$D^{2-a}(G,P)=0. \tag{9}$$

If $ord_{x\to x_0}|f(x)-f(x_0)|\le\frac{1}{2-a}$ *then in general we have that:*

$$\overline{D^{2-a}}(G,P)<\infty. \tag{10}$$

If f is uniformly essentially a-Hölder continuous and if $ord_{x\to x_o}|f(x)-f(x_o)|>\dfrac{1}{2-a}$ *it is:*

$$\Theta^{2-a}(P^{2-a}_{\lfloor G},P)=+\infty, \tag{11}$$

while, if $ord_{x\to x_o}|f(x)-f(x_o)|\ge\dfrac{1}{2-a}$ *it is:*

$$\Theta^{2-a}_{*}(P^{2-a}_{\lfloor G},P)>0; \tag{12}$$

In particular if f is uniformly essentially a-Hölder continuous and if $ord_{x\to x_0}|f(x)-f(x_o)|=\dfrac{1}{2-a}$ *then both* (10) *and* (12) *hold.*

PROOF. – It is not restrictive to assume $[a,b]=[-1,1]$, $x_o=0, f(x_o)=0$ and therefore $P=(0,0)$. By (8) for every $M>0$ there exists $\delta>0$ such that if $0<|x|<\delta$ it is $|f(x)|>M|x|^{\frac{1}{2-a}}$.

Let $0<r<\delta$ be such that $x^*=\left(\dfrac{r}{M}\right)^{2-a}<r$. The value x^* is the abscissa of the intersection point in the first quadrant of the right line whose equation is $y=r$ and the curve whose equation is $|y|=M|x|^{\frac{1}{2-a}}$. If $x>x*$ the corresponding points of G do not lie in $Q_r(0)$ and therefore $G\cap Q_r(0)\subseteq G\cap T_{x^*}(0)$. By Theorem 2.3 i), there exists $C>0$ such that:

$$\frac{H^{2-a}(G\cap Q_r(0))}{(2r)^{2-a}}\le\frac{H^{2-a}(G\cap T_{x*}(0))}{(2r)^{2-a}}\le\frac{C}{M^{2-a}}$$

whence:

$$\overline{lim}_{r\to 0}\frac{H^{2-a}(G\cap Q_r(0))}{(2r)^{2-a}}\le\frac{C}{M^{2-a}};$$

since M can be arbitrarily large, (9) is proved.

If $ord_{x\to 0}|f(x)|\le\dfrac{1}{2-a}$, then there exists $m>0$ such that $m<\underline{lim}_{x\to 0}\dfrac{|f(x)|}{|x^{\frac{1}{2-a}}}$

and therefore there exists $\delta > 0$ such that if $0<|x|<\delta$ then it is $|f(x)| > m|x|^{\frac{1}{2-a}}$. Let $0<r<\delta$ be such that $x^* = \left(\frac{r}{m}\right)^{2-a} < r$: then it is: $G \cap Q_r(0) \subseteq G \cap T_{x*}(0)$ for such values of r and therefore, by Theorem 2.3 i) there exists $C > 0$ such that

$$\frac{H^{2-a}(G \cap Q_r(0))}{(2r)^{2-a}} \leq C$$

whence (10) follows.

If f is uniformly essentially a-Hölder continuous and if $ord_{x\to o}|f(x)| > \frac{1}{2-a}$, then: $lim_{x\to 0} \frac{|f(x)|}{|x|^{\frac{1}{2-a}}} = 0$ and therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0<|x|<\delta$ it is $|f(x)|<\varepsilon|x|^{\frac{1}{2-a}}$. Let $0<r<\delta$ be such that $x^* = \left(\frac{r}{\varepsilon}\right)^{2-a} < r$; $x*$ is the abscissa of the intersection point in the first quadrant of the right line of equation $y = r$ and the curve whose equation is $|y| = \varepsilon|x|^{\frac{1}{2-a}}$. Therefore, for enough small r, $G \cap Q_r(0)$ contains $G \cap T_x * (0)$ and, by Theorem 2.3 ii), there exists a constant $C > 0$ such that

$$\frac{P^{2-a}(G \cap Q_r(0))}{(2r)^{2-a}} \geq \frac{P^{2-a}(G \cap T_{x^*}(0))}{(2r)^{2-a}} \geq \frac{C}{\varepsilon^{2-a}}.$$

Since ε can be arbitrarily small, by this inequality we obtain (11).

Finally, if $ord_{x\to o}|f(x)| \geq \frac{1}{2-a}$, then there exists $M > 0$ such that $\overline{lim}_{x\to 0} \frac{|f(x)|}{|x|^{\frac{1}{2-a}}} < M$ and therefore there exists $\delta > 0$ such that if $0<|x|<\delta$ then it is $|f(x)|<M|x|^{\frac{1}{2-a}}$. Let $0<r<\delta$ be such that $x^* = \left(\frac{r}{M}\right)^{2-a} < r$. For such values of r $G \cap Q_r(0)$ contains $G \cap T_x * (0)$, and therefore, by Theorem 2.3 ii), there exists $C > 0$ such that

$$\frac{P^{2-a}(G \cap Q_r(0)}{(2r)^{2-a}} \geq C,$$

whence the inequality (12) follows. □

If $f : [a, b] \to R$ is an a-Hölder continuous function then $\overline{lim}_{\delta\to 0} N_\delta(G)\delta^{2-a} < +\infty$ and therefore, by Theorem 2.1, it is $H^{2-a}(G) < +\infty$ and, by Theorem 6.2 of [16], $2^{-2+a} \leq \overline{D^{2-a}}(G, P) \leq 1$ for H^{2-a} almost all P of G. Thus, by Theorem 3.1, the following holds:

COROLLARY 3.2. – *Let $f : [a, b] \to R$ be an a-Hölder continuous function, and let $0<a<1$; then the points $(x_o, f(x_o))$ such that $ord_{x\to x_o}|f(x) - f(x_o)| < \frac{1}{2-a}$ constitute a subset of G of zero H^{2-a}-measure.*

Recall that in [14] J. M. Marstrand defined the *upper and lower angular densities* of plane s-sets, analogously to the spherical ones, by:

$$\overline{D^s}(E,P,\theta,\varphi) = \overline{lim}_{r\to 0}\,(2r)^{-s}H^s(E\cap S_r(P,\theta,\varphi))$$

and

$$\underline{D^s}(E,P,\theta,\varphi) = \underline{lim}_{r\to 0}\,(2r)^{-s}H^s(E\cap S_r(P,\theta,\varphi)).$$

In this case the ball $B_r(P)$ has been replaced by the spherical sector of radius r:

$$S_r(P,\theta,\varphi) = B_r(P)\cap S(P,\theta,\varphi)$$

where $S(P,\theta,\varphi)$ is the closed one-way infinite cone with vertex P and axis in direction θ consisting of those points Q for which the vector PQ makes an angle of at most φ with the axis. An $s-$set E is called by Marstrand to have a *weak tangent in direction θ at P* if $\underline{D^s}(E,P) > 0$ and for every $\varphi \in]0,\frac{\pi}{2}[$ it is

$$\underline{lim}_{r\to 0}\,r^{-s}H^s(E\cap(B_r(P) - S_r(P,\theta,\varphi) - S_r(P,-\theta,\varphi))) = 0$$

the weak tangent at P being the line through P which lies in the direction θ. If the last equality holds with $lim_{r\to 0}$ instead of $\underline{lim}_{r\to 0}$, then, according with the original definition by Besicovitch (see [2]), the $s-$set E is said to possess a tangent in direction θ. Then for a set whose Hausdorff dimension is fractional the tangent at a point (in the sense of [2] or [13]) is defined only if the lower density at the point is different from zero. Now, by Theorem 3.1, for the points $P = (x_o, f(x_o))$ such that $ord_{x\to x_0}|f(x) - f(xo)| = a$ or such that $ord_{x\to x_0}|f(x) - f(xo)| > a$ but less than $\frac{1}{2-a}$ the lower density with respect to the Hausdorff measure is zero, hence the problem of the existence of the tangent cannot be posed at all; on the other hand, for these points, the proof of Theorem 3.1 implies the existence of a vertical tangent in an obvious way.

REFERENCES

[1] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Annalen, **98** (1927) 422-65.

[2] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points(II)*, Math. Annalen, **115** (1938) 296-329.

[3] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points (III)*, Math. Annalen, **116** (1939) 349-57.

[4] A. S. Besicovitch, *On tangents to general sets of points*, Fundamenta Mathematicae, **22** (1934) 49-53.

[5] A. S. Besicovitch - H. D. Ursell, *Sets of fractional dimensions (V): on dimensional numbers of some continuous curves*, Journal London Math. Soc., **12** (1937) 18-25.

[6] A. S. BESICOVITCH, *On the existence of tangents to rectifiable curves*, Journal London Math. Soc., **19** (1945) 205-207.
[7] L. BIACINO, *Derivatives of fractional order of continuous functions*, Ricerche Mat., **LIII** (2004) 231-254.
[8] L. BIACINO, *Hausdorff dimension of the diagram of a-Hölder continuous functions*, Ricerche Mat., **LIV** (2005) 229-243.
[9] L. BIACINO, *A note on the box dimension of the graph of a continuous function*, submitted for publication.
[10] K. J. FALCONER, *The geometry of fractal sets*, Cambridge University Press, 1985.
[11] K. J. FALCONER, Fractal Geometry, *Mathematical Foundations and Applications*, John Wiley and Sons Ltd., New-York, 1990.
[12] Y. HEURTEAUX, *Weierstrass functions with random phases*, Trans. Am. Math. Soc., **355**, n. 8 (2003) 3065-3077.
[13] M. A. MARTIN - P. MATTILA, *Hausdorff measures, Hölder continuous maps and self similar fractals*, Math. Proc. Cambridge Philos. Soc., **114** (1993) 37-42.
[14] J. M. MARSTRAND, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proceedings London Math. Soc., **4** (1954) 257-302.
[15] J. M. MARSTRAND, *Circular density of plane sets*, Journal London Math. Soc., **30** (1954) 238-46.
[16] P. MATTILA, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.
[17] D. PREISS, *Geometry of measures in R^n : Distribution, rectifiability, and densities*, Annals of Mathematics, **125** (1987) 537-643.
[18] F. PRZYTYCKI - M. URBANSKI, *On the Hausdorff dimension of some fractal sets*, Studia Math., **93** (1989) 155-186.
[19] C. TRICOT, *Two definitions of fractional dimension*, Math. Proc. Cambridge Philos. Soc., **91** (1982) 57-74.

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”,
Via Cinzia, Monte Sant’Angelo, 80126 Napoli
E-mail: loredana.biacino2@unina.it

Received March 19, 2010 and in revised form June 11, 2010