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The Existence and Multiplicity of Heteroclinic and Homoclinic Orbits for a Class of Singular Hamiltonian Systems in $\mathbb{R}^2$

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Abstract. – *In this work we consider a class of planar second order Hamiltonian systems: $\ddot{q} + \nabla V(q) = 0$, where a potential V has a singularity at a point $\xi \in \mathbb{R}^2$: $V(q) \to -\infty$, as $q \to \xi$ and the unique global maximum $0 \in \mathbb{R}$ that is achieved at two distinct points $a, b \in \mathbb{R}^2 \setminus \{\xi\}$. For a class of potentials that satisfy a strong force condition introduced by W. B. Gordon [Trans. Amer. Math. Soc. 204 (1975)], via minimization of action integrals, we establish the existence of at least two solutions which wind around ξ and join $\{a, b\}$ to $\{a, b\}$. One of them, Q, is a heteroclinic orbit joining a to b. The second is either homoclinic or heteroclinic possessing a rotation number (a winding number) different from Q.*

1. – Introduction.

In this work we will be concerned with the existence and multiplicity of heteroclinic and homoclinic orbits for a class of autonomous second order Hamiltonian systems in $\mathbb{R}^2$,

$$\ddot{q} + \nabla V(q) = 0, \tag{1.1}$$

where a potential V satisfies the following conditions:

(V_1) there exists $\xi \in \mathbb{R}^2$ such that $V \in C^1(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$,

(V_2) $\lim_{x \to \xi} V(x) = -\infty$,

(V_3) there is a neighbourhood $\mathcal{N}$ of the point ξ and there is a function $U \in C^1(\mathcal{N} \setminus \{\xi\}, \mathbb{R})$ such that $|U(x)| \to \infty$, as $x \to \xi$ and $|\nabla U(x)|^2 \leq -V(x)$ for all $x \in \mathcal{N} \setminus \{\xi\}$,

(V_4) $V(x) \leq 0$ and there are two distinct points $a, b \in \mathbb{R}^2 \setminus \{\xi\}$ such that $V(x) = 0$ if and only if $x \in \{a, b\}$,

(V_5) there is a negative constant V_0 such that $\limsup_{|x| \to \infty} V(x) \leq V_0$.

In the literature, (V_3) is known as a strong force condition introduced by Gordon (see [9]). Here and subsequently, $|\cdot| \colon \mathbb{R}^n \to [0, \infty)$ is the norm in $\mathbb{R}^n$ induced by the standard inner product.

A solution $q\colon \mathbb{R} \to \mathbb{R}^2$ of (1.1) is said to be *homoclinic* if $q(-\infty) = a = q(\infty)$ or $q(-\infty) = b = q(\infty)$, where

$$q(\pm\infty) = \lim_{t\to\pm\infty} q(t).$$

We call a solution $q\colon \mathbb{R} \to \mathbb{R}^2$ of (1.1) *heteroclinic* if $q(-\infty) = a$ and $q(\infty) = b$ or conversely.

Let E be given by

$$E = \left\{ q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)\colon \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt < \infty \right\}.$$

It is known that E is the Hilbert space under the norm:

$$\|q\|_E^2 = \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt + |q(0)|^2.$$

We will consider the families of paths that omit ξ defined as follows:

$$\Lambda = \{q \in E\colon q(t) \neq \xi \text{ for all } t \in \mathbb{R}\},$$

$$\Gamma = \{q \in \Lambda\colon q(-\infty) = a \ \wedge \ q(\infty) = b\},$$

$$\Upsilon = \{q \in \Lambda\colon q(-\infty) = b \ \wedge \ q(\infty) = a\},$$

$$\Omega_a = \{q \in \Lambda\colon q(-\infty) = a = q(\infty)\},$$

$$\Omega_b = \{q \in \Lambda\colon q(-\infty) = b = q(\infty)\}.$$

Let's introduce the polar coordinate system in $\mathbb{R}^2$ with the pole ξ and the polar axis

$$l = \{x \in \mathbb{R}^2\colon x = \xi + t \cdot \vec{\xi a},\ t \geq 0\},$$

where polar angles are measured counterclockwise from the axis.

In this polar coordinate system one has $q(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t))$ for all $q \in \Lambda$. There is no uniqueness of a function $\varphi(t)$. If $q(t)$ is continuous then we can assume that $r(t)$ and $\varphi(t)$ are continuous, too.

DEFINITION 1.1. – *For each $q \in \Lambda$ such that $q(\pm\infty) \in \mathbb{R}^2 \setminus \{\xi\}$ we can determine the rotation number $rot(q)$ (the winding number) as follows:*

$$rot(q) = \left[\frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}\right],$$

where $[s]$ denotes the integral part of $s \in \mathbb{R}$.

This definition is independent of the choice of a function $\varphi(t)$.

Set

$$R = \frac{1}{3}\min\{|b-a|, |b-\xi|, |a-\xi|\}.$$

From now on, $B_r(x)$ stands for an open ball in $\mathbb{R}^2$ of radius $r > 0$, centered at a point $x \in \mathbb{R}^2$.

REMARK 1.1. – Let $0 < \varepsilon \le R$. Assume that $q \in \Gamma$ and there is $T \in \mathbb{R}$ such that $q(T) \in B_\varepsilon(b)$. Then, by $rot(q|_{(-\infty,T]})$ and $rot(q|_{[T,\infty)})$ we mean the rotation numbers of appropriate paths in Γ and Ω_b, resp., that arise from $q|_{(-\infty,T]}$ and $q|_{[T,\infty)}$, resp., by connecting $q(T)$ to b by a line segment.

It is justified by elementary homotopy arguments that

$$rot(q) = rot(q|_{(-\infty,T]}) + rot(q|_{[T,\infty)}).$$

Moreover, if $q([T,\infty)) \subset B_\varepsilon(b)$ then

$$rot(q) = rot(q|_{(-\infty,T]}).$$

We can also introduce similar notation for q in Υ, Ω_a or Ω_b.

REMARK 1.2. – If $q_1, q_2 \in \Gamma$ and there are $t_1, t_2 \in \mathbb{R}$ and $0 < \varepsilon \le R$ such that $q_1((-\infty, t_1]) \cup q_2((-\infty, t_1]) \subset B_\varepsilon(a)$, $q_1([t_2,\infty)) \cup q_2([t_2,\infty)) \subset B_\varepsilon(b)$ and $q_1(t) = q_2(t)$ for all $t \in [t_1, t_2]$ then $rot(q_1) = rot(q_2)$.

Analogous observations take place for q_1, q_2 belonging to one of the sets: Υ, Ω_a and Ω_b.

To exam rotation numbers of homoclinic and heteroclinic solutions of (1.1) we introduce the sets:

$$\Gamma^- = \{q \in \Gamma : rot(q) < 0\}, \qquad \Gamma^+ = \{q \in \Gamma : rot(q) \ge 0\},$$

$$\Omega_a^n = \{q \in \Omega_a : rot(q) \ge n\}, \qquad \Omega_a^{-n} = \{q \in \Omega_a : rot(q) \le -n\}$$

and

$$\Omega_b^n = \{q \in \Omega_b : rot(q) \ge n\}, \qquad \Omega_b^{-n} = \{q \in \Omega_b : rot(q) \le -n\},$$

where $n \in \mathbb{N}$. For $q \in \Lambda$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt. \tag{1.2}$$

Let

$$\gamma^\pm = \inf\{I(q) : q \in \Gamma^\pm\},$$

$$\omega_a^{\pm n} = \inf\{I(q) : q \in \Omega_a^{\pm n}\}$$

and

$$\omega_b^{\pm n} = \inf\{I(q): q \in \Omega_b^{\pm n}\},$$

where $n \in \mathbb{N}$. From now on, we will assume that

$$\gamma^- \le \gamma^+.$$

This involves no loss of generality.

Let us remark that if q is a member of one of the sets: $\Gamma^\pm$, Γ, $\Omega_a^{\pm n}$, Ω_a, $\Omega_b^{\pm n}$, Ω_b then $q + s\psi$ is a member of the same set for $s \in \mathbb{R}$ small enough and $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$. Moreover, if q is a minimizer of I on one of these families then

$$\frac{d}{ds} I(q + s\psi)_{|s=0} = 0 = \int_{-\infty}^{\infty} ((\dot{q}(t), \dot{\psi}(t)) - (\nabla V(q(t)), \psi(t)))dt,$$

and consequently, q is a weak solution of (1.1). Analysis similar to that in the proof of Proposition 3.18 in [13] shows that q is a classical solution of (1.1).

The goal of this paper is to exam the existence of solutions of the Hamiltonian system (1.1) that are not homotopic in $\mathbb{R}^2 \setminus \{\xi\}$ rel the endpoints. More precisely, we are going to prove the following theorems.

THEOREM 1.3. – *Assume that* $V: \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ *satisfies conditions* $(V_1) - (V_5)$ *and* $\gamma^- \le \gamma^+$. *Then there exists* $Q \in \Gamma^-$ *such that* Q *is a classical solution of the Hamiltonian system* (1.1) and $\dot{Q}(t) \to 0$, *as* $t \to \pm\infty$.

THEOREM 1.4. – *Assume that* $V: \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ *satisfies conditions* $(V_1) - (V_5)$ *and* $\gamma^- \le \gamma^+$. *Then one of the following theses holds.*

(*i*) *There is* $Q \in \Gamma^+$ *such that* Q *is a classical solution of* (1.1) and $\dot{Q}(t) \to 0$, as $t \to \pm\infty$.

(*ii*) *There is* $Q \in \Omega_a$ *such that* Q *is a classical solution of* (1.1) *with* $rot(Q) > 0$ *and* $\dot{Q}(t) \to 0$, *as* $t \to \pm\infty$.

(*iii*) *There is* $Q \in \Gamma^-$ *such that* Q *is a classical solution of* (1.1), $\dot{Q}(t) \to 0$, *as* $t \to \pm\infty$ *and*

$$\gamma^+ = I(Q) + \omega_b^n,$$

where $n = -rot(Q)$. *Moreover, the Hamiltonian system* (1.1) *possesses either a classical solution* $p \in \Omega_b$ *such that* $rot(p) > 0$ *and* $\dot{p}(t) \to 0$, *as* $t \to \pm\infty$ *or a classical solution* $Q_0 \in \Gamma^-$ *such that* $rot(Q_0) < rot(Q)$ *and* $\dot{Q}_0(t) \to 0$, *as* $t \to \pm\infty$.

In the case where $\gamma^- > \gamma^+$ we get the analogous theorems. Summarizing.

CONCLUSION 1.5. – *Under the assumptions* $(V_1) - (V_5)$, *the Hamiltonian system* (1.1) *possesses at least two solutions which wind around* ξ *and join* $\{a, b\}$

to $\{a, b\}$. *One of them is a heteroclinic orbit joining a to b. The second is either heteroclinic with a rotation number different from the first or homoclinic.*

There are some works on periodic, homoclinic and heteroclinic solutions for Hamiltonian systems with singularities. We refer the reader to: [1-6, 9, 10, 14, 15, 18] and the references given there. For a treatment of the existence of other types of solutions we refer to: [7, 8, 16, 17].

We are motivated by [14] of P. H. Rabinowitz. He studied the existence of homoclinic (to 0) solutions for a family of singular Hamiltonian systems which are periodically forced:

$$\ddot{q} + V_q(t, q) = 0,$$

where $V \in C^1(\mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}), \mathbb{R})$ $(\xi \neq 0)$ is T-periodic with respect to t. Moreover, for each $t \in \mathbb{R}$, $V(t, \cdot)$ satisfies (V_2), (V_3) and (V_5) uniformly in t, $V(t, x) \leq 0$ and $V(t, x) = 0$ iff $x = 0$. Under these assumptions, he proved the existence of two homoclinic orbits $q^{\pm}: \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$ such that the rotation number of $q^{\pm}$ is positive and negative, respectively. In the same work he also treated a more general situation in which V has strong force singularities at $\xi_1, \ldots, \xi_k$. In this case he established the existence of at least k geometrically distinct solutions homoclinic to 0.

Finally, I would like to mention the paper [4] of M. J. Borges. She considered the Hamiltonian system (1.1) with the potential V possessing a global maximum at 0 and strong force singularities at two points: ξ_1, ξ_2. Using variational methods she found homoclinic solutions winding around each singularity and around both singularities, periodic solutions and heteroclinic solutions joining 0 to periodic solutions.

Our paper is organized as follows. In Section 2 we discuss some properties of the action integral I. In Section 3 we prove Theorem 1.3. Section 4 provides a detailed proof of Theorem 1.4.

The problem is studied by variational methods. We look for connecting orbits by minimizing I on suitable classes of maps $q: \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$. Although some ideas are from [14], there are many new tricks involved in this work (see Lemmas 4.1-4.6).

2. – Some properties of the action integral.

In this section we present some properties of the action functional I given by (1.2). We will use them in our studies.

Define

$$\mathcal{M} = \{x \in \mathbb{R}^2 : V(x) = 0\} = \{a, b\},$$

$$a_\varepsilon = \inf\{-V(x) : x \notin B_\varepsilon(\mathcal{M})\},$$

where $0 < \varepsilon \leq R$ and $B_\varepsilon(\mathcal{M}) = B_\varepsilon(a) \cup B_\varepsilon(b)$. By (V_2), (V_4) and (V_5) it follows that $a_\varepsilon > 0$.

LEMMA 2.1. – *Suppose that* $q \in \Lambda$ *and* $q(t) \notin B_\varepsilon(\mathcal{M})$ *for each* $t \in \bigcup_{i=1}^{k} [r_i, s_i]$, *where* $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ *for* $i \neq j$. *Then*

$$I(q) \geq \sqrt{2a_\varepsilon} \sum_{i=1}^{k} |q(s_i) - q(r_i)|. \tag{2.1}$$

The proof of Lemma 2.1 is the same as that of Lemma 3.6 in [13] or Lemma 2.1 in [12].

LEMMA 2.2. – *If* $q \in \Lambda$ *and* $I(q) < \infty$ *then* $q \in L^\infty(\mathbb{R}, \mathbb{R}^2)$.

LEMMA 2.3. – *If* $q \in \Lambda$ *and* $I(q) < \infty$ *then* $q(\pm\infty) \in \mathcal{M}$.

We can easily prove these two lemmas by the use of Lemma 2.1. For more details we refer the reader to [13] (see Remark 3.10 and Proposition 3.11) and [12] (see Corollary 2.2 and Lemma 2.4).

PROPOSITION 2.4. – *If* $\{q_m\}_{m\in\mathbb{N}}$ *is a sequence that belongs to one of the families:* Γ, Υ, Ω_a *or* Ω_b *and* $\{I(q_m)\}_{m\in\mathbb{N}}$ *is a bounded sequence in* $\mathbb{R}$, *then* $\{q_m\}_{m\in\mathbb{N}}$ *possesses a subsequence that converges weakly in* E *and strongly in* $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$.

PROOF. – Assume that $\{q_m\}_{m\in\mathbb{N}} \subset \Gamma$. It is sufficient to show that $\{q_m\}_{m\in\mathbb{N}}$ is a bounded sequence in E. By assumption, there is $M > 0$ such that for all $m \in \mathbb{N}$,

$$0 < I(q_m) \leq M.$$

From this and (1.2) we get

$$\|\dot{q}_m\|^2_{L^2} \leq 2M.$$

Moreover, from Lemma 2.2 it follows that $q_m \in L^\infty(\mathbb{R}, \mathbb{R}^2)$ for all $m \in \mathbb{N}$.

Fix $0 < \varepsilon \leq R$. Then for each $m \in \mathbb{N}$ there are $\tau_m, t_m \in \mathbb{R}$ such that $q_m(\tau_m) \in \partial B_\varepsilon(a)$, $q_m(t) \in B_\varepsilon(a)$ for all $t < \tau_m$, $q_m(t_m) \in \partial B_\varepsilon(b)$ and $q_m(t) \in B_\varepsilon(b)$ for all $t > t_m$. Finally, for $q_m|_{[\tau_m, t_m]}$ there is $s_m \in [\tau_m, t_m]$ such that

$$|q_m(s_m)| = \max_{t\in[\tau_m, t_m]} |q_m(t)|.$$

Applying Lemma 2.1 we conclude that the sequence $\{q_m(s_m)\}_{m\in\mathbb{N}}$ is bounded. Hence $\{q_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^2)$.

In consequence, $\{q_m\}_{m\in\mathbb{N}}$ is bounded in E. By the reflexivity of E there is $Q \in E$ such that going to a subsequence $q_m \rightharpoonup Q$ in E, which implies that $q_m \to Q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$.

In the rest of cases, the proof is similar. □

LEMMA 2.5. – *If* $q \in \Lambda$ *and* $q(t) \in \mathcal{N}$ *for all* $t \in [\sigma, \mu]$ *then*

$$|U(q(\mu))| \leq |U(q(\sigma))| + \left(\int_{\sigma}^{\mu} -V(q(t))dt\right)^{\frac{1}{2}} \cdot \left(\int_{\sigma}^{\mu} |\dot{q}(t)|^2 dt\right)^{\frac{1}{2}}.$$

The proof of this lemma can be found in [14] (see (2.21), p. 271). It is based on the strong force condition (V_3).

Applying the above inequality and (1.2), for $q \in \Lambda$ such that $q(t) \in \mathcal{N}$ for all $t \in [\sigma, \mu]$ we get

$$|U(q(\mu))| \leq |U(q(\sigma))| + \sqrt{2} I(q).$$

PROPOSITION 2.6. – *Let* $\{q_m\}_{m \in \mathbb{N}} \subset \Lambda$ *be a sequence such that* $\{I(q_m)\}_{m \in \mathbb{N}}$ *is bounded. Then there is* $r > 0$ *such that* $q_m(t) \cap B_r(\xi) = \emptyset$ *for all* $t \in \mathbb{R}$ *and* $m \in \mathbb{N}$.

PROOF. – By Lemma 2.3, $q_m(\pm\infty) \in \mathcal{M}$ for each $m \in \mathbb{N}$.

On the contrary, suppose that there exists a sequence $\{q_m(\mu_m)\}_{m \in \mathbb{N}}$ such that $q_m(\mu_m) \to \xi$, as $m \to \infty$. Fix $0 < \delta \leq R$ such that $\overline{B_\delta(\xi)} \subset \mathcal{N}$. There is $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $|q_m(\mu_m) - \xi| < \delta$. For each $m \geq m_0$ there exists $\sigma_m < \mu_m$ such that $q_m(\sigma_m) \in \partial B_\delta(\xi)$ and $q_m(t) \in B_\delta(\xi)$ for all $t \in (\sigma_m, \mu_m)$. Then

$$|U(q_m(\mu_m))| \leq |U(q_m(\sigma_m))| + \sqrt{2} I(q_m).$$

As $\{U(q_m(\sigma_m))\}_{m \in \mathbb{N}}$ and $\{I(q_m)\}_{m \in \mathbb{N}}$ are bounded, we get $\{U(q_m(\mu_m))\}_{m \in \mathbb{N}}$ is bounded, too. On the other hand, by (V_3), we receive $|U(q_m(\mu_m))| \to \infty$, as $m \to \infty$, a contradiction. □

FACT 2.7. – *If* $q: \mathbb{R} \to \mathbb{R}^n$ *is a continuous function such that* $\dot{q} \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ *then for each* $t \in \mathbb{R}$,

$$|q(t)| \leq \sqrt{2} \left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|q(s)|^2 + |\dot{q}(s)|^2) ds \right)^{\frac{1}{2}}.$$

This inequality was proved for example in [11] (see Fact 2.8).

PROPOSITION 2.8. – *If* $Q \in \Lambda$ *is a homoclinic or heteroclinic orbit of the Hamiltonian system* (1.1) *then*

$$\dot{Q}(t) \to 0, \text{ as } t \to \pm\infty.$$

PROOF. – From Fact 2.7 we get

$$|\dot{Q}(t)|^2 \leq 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{Q}(s)|^2 ds + 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{Q}(s)|^2 ds$$

for each $t \in \mathbb{R}$. As $Q(t)$ satisfies the Hamiltonian system (1.1) we have

$$|\dot{Q}(t)|^2 \leq 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{Q}(s)|^2 ds + 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(Q(s))|^2 ds.$$

Let $\eta > 0$. By (V_4), there is $L_1 > 0$ such that if $|s| > L_1$ then $|\nabla V(Q(s))|^2 < \frac{\eta}{4}$. Since $Q \in \Lambda \subset E$, there is $L_2 > 0$ such that if $|t| > L_2$ then

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{Q}(s)|^2 ds < \frac{\eta}{4}.$$

Set $L = \max\{L_1, L_2\}$. If $|t| > L + \frac{1}{2}$ then $|\dot{Q}(t)|^2 < \eta$. Hence $\dot{Q}(t) \to 0$, as $t \to \pm\infty$. □

3. – Proof of Theorem 1.3.

Let $\{q_m\}_{m\in\mathbb{N}} \subset \Gamma^-$ be a sequence such that

$$\lim_{m\to\infty} I(q_m) = \gamma^-.$$

From Proposition 2.4 it follows that there is $Q \in E$ such that going to a subsequence if necessary $q_m \rightharpoonup Q$ in E and $q_m \to Q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. By Proposition 2.6 we conclude that $Q \in \Lambda$.

REMARK 3.1. – For all $T_1, T_2 \in \mathbb{R}$ such that $T_1 < T_2$ a functional given by

$$E \ni q \longrightarrow \int_{T_1}^{T_2} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt$$

is weakly lower semi-continuous.

Hence for each $l \in \mathbb{N}$,

$$\begin{aligned} \int_{-l}^{l} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt &\leq \liminf_{m\to\infty} \int_{-l}^{l} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt \\ &\leq \lim_{m\to\infty} I(q_m) = \gamma^-. \end{aligned}$$

Letting $l \to \infty$ we receive

$$I(Q) \leq \gamma^-.$$

By Lemma 2.3, $Q(\pm\infty) \in \mathcal{M}$. We will show that $Q(-\infty) = a$ and $Q(\infty) = b$. Fix

$0<\varepsilon\leq R$. Since $q_m(-\infty)=a$, there is $\tau_m\in\mathbb{R}$ such that $q_m(\tau_m)\in\partial B_\varepsilon(a)$ and $q_m(t)\in B_\varepsilon(a)$ for all $t<\tau_m$.

REMARK 3.2. – If $q\in\Lambda$ then for each $\theta\in\mathbb{R}$, $\theta q=q(\cdot-\theta)\in\Lambda$ and $I(q)=I(\theta q)$. Moreover, if $\bar{q}(t)=q(-t)$ then $I(q)=I(\bar{q})$.

Therefore, without loss of generality, we can assume that $\tau_m=0$ for each $m\in\mathbb{N}$. In consequence,

$$|q_m(t)-a|\leq\varepsilon$$

for all $m\in\mathbb{N}$ and $t\leq 0$, and so

$$|Q(t)-a|\leq\varepsilon$$

for all $t\leq 0$. As $Q(t)\in\overline{B_\varepsilon(a)}$ for all $t\leq 0$ and $Q(-\infty)\in\mathcal{M}$, we get $Q(-\infty)=a$. Suppose, contrary to our claim, that $Q(\infty)=a$. Let $\delta>0$ be sufficiently small such that $4\delta<\varepsilon$ and

$$2\delta^2+\max\{-V(x)\colon |x-a|\leq 2\delta\}<\frac{\varepsilon}{4}\sqrt{2a_{\frac{\varepsilon}{2}}}.$$

There is $t_\delta>0$ such that $Q(t_\delta)\in\partial B_\delta(a)$ and $Q(t)\in B_\delta(a)$ for all $t>t_\delta$. Since $q_m(t_\delta)\to Q(t_\delta)$, there exists $m_0\in\mathbb{N}$ such that for all $m\geq m_0$,

$$|q_m(t_\delta)-Q(t_\delta)|<\delta.$$

From this,

$$|q_m(t_\delta)-a|<2\delta$$

for all $m\geq m_0$. Take $s_\delta^m\in[0,t_\delta]$ such that $q_m(t)\notin B_{\frac{\varepsilon}{2}}(a)$ for all $t\in[0,s_\delta^m]$ and $q_m(s_\delta^m)\in\partial B_{\frac{\varepsilon}{2}}(a)$.

Then

$$\begin{aligned}I(q_m)&\geq\int_0^{s_\delta^m}\left(\frac{1}{2}|\dot{q}_m(t)|^2-V(q_m(t))\right)dt+\int_{t_\delta}^{\infty}\left(\frac{1}{2}|\dot{q}_m(t)|^2-V(q_m(t))\right)dt\\&\geq\frac{\varepsilon}{2}\sqrt{2a_{\frac{\varepsilon}{2}}}+\int_{t_\delta}^{\infty}\left(\frac{1}{2}|\dot{q}_m(t)|^2-V(q_m(t))\right)dt,\end{aligned}$$

by Lemma 2.1. Define

$$Q_m(t)=\begin{cases}a \text{ if } \; t\leq t_\delta-1,\\(t-t_\delta+1)q_m(t_\delta)+(t_\delta-t)a \text{ if } \; t\in[t_\delta-1,t_\delta],\\q_m(t) \text{ if } \; t\geq t_\delta,\end{cases}$$

where $m \geq m_0$. By the above, we have $Q_m \in \Gamma$ and

$$\begin{aligned} I(Q_m) =& \int_{t_\delta - 1}^{t_\delta} \left(\frac{1}{2} |q_m(t_\delta) - a|^2 - V(Q_m(t)) \right) dt \\ &+ \int_{t_\delta}^{\infty} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt \\ \leq& 2\delta^2 + \max\{-V(x) \colon |x - a| \leq 2\delta\} \\ &+ I(q_m) - \frac{\varepsilon}{2} \sqrt{2a_{\frac{\varepsilon}{2}}} \\ <& I(q_m) - \frac{\varepsilon}{4} \sqrt{2a_{\frac{\varepsilon}{2}}} \end{aligned}$$

for all $m \geq m_0$. In consequence,

$$\gamma^- \leq \liminf_{m \to \infty} I(Q_m) \leq \liminf_{m \to \infty} I(q_m) - \frac{\varepsilon}{4} \sqrt{2a_{\frac{\varepsilon}{2}}} = \gamma^- - \frac{\varepsilon}{4} \sqrt{2a_{\frac{\varepsilon}{2}}},$$

a contradiction. Thus $Q(\infty) = b$.

By the above, $Q \in \Gamma$. Suppose, contrary to our claim, that $Q \in \Gamma^+$. If $\gamma^- < \gamma^+$ then $I(Q) \geq \gamma^+ > \gamma^-$, a contradiction. Assume now that $\gamma^- = \gamma^+$. Let β be a positive constant such that $\beta < \frac{1}{2} \sqrt{2a_\varepsilon}(|b - \xi| - 2\varepsilon)$. Choose $T > 0$ such that $Q([T, \infty)) \subset B_\varepsilon(b)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2} |\dot{Q}(t)|^2 - V(Q(t)) dt \right) dt > \gamma^- - \beta.$$

Since $q_m \to Q$ uniformly on $[0, T]$, there is $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $rot(q_m|_{(-\infty, T]}) = rot(Q) \geq 0$, and so $rot(q_m|_{[T, \infty)}) < -rot(Q) \leq 0$. Moreover, by Lemma 2.1,

$$\int_{T}^{\infty} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt \geq \sqrt{2a_\varepsilon}(|b - \xi| - 2\varepsilon)$$

for all $m \geq m_0$. From Remark 3.1 we get

$$\liminf_{m \to \infty} \int_{-\infty}^{T} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt \geq \int_{-\infty}^{T} \left(\frac{1}{2} |\dot{Q}(t)|^2 - V(Q(t)) \right) dt.$$

Hence there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt > \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt - \frac{1}{4}\sqrt{2a_\varepsilon}(|b-\xi| - 2\varepsilon).$$

Consequently, for $m \in \mathbb{N}$ large enough,

$$\begin{aligned} I(q_m) &= \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt + \int_{T}^{\infty} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt \\ &> \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))dt\right) dt + \frac{3}{4}\sqrt{2a_\varepsilon}(|b-\xi| - 2\varepsilon) \\ &> \gamma^- - \beta + \frac{3}{4}\sqrt{2a_\varepsilon}(|b-\xi| - 2\varepsilon) \\ &> \gamma^- + \frac{1}{4}\sqrt{2a_\varepsilon}(|b-\xi| - 2\varepsilon), \end{aligned}$$

which contradicts the assumption $\lim_{m\to\infty} I(q_m) = \gamma^-$. Therefore $Q \in \Gamma^-$, and in consequence, $I(Q) = \gamma^-$. Thus Q is a classical solution of (1.1).

From Proposition 2.8 we get $\dot{Q}(t) \to 0$, as $t \to \pm\infty$.

4. – Proof of Theorem 1.4.

Let $\{q_m\}_{m\in\mathbb{N}} \subset \Gamma^+$ be a sequence such that

$$\lim_{m\to\infty} I(q_m) = \gamma^+.$$

From Proposition 2.4 it follows that there is a subsequence of $\{q_m\}_{m\in\mathbb{N}}$ which converges weakly in E and strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. For abbreviation, we write

$$q_m \rightharpoonup Q \text{ in } E \text{ and } q_m \to Q \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2).$$

Moreover, by Proposition 2.6, $Q \in \Lambda$, by Remark 3.1, $I(Q) \leq \gamma^+$, and by Lemma 2.3, $Q(\pm\infty) \in \mathcal{M}$.

Fix $0 < \varepsilon \leq R$. Remark 3.2 implies that we can assume that $q_m(0) \in \partial B_\varepsilon(a)$ and $q_m(t) \in B_\varepsilon(a)$ for all $t < 0$ and $m \in \mathbb{N}$. Then $Q((-\infty, 0]) \subset \overline{B_\varepsilon(a)}$, and in consequence, $Q(-\infty) = a$. Now the proof falls naturally into three parts, because Q may belong to Γ^+, Γ^- or Ω_a.

We have divided the proof into a sequence of lemmas. The proofs of Lemmas 4.3-4.6 are similar in spirit. However, for convenience of the reader, we have decided to prove them all.

LEMMA 4.1. – *For each* $\eta > 0$ *there is* $0 < r \leq R$ *such that for all* $x, y \in B_r(a)$ *(resp.* $x, y \in B_r(b)$*) and* $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2}|y-x|^2 - V(l_{x,y}(t))\right) dt < \eta,$$

where $l_{x,y}(t) = (T+1-t)x + (t-T)y$ *for each* $t \in [T, T+1]$.

The proof of Lemma 4.1 is straightforward. Therefore we omit it.

LEMMA 4.2. – *If* $Q \in \Gamma^+$ *then* Q *is a classical solution of* (1.1).

PROOF. – Since $Q \in \Gamma^+$, we get $I(Q) \geq \gamma^+$. Consequently, $I(Q) = \gamma^+$, and so Q is a minimizer of I on Γ^+. Thus Q is a classical solution of (1.1). □

LEMMA 4.3. – *If* $Q \in \Omega_a$ *then* Q *is a classical solution of* (1.1) *and* $rot(Q) > 0$.

PROOF. – Set $k = rot(Q)$. It is sufficient to show that

$$I(Q) = \inf\{I(q) : q \in \Omega_a \ \wedge \ rot(q) = k\}.$$

Suppose, by contradiction, that there is $\tilde{q} \in \Omega_a$ such that $rot(\tilde{q}) = k$ and $I(\tilde{q}) < I(Q)$. Put $d = I(Q) - I(\tilde{q})$. Fix $0 < \eta < \frac{d}{2}$. By Lemma 4.1, there is $0 < \delta \leq R$ such that for all $x, y \in B_\delta(a)$ and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2}|y-x|^2 - V(l_{x,y}(t))\right) dt < \frac{\eta}{2}.$$

Choose $T > 0$ such that $\tilde{q}([T, \infty)) \subset B_\delta(a)$, $Q([T, \infty)) \subset B_\delta(a)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt > I(Q) - \frac{\eta}{4}.$$

Since $q_m \to Q$ uniformly on $[0, T+1]$, there is $m_0 \in \mathbb{N}$ such that $q_m([T, T+1]) \subset B_\delta(a)$ and $rot(q_m|_{(-\infty,T]}) = rot(Q)$ for all $m \geq m_0$.

Let

$$\tilde{q}_m(t) = \begin{cases} \tilde{q}(t) \ \text{ if } \ t \leq T, \\ (T+1-t)\tilde{q}(T) + (t-T)q_m(T+1) \ \text{ if } \ t \in [T, T+1], \\ q_m(t) \ \text{ if } \ t \geq T+1, \end{cases}$$

where $m \geq m_0$. Then $\tilde{q}_m(-\infty) = a$, $\tilde{q}_m(\infty) = b$ and

$$\begin{aligned} rot(\tilde{q}_m) &= rot(\tilde{q}_{|(-\infty,T]}) + rot(q_{m|[T+1,\infty)}) \\ &= rot(Q_{|(-\infty,T]}) + rot(q_{m|[T+1,\infty)}) \\ &= rot(q_{m|(-\infty,T]}) + rot(q_{m|[T+1,\infty)}) = rot(q_m), \end{aligned}$$

and hence $\tilde{q}_m \in \Gamma^+$. From Remark 3.1 we deduce that there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt > \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt - \frac{\eta}{4}.$$

Using the above inequalities we get

$$\begin{aligned} I(q_m) - I(\tilde{q}_m) \geq & \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt \\ & - \int_{T}^{T+1} \left(\frac{1}{2}|\dot{\tilde{q}}_m(t)|^2 - V(\tilde{q}_m(t))\right) dt \\ & - \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{\tilde{q}}(t)|^2 - V(\tilde{q}(t))\right) dt \\ & > I(Q) - I(\tilde{q}) - \eta > \frac{d}{2} \end{aligned}$$

for $m \in \mathbb{N}$ large enough, and so

$$\gamma^+ = \lim_{m\to\infty} I(q_m) \geq \liminf_{m\to\infty} I(\tilde{q}_m) + \frac{d}{2} \geq \gamma^+ + \frac{d}{2},$$

a contradiction. Consequently, Q is a classical solution of (1.1).

To complete the proof, we have to show that $k > 0$. Suppose, on the contrary, that $k \leq 0$. For $\eta > 0$ we choose δ, T, m_0 and m_1 as above.

For $m \geq m_0$, let

$$u_m(t) = \begin{cases} a \text{ if } \; t \leq T, \\ (T+1-t)a + (t-T)q_m(T+1) \text{ if } \; t \in [T, T+1], \\ q_m(t) \text{ if } \; t \geq T+1. \end{cases}$$

Since $rot(q_m) \geq 0$ and $rot(q_{m|(-\infty,T+1]}) = rot(Q_{|(-\infty,T+1]}) = k$, we get

$$rot(u_m) = rot(q_{m|[T+1,\infty)}) \geq 0.$$

Thus $u_m \in \Gamma^+$. Furthermore, for $m \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} I(q_m) - I(u_m) \geq & \int_{-\infty}^{T} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt \\ & - \int_{T}^{T+1} \left(\frac{1}{2} |\dot{u}_m(t)|^2 - V(u_m(t)) \right) dt \\ > & I(Q) - \eta. \end{aligned}$$

Passing to a limit we get

$$\gamma^+ = \lim_{m\to\infty} I(q_m) \geq \liminf_{m\to\infty} I(u_m) + I(Q) - \eta \geq \gamma^+ + I(Q) - \eta.$$

Letting $\eta \to 0^+$,

$$\gamma^+ \geq \gamma^+ + I(Q) > \gamma^+,$$

a contradiction. Therefore $k > 0$. □

LEMMA 4.4. – *If $Q \in \Gamma^-$ then $\gamma^+ = I(Q) + \omega_b^n$, where $n = -\,rot(Q)$.*

PROOF. – Let $\eta > 0$. From Lemma 4.1 it follows that there is $0 < \delta \leq R$ such that for all $x, y \in B_\delta(b)$ and $T \in \mathbb{R}$,

$$\int_{T}^{T+1} \left(\frac{1}{2} |y - x|^2 - V(l_{x,y}(t)) \right) dt < \frac{\eta}{2}.$$

By assumption, there is $T > 0$ such that $Q([T, \infty)) \subset B_\delta(b)$. By the definition of infimum, there exists $p \in \Omega_b^n$ such that

$$I(p) < \omega_b^n + \frac{\eta}{2}.$$

Additionally, by Remark 3.2, we can assume that $p((-\infty, T+1]) \subset B_\delta(b)$.

Define

$$\hat{Q}(t) = \begin{cases} Q(t) \text{ if } & t \leq T, \\ (T+1-t)Q(T) + (t-T)p(T+1) \text{ if } & t \in [T, T+1], \\ p(t) \text{ if } & t \geq T+1. \end{cases}$$

We have $\hat{Q}(-\infty) = a$, $\hat{Q}(\infty) = b$ and $I(\hat{Q}) < I(Q) + \omega_b^n + \eta$. Moreover,

$$rot(\hat{Q}) = rot(Q) + rot(p) \geq 0.$$

Hence $\hat{Q} \in \Gamma^+$ and $I(\hat{Q}) \geq \gamma^+$. In consequence,

$$\gamma^+ < I(Q) + \omega_b^n + \eta,$$

and letting $\eta \to 0^+$, we get

$$\gamma^+ \le I(Q) + \omega_b^n. \tag{4.1}$$

In order to complete the proof, we have to show that $\gamma^+ \ge I(Q) + \omega_b^n$.

To this aim, fix $\eta > 0$. By Lemma 4.1, there is $0 < \delta \le R$ such that for each $x \in B_\delta(b)$ and $T \in \mathbb{R}$,

$$\int_{T-1}^{T} \left(\frac{1}{2}|x-b|^2 - V(l_{b,x}(t))\right) dt < \frac{\eta}{2},$$

where $l_{b,x}(t) = (T-t)b + (t-T+1)x$ for $t \in [T-1, T]$. By assumption, there is $T > 0$ such that $Q([T,\infty)) \subset B_\delta(b)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt > I(Q) - \frac{\eta}{4}.$$

Let $\{p_m\}_{m\in\mathbb{N}} \subset \Omega_b$ be given by

$$p_m(t) = \begin{cases} b \text{ if } \ t \le T-1, \\ (T-t)b + (t-T+1)q_m(T) \text{ if } \ t \in [T-1, T], \\ q_m(t) \text{ if } \ t \ge T. \end{cases}$$

Then for $m \in \mathbb{N}$ large enough,

$$I(p_m) < \int_{T}^{\infty} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt + \frac{\eta}{2}.$$

By the strong convergence of $\{q_m\}_{m\in\mathbb{N}}$ in $L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$, there is $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$, $q_m(T) \in B_\delta(b)$ and $rot(q_m|_{(-\infty,T]}) = rot(Q)$.

Hence

$$rot(Q) + rot(p_m) = rot(q_m) \ge 0.$$

From this it follows that for all $m \ge m_0$ we have

$$rot(p_m) \ge n,$$

and so $p_m \in \Omega_b^n$. Remark 3.1 implies the existence of $m_1 \in \mathbb{N}$ such that

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt > \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt - \frac{\eta}{4}$$

for all $m \ge m_1$. By the above, we conclude that for $m \in \mathbb{N}$ sufficiently large,

$$I(q_m) > I(Q) + I(p_m) - \eta \ge I(Q) + \omega_b^n - \eta.$$

Hence

$$\gamma^+ \geq I(Q) + \omega_b^n - \eta.$$

Letting $\eta \to 0^+$, we receive

$$\gamma^+ \geq I(Q) + \omega_b^n. \tag{4.2}$$

Combining (4.1) with (4.2), we get our claim. □

LEMMA 4.5. – *If* $Q \in \Gamma^-$ *then* Q *is a classical solution of* (1.1).

PROOF. – Put $n = -rot(Q)$. It suffices to prove that

$$I(Q) = \inf\{I(q): q \in \Gamma^- \ \wedge \ rot(q) = -n\}.$$

On the contrary, suppose that there exists $\hat{q} \in \Gamma^-$ such that $rot(\hat{q}) = -n$ and $I(\hat{q}) < I(Q)$. Define $d = I(Q) - I(\hat{q})$. Fix $0 < \eta < \frac{d}{2}$. From Lemma 4.1 it follows that there is $0 < \delta \leq R$ such that for all $x, y \in B_\delta(b)$ and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2}|y - x|^2 - V(l_{x,y}(t))\right) dt < \frac{\eta}{2}.$$

We can choose $T > 0$ such that $\hat{q}([T, \infty)) \subset B_\delta(b)$, $Q([T, \infty)) \subset B_\delta(b)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt > I(Q) - \frac{\eta}{4}.$$

Since $q_m \to Q$ uniformly on $[0, T+1]$, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $q_m([T, T+1]) \subset B_\delta(b)$ and $rot(q_m|_{(-\infty,T]}) = rot(Q)$. For each $m \geq m_0$, let $\hat{q}_m: \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\hat{q}_m(t) = \begin{cases} \hat{q}(t) \text{ if } \ t \leq T, \\ (T + 1 - t)\hat{q}(T) + (t - T)q_m(T + 1) \text{ if } \ t \in [T, T + 1], \\ q_m(t) \text{ if } \ t \geq T + 1. \end{cases}$$

By definition, it follows that $\hat{q}_m \in \Gamma$. What is more,

$$\begin{aligned} rot(\hat{q}_m) &= rot(\hat{q}|_{(-\infty,T]}) + rot(q_m|_{[T+1,\infty)}) \\ &= rot(Q|_{(-\infty,T]}) + rot(q_m|_{[T+1,\infty)}) \\ &= rot(q_m|_{(-\infty,T]}) + rot(q_m|_{[T+1,\infty)}) = rot(q_m), \end{aligned}$$

and so $\hat{q}_m \in \Gamma^+$. Applying Remark 3.1 we conclude that there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}_m(t)|^2 - V(q_m(t))\right) dt > \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{Q}(t)|^2 - V(Q(t))\right) dt - \frac{\eta}{4}.$$

Combining the above inequalities we receive

$$I(q_m) - I(\hat{q}_m) > I(Q) - I(\hat{q}) - \eta$$

for $m \in \mathbb{N}$ sufficiently large, and hence

$$I(q_m) - I(\hat{q}_m) > \frac{d}{2}.$$

In consequence,

$$\gamma^+ = \lim_{m\to\infty} I(q_m) \geq \liminf_{m\to\infty} I(\hat{q}_m) + \frac{d}{2} \geq \gamma^+ + \frac{d}{2},$$

a contradiction. Therefore Q is a classical solution of (1.1). □

LEMMA 4.6. – *If* $Q \in \Gamma^-$ *then* (1.1) *has either a homoclinic solution* $p \in \Omega_b$ *such that* $rot(p) > 0$ *or a heteroclinic solution* $Q_0 \in \Gamma^-$ *such that* $rot(Q_0) < rot(Q)$.

PROOF. – By Lemma 4.4,

$$\gamma^+ = I(Q) + \omega_b^n,$$

where $n = -rot(Q)$. Let $\{p_m\}_{m\in\mathbb{N}} \subset \Omega_b^n$ be a sequence such that

$$\lim_{m\to\infty} I(p_m) = \omega_b^n.$$

By Proposition 2.4, going to a subsequence if necessary, there is $p \in E$ such that

$$p_m \rightharpoonup p \text{ in } E \quad \wedge \quad p_m \to p \text{ in } L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^2).$$

From Proposition 2.6 it follows that $p \in \Lambda$. Applying Remark 3.1 we obtain

$$I(p) \leq \omega_b^n.$$

Lemma 2.3 implies $p(\pm\infty) \in \mathcal{M}$.

By Remark 3.2, we can assume that for all $m \in \mathbb{N}$, $p_m(0) \in \partial B_\varepsilon(b)$ and $p_m((-\infty, 0)) \subset B_\varepsilon(b)$. Then $p((-\infty, 0]) \subset \overline{B_\varepsilon(b)}$ and $p(-\infty) = b$.

We have to consider now two cases: $p(\infty) = b$ or $p(\infty) = a$.

CASE 1. – $p(\infty) = b$.

Put $k = rot(p)$. We will prove that

$$I(p) = \inf\{I(q) : q \in \Omega_b \ \wedge \ rot(q) = k\}.$$

Suppose, contrary to our claim, that there is $\tilde{p} \in \Omega_b$ such that $rot(\tilde{p}) = k$ and $I(\tilde{p}) < I(p)$. Set $d = I(p) - I(\tilde{p})$ and take $0 < \eta < \frac{d}{2}$. By Lemma 4.1, there exists $0 < \delta \leq R$ such that

$$\int_T^{T+1} \left(\frac{1}{2}|y-x|^2 - V(l_{x,y}(t))\right) dt < \frac{\eta}{2}$$

for all $x, y \in B_\delta(b)$ and $T \in \mathbb{R}$. Choose $T > 0$ such that $\tilde{p}([T,\infty)) \subset B_\delta(b)$, $p([T,\infty)) \subset B_\delta(b)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{p}(t)|^2 - V(p(t))\right) dt > I(p) - \frac{\eta}{4}.$$

By the almost uniformly convergence of $\{p_m\}_{m\in\mathbb{N}}$, there is $m_0 \in \mathbb{N}$ such that $p_m([T, T+1]) \subset B_\delta(b)$ and $rot(p_m|_{(-\infty,T]}) = rot(p)$ for all $m \geq m_0$.

For each $m \geq m_0$, let $\tilde{p}_m$ be given by

$$\tilde{p}_m(t) = \begin{cases} \tilde{p}(t) \text{ if } \; t \leq T, \\ (T+1-t)\tilde{p}(T) + (t-T)p_m(T+1) \;\text{ if }\; t \in [T, T+1], \\ p_m(t) \;\text{ if }\; t \geq T+1. \end{cases}$$

We have $\tilde{p}_m(-\infty) = b = \tilde{p}_m(\infty)$ and $rot(\tilde{p}_m) = rot(p_m)$. From Remark 3.1 it follows that there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{p}_m(t)|^2 - V(p_m(t))\right) dt > \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{p}(t)|^2 - V(p(t))\right) dt - \frac{\eta}{4}.$$

From what has already been proved, we obtain

$$I(p_m) > I(\tilde{p}_m) + \frac{d}{2}$$

for $m \in \mathbb{N}$ large enough. Hence

$$\omega_b^n = \lim_{m\to\infty} I(p_m) \geq \liminf_{m\to\infty} I(\tilde{p}_m) + \frac{d}{2} \geq \omega_b^n + \frac{d}{2},$$

a contradiction. In consequence, p is a classical solution of (1.1).

We observe that $k > 0$. By contradiction, assume that $k \leq 0$. Fix $\eta > 0$ and choose δ, T, m_0 and m_1 as above.

Define

$$v_m(t) = \begin{cases} b \text{ if } \; t \leq T, \\ (T+1-t)b + (t-T)p_m(t) \text{ if } \; t \in [T, T+1], \\ p_m(t) \text{ if } \; t \geq T+1 \end{cases}$$

for $m \geq m_0$. Since $rot(p_m) \geq n$ and $rot(p_{m|(-\infty,T+1]}) = k$, we get

$$rot(v_m) = rot(p_{m|[T+1,\infty)}) \geq n - k \geq n.$$

What is more, for $m \in \mathbb{N}$ large enough,

$$I(p_m) - I(v_m) > I(p) - \eta,$$

and so

$$\omega_b^n = \lim_{m\to\infty} I(p_m) \geq \liminf_{m\to\infty} I(v_m) + I(p) - \eta \geq \omega_b^n + I(p) - \eta.$$

Letting $\eta \to 0^+$, we receive

$$\omega_b^n \geq \omega_b^n + I(p) > \omega_b^n,$$

a contradiction.

Case 2. – $p(\infty) = a$.

Set $k = rot(p)$. We will prove that

$$I(p) = \inf\{I(q): q \in \Upsilon \ \wedge \ rot(q) = k\}.$$

On the contrary, assume that there exists $\hat{p} \in \Upsilon$ such that $rot(\hat{p}) = k$ and $I(\hat{p}) < I(p)$. Put $d = I(p) - I(\hat{p})$. Fix $0 < \eta < \frac{d}{2}$. By Lemma 4.1, there is $0 < \delta \leq R$ such that

$$\int_T^{T+1} \left(\frac{1}{2}|y - x|^2 - V(l_{x,y}(t))\right) dt < \frac{\eta}{2}$$

for all $x, y \in B_\delta(a)$ and $T \in \mathbb{R}$. Choose $T > 0$ such that $\tilde{p}([T,\infty)) \subset B_\delta(a)$, $p([T,\infty)) \subset B_\delta(a)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2}|\dot{p}(t)|^2 - V(p(t))\right) dt > I(p) - \frac{\eta}{4}.$$

There is $m_0 \in \mathbb{N}$ such that $p_m([T, T+1]) \subset B_\delta(a)$ and $rot(p_m|_{(-\infty,T]}) = rot(p)$ for all $m \geq m_0$.

For each $m \geq m_0$, we define $\hat{p}_m$ as follows:

$$\hat{p}_m(t) = \begin{cases} \hat{p}(t) \text{ if } \ t \leq T, \\ (T+1-t)\hat{p}(T) + (t-T)p_m(T+1) \text{ if } \ t \in [T, T+1], \\ p_m(t) \text{ if } \ t \geq T+1. \end{cases}$$

By definition, $\hat{p}_m(-\infty) = b = \hat{p}_m(\infty)$ and $rot(\hat{p}_m) = rot(p_m)$. From Remark 3.1

we see that there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2} |\dot{p}_m(t)|^2 - V(p_m(t)) \right) dt > \int_{-\infty}^{T} \left(\frac{1}{2} |\dot{p}(t)|^2 - V(p(t)) \right) dt - \frac{\eta}{4}.$$

Applying the above inequalities we immediately check that

$$I(p_m) > I(\hat{p}_m) + \frac{d}{2}.$$

Going to a limit we get

$$\omega_b^n = \lim_{m \to \infty} I(p_m) \geq \liminf_{m \to \infty} I(\hat{p}_m) + \frac{d}{2} \geq \omega_b^n + \frac{d}{2},$$

a contradiction. Hence p minimizes I on $\{q \in \Upsilon : rot(q) = k\}$.

We now observe that $k > n$. To this end, we first exclude the case $k < 0$. Set $\bar{p}(t) = p(-t)$. Of course, $\bar{p} \in \Gamma^+$. We obtain

$$I(\bar{p}) \geq \gamma^+ = I(Q) + \omega_b^n > \omega_b^n \geq I(p) = I(\bar{p}),$$

a contradiction.

We now exclude the case $0 \leq k \leq n$. For this purpose, take $\eta > 0$ and choose δ, T and m_0 as above.

Let P_m for $m \geq m_0$ be given by

$$P_m(t) = \begin{cases} a \text{ if } \; t \leq T, \\ (T+1-t)a + (t-T)p_m(T+1) \text{ if } \; t \in [T, T+1], \\ p_m(t) \text{ if } \; t \geq T+1. \end{cases}$$

Then $P_m(-\infty) = a$, $P_m(\infty) = b$ and $rot(P_m) = rot(p_m|_{[T+1,\infty)}) \geq n - k \geq 0$, and so $P_m \in \Gamma^+$. Moreover,

$$I(P_m) \leq \int_{T}^{T+1} \left(\frac{1}{2} |\dot{P}_m(t)|^2 + V(P_m(t)) \right) dt + I(p_m) < \frac{\eta}{2} + I(p_m).$$

Hence $\gamma^+ \leq \frac{\eta}{2} + \omega_b^n$, and consequently, $\gamma^+ \leq \omega_b^n$, a contradiction. By the above, we have $k > n$.

Set $Q_0(t) = p(-t)$ for $t \in \mathbb{R}$. Then $Q_0 \in \Gamma^-$ and

$$rot(Q_0) = -rot(p) - 1 = -k - 1 < -n - 1 < -n = rot(Q).$$

Thus $Q_0 \neq Q$. Finally, by Remark 3.2, we get

$$\begin{aligned} I(Q_0) &= I(p) = \inf\{I(q) : q \in \Upsilon \ \wedge \ rot(q) = k\} \\ &= \inf\{I(q) : q \in \Gamma^- \ \wedge \ rot(q) = -k - 1\}. \end{aligned}$$

Therefore Q_0 is a classical solution of (1.1). □

Applying Proposition 2.8 we complete the proof of Theorem 1.4.

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REFERENCES

[1] A. Ambrosetti - V. Coti Zelati, *Periodic solutions of singular Lagrangian systems*, Progress in Nonlinear Differential Equations and their Applications, Vol. **X**, Birkhäuser Boston, Inc., Boston, MA (1993).
[2] M. L. Bertotti - L. Jeanjean, *Multiplicity of homoclinic solutions for singular second-order conservative systems*, Proc. Roy. Soc. Edinburgh Sect. A, **126** (1996), 1169-1180.
[3] S. V. Bolotin, *Shadowing chains of collision orbits*, Discrete Contin. Dyn. Syst., **14** (2006), 235-260.
[4] M. J. Borges, *Heteroclinic and homoclinic solutions for a singular Hamiltonian system*, European J. Appl. Math., **17** (2006), 1-32.
[5] P. Caldiroli - L. Jeanjean, *Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems*, J. Differential Equations, **136** (1997), 76-114.
[6] P. Caldiroli - M. Nolasco, *Multiple homoclinic solutions for a class of autonomous singular systems in* $\mathbb{R}^2$, Ann. Inst. H. Poincaré Anal. Non Linéaire, **15** (1998), 113-125.
[7] P. Felmer - K. Tanaka, *Hyperbolic-like solutions for singular Hamiltonian systems*, NoDEA Nonlinear Differential Equations Appl., **7** (2000), 43-65.
[8] P. Felmer - K. Tanaka, *Scattering solutions for planar singular Hamiltonian systems via minimization*, Adv. Differential Equations, **5** (2000), 1519-1544.
[9] W. B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc., **204** (1975), 113-135.
[10] C. Greco, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlinear Anal., **12** (1988), 259-269.
[11] M. Izydorek - J. Janczewska, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations, **219** (2005), 375-389.
[12] M. Izydorek - J. Janczewska, *Heteroclinic solutions for a class of the second order Hamiltonian systems*, J. Differential Equations, **238** (2007), 381-393.
[13] P. H. Rabinowitz, *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **6** (1989), 331-346.
[14] P. H. Rabinowitz, *Homoclinics for a singular Hamiltonian system*, in *Geometric analysis and the calculus of variations*, Int. Press, Cambridge, MA, (1996), 267-296.
[15] E. Serra - S. Terracini, *Noncollision solutions to some singular minimization problems with Keplerian-like potentials*, Nonlinear Anal., **22** (1994), 45-62.
[16] E. Serra, *Homoclinic orbits at infinity for second order conservative systems*, NoDEA Nonlinear Differential Equations Appl., **1** (1994), 249-266.
[17] E. Serra, *Heteroclinic orbits at infinity for two classes of Hamiltonian systems*, Boll. Un. Mat. Ital. Sect. B (7), **8** (1994), 615-639.
[18] K. Tanaka, *Homoclinic orbits for a singular second order Hamiltonian system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **7** (1990), 427-438.

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