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# Central Elements in Pseudo-D-Lattices and Hahn Decomposition Theorem

ANNA AVALLONE - GIUSEPPINA BARBIERI - PAOLO VITOLO

**Abstract.** – *We prove a Hahn decomposition theorem for modular measures on pseudo-D-lattices. As a consequence, we obtain a Uhl type theorem and a Kadets type theorem concerning compactness and convexity of the closure of the range.*

## 1. – Introduction.

Effect algebras (alias D-posets) have been independently introduced in 1994 by D. J. Foulis and M. K. Bennett in [6] and by F. Chovanek and F. Kopka in [8] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [10] and in Mathematical Economics [12, 7], in particular they are a generalization of orthomodular posets and MV-algebras.

G. Georgescu and A. Iorgulescu in [13] introduced the concept of a pseudo-MV-algebra, which is a non-commutative generalization of an MV-algebra, and A. Dvurecenskij and T. Vetterlein in [11] introduced the more general structure of a pseudo-effect algebra, which is a non-commutative generalization of an effect algebra. The investigation of these structures is motivated by quantum mechanical experiments. For a study see for example [11, 14, 16].

In this paper we prove a Hahn decomposition theorem for modular measures on pseudo-D-lattices (i.e. lattice-ordered pseudo-effect algebras). To prove this theorem, a crucial result is the following: If $\mu$ is a $\sigma$-additive modular measure on a $\sigma$-complete pseudo-D-lattice $L$, and there are no nonzero negligible elements, then $\mu$ attains its supremum in a central element. For this, an essential tool is a characterization of central elements given in the first part of the paper.

As a consequence of Hahn theorem, we obtain a Uhl type theorem, namely, we prove that, if $X$ is a Banach space with the Radon-Nykodym property and $\mu: L \to X$ is a nonatomic modular measure of bounded variation, then $\overline{\mu(L)}$ is convex and compact. Moreover, we prove that the previous result also holds whenever $X$ is a $B$-convex space.

## 2. – Preliminaries.

DEFINITION 2.1. – *A partial algebra* $(E, +, 0, 1)$*, where* $+$ *is a partial binary operation and* $0$, $1$ *are constants, is called a pseudo-effect-algebra if, for all* $a$, $b$, $c \in E$*, the following properties hold:*

(P1) *The sums* $a + b$ *and* $(a + b) + c$ *exist if and only if* $b + c$ *and* $a + (b + c)$ *exist and in this case* $(a + b) + c = a + (b + c)$.

(P2) *For any* $a \in E$*, there exist exactly one* $d \in E$ *and exactly one* $e \in E$ *such that* $a + d = e + a = 1$.

(P3) *If* $a + b$ *exists, there are* $d$, $e \in E$ *such that* $a + b = d + a = b + e$.

(P4) *If* $1 + a$ *or* $a + 1$ *exists, then* $a = 0$.

We note that, if $+$ is commutative, then $E$ becomes an effect algebra.

If we define $a \leq b$ if and only if there exists $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on $E$ such that $0 \leq a \leq 1$ for any $a \in E$. If $E$ is a lattice with respect to this order, then we say tha $E$ is a *lattice pseudo-effect-algebra* or a *pseudo-D-lattice*.

If $E$ is a pseudo-effect algebra, we can define two partial binary operations on $E$ such that, for $a, b \in E$, $a/b$ is defined if and only if $b \backslash a$ is defined if and only if $a \leq b$ and in this case we have $(b \backslash a) + a = a + (a/b) = b$. In particular, we set ${}^{\perp}a = 1 \backslash a$ and $a^{\perp} = a/1$.

*In the sequel, we denote by* $E$ *a pseudo-effect algebra,* $L$ *a pseudo-D-lattice and* $(G, +)$ *a topological Abelian group.*

If $a, b \in E$ and $a \leq b$, we set $[a, b] = \{c \in E : a \leq c \leq b\}$.

If $a, b \in E$, we write $a \perp b$ to mean that the sum $a + b$ is defined.

If $a_1, \ldots, a_n \in E$, we inductively define $a_1 + \cdots + a_n = (a_1 + \cdots + a_{n-1}) + a_n$, provided that the right hand side exists. We say that the finite sequence $(a_1, \ldots, a_n)$ of $E$ is *orthogonal* if $a_1 + \cdots + a_n$ exists. Given an infinite sequence $(a_n)_{n \in \mathbb{N}}$, we say that it is *orthogonal* if, for every positive integer $n$, the sum $a_1 + \cdots + a_n$ exists.

The following properties of pseudo-effect algebras will be used:

PROPOSITION 2.2. – *For every* $a, b, c \in E$*, we have:*

(i) $a \perp b$ *if and only if* $a \leq {}^{\perp}b$ *if and only if* $b \leq a^{\perp}$.

(ii) ${}^{\perp}(a^{\perp}) = ({}^{\perp}a)^{\perp} = a$.

(iii) *If* $a \perp c$ *and* $b \perp c$*, then* $a + c = b + c$ *implies* $a = b$*; similarly, if* $c \perp a$ *and* $c \perp b$*, then* $c + a = c + b$ *implies* $a = b$.

(iv) *If* $a \leq b$*, then* $b \perp c$ *implies* $a + c \leq b + c$ *and* $c \perp b$ *implies* $c + a \leq c + b$.

(v) *If* $a \leq b \leq c$ *then* $b \backslash a \leq c \backslash a$ *and* $a/b \leq a/c$.
(vi) *If* $a \leq b \leq c$ *then* $c \backslash b \leq c \backslash a$ *and* $b/c \leq a/c$.
(vii) *If* $a \leq b \leq c$ *then* $(c \backslash a) \backslash (b \backslash a) = c \backslash b$ *and* $(a/b)/(a/c) = b/c$.
(viii) *If* $a \leq b$ *then* $b \backslash a = {}^{\perp}b/{}^{\perp}a$ *and* $a/b = a^{\perp} \backslash b^{\perp}$.
(ix) *If* $a \perp b$ *and* $a \vee b$ *exists, then* $a \vee b \leq a + b$.
(x) *If* $a \leq b$, $a \leq c$ *and* $b \wedge c$ *exists, then* $(b \backslash a) \wedge (c \backslash a)$ *exists and equals* $(b \wedge c) \backslash a$.
(xi) *If* $a \leq b$, $a \leq c$ *and* $b \wedge c$ *exists, then* $(a/b) \wedge (a/c)$ *exists and equals* $a/(b \wedge c)$.

PROPOSITION 2.3. – *Let* $a, b \in E$, *and suppose that* $a \wedge b$ *exists. Then* $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$ *and* ${}^{\perp}(a \wedge b) = {}^{\perp}a \vee {}^{\perp}b$.

PROPOSITION 2.4. – *Let* $a, b \in E$ *such that* $a \vee b$ *exists. For every* $c \in E$ *with* $a \vee b \leq c$, *we have* $c \backslash (a \vee b) = (c \backslash a) \wedge (c \backslash b)$ *and* $(a \vee b)/c = (a/c) \wedge (b/c)$.
*In particular (if* $c = 1$*) we have* ${}^{\perp}(a \vee b) = {}^{\perp}a \wedge {}^{\perp}b$ *and* $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$.

Following Dvurečenskij [9, Def. 2.1], we give the following definition.

DEFINITION 2.5. – *We say that* $p \in E$ *is central if there exists an isomorphism* $f : E \to [0, p] \times [0, p^{\perp}]$ *such that*

(C1) $f(p) = (p, 0)$;
(C2) *for every* $a \in E$, *if* $f(a) = (a_1, a_2)$, *then* $a = a_1 + a_2$.

*The set of all central elements of* $E$ *is called the centre of* $E$, *and denoted by* $C(E)$.

PROPOSITION 2.6. – *Let* $p \in E$ *be central. The following hold:*

(i) *For every* $a \in E$, *both* $a \wedge p$ *and* $a \wedge p^{\perp}$ *exist.*
(ii) *The mapping* $f$ *of Definition* 2.5 *is unique and, for every* $a \in E$, *we have* $f(a) = (a \wedge p, a \wedge p^{\perp})$.
(iii) *The mapping* $f_p : E \to [0, p]$ *defined by* $a \mapsto a \wedge p$ *is a homomorphism.*

PROOF. – (i) See [9, Prop. 2.2(vi)].
(ii) For every $a \in E$, we have $f(a) = (a \wedge p, a \wedge p^{\perp})$ by [9, Prop. 2.2(vi)]. Hence $f$ is unique.
(iii) Let $\pi$ denote the projection of $[0, p] \times [0, p^{\perp}]$ onto $[0, p]$. By (ii), we have that $f_p = \pi \circ f$ and hence $f_p$ is a homomorphism. □

The following properties of the centre also will be used in the sequel.

PROPOSITION 2.7. – (i) *If* $p, q \in C(E)$ *then* $p \wedge q \in C(E)$*, and* $C(E)$ *is a Boolean algebra.*

(ii) *If* $\{p_k : k = 1, \dots, n\}$ *is a subset of* $C(E)$ *and* $a \in E$*, then* $a \wedge \bigvee_{k=1}^{n} p_k = \bigvee_{k=1}^{n} (a \wedge p_k)$.

PROOF. – (i) See [9, Theor. 2.3].

(ii) See [9, Prop. 2.7(iii)]. □

A function $\mu: E \to G$ is said to be a *measure* if, for every $a, b \in E$ with $a \leq b$, $\mu(b) - \mu(a) = \mu(b \backslash a) = \mu(a/b)$. The map $\mu$ is a measure if and only if, for every $a, b \in E$ such that the sum $a + b$ exists, $\mu(a + b) = \mu(a) + \mu(b)$.

If $\mu: L \to G$, we say that $\mu$ is *modular* if, for every $a, b \in L$, $\mu(a \wedge b) + \mu(a \vee b) = \mu(a) + \mu(b)$.

A uniformity $\mathcal{U}$ on $L$ is said to be a *lattice uniformity* if the lattice operations are uniformly continuous with respect to $\mathcal{U}$. A lattice uniformity $\mathcal{U}$ on a pseudo-D-lattice is said to be a *D-uniformity* if all the pseudo-D-lattice operations are uniformly continuous with respect to $\mathcal{U}$.

By [3, Theor. 2.9] every modular measure on $L$ generates a D-uniformity $\mathcal{U}(\mu)$.

A lattice uniformity $\mathcal{U}$ on $L_1$ is said to be *exhaustive* if every monotone sequence is a Cauchy sequence, *σ-order continuous* (σ-o.c. for short) if every monotone sequence is convergent and *order continuous* (o.c. for short) if the same holds for nets.

If $\mu: L_1 \to G$ is a modular function, $\mu$ is said to be *exhaustive* (*σ-o.c.* or *o.c.*, respectively) if $\mathcal{U}(\mu)$ is exhaustive (σ-o.c. or o.c., respectively).

## 3. – Pseudo-central and central elements.

In this section we will define pseudo-central elements in pseudo-effect algebras, generalizing the definition of central elements in effect algebras. We will prove the crucial fact (see Theorem 3.17) that pseudo-central is equivalent to central. In order to prove that fact, we will also prove several properties of pseudo-central elements. These results are essential tools in Section 4.

We begin by generalizing to pseudo-effect algebras the notions of sharp and principal elements. We will also show that every pseudo-central element is principal and every principal element is sharp.

DEFINITION 3.1. – *We say that* $p \in E$ *is sharp if* $p \wedge p^{\perp} = 0$.

Note that this definition looks exactly the same as in effect algebras. In particular it seems that $^{\perp}p$ is not taken care of.

But this does not cause any problem, as we are going to see.

PROPOSITION 3.2. – *Let $p \in E$. The following are equivalent:*

(1) $p$ is sharp;
(2) ${}^{\perp}p \vee p = 1$;
(3) $p \wedge^{\perp} p = 0$;
(4) $p^{\perp} \vee p = 1$.

PROOF. – (1) $\Leftrightarrow$ (2) By Proposition 2.3 and Proposition 2.2(ii), we have ${}^{\perp}(p \wedge p^{\perp}) = {}^{\perp}p \vee {}^{\perp}(p^{\perp}) = {}^{\perp}p \vee p$. Hence $p \wedge p^{\perp} = 0$ (i.e. $p$ is sharp) if and only if ${}^{\perp}p \vee p = 1$.

(1) $\Rightarrow$ (3) Suppose that $p$ is sharp, and let $e \in E$ with $e \leq p$ and $e \leq {}^{\perp}p$. Setting $h = (e/p)/p$, we clearly have $h \leq p$ and, since $p = ({}^{\perp}p)^{\perp} \leq e^{\perp}$, we also have, applying Proposition 2.2(v) and (vii),

$$h = (e/p)/p \leq (e/p)/e^{\perp} = (e/p)/(e/1) = p/1 = p^{\perp},$$

so that $h \leq p \wedge p^{\perp} = 0$. Therefore $e/p = p$, and hence $e = 0$.

(3) $\Rightarrow$ (1) Similar to (1) $\Rightarrow$ (3).

(3) $\Leftrightarrow$ (4) Similar to (1) $\Leftrightarrow$ (2). □

COROLLARY 3.3. – *If $p \in E$ is sharp, then ${}^{\perp}p$ and $p^{\perp}$ are sharp, too.*

PROOF. – Indeed ${}^{\perp}p \wedge ({}^{\perp}p)^{\perp} = {}^{\perp}p \wedge p = 0$ and similarly for $p^{\perp}$. □

In order to introduce principal elements, the notion of ideal is useful (see [11, Def. 3.4(i)]). We include it in the definition below for convenience.

DEFINITION 3.4. – *An ideal of $E$ is a nonempty subset $I \subseteq E$ with the following properties:*

(I1) *If $a \in I$ and $b \leq a$, then $b \in I$.*
(I2) *If $a, b \in I$ and $a \perp b$, then $a + b \in I$.*

*We say that an element $p \in E$ is principal if* $[0, p]$ *is an ideal* (*i.e.* $I = [0, p]$ *satisfies* (I2)).

PROPOSITION 3.5. – *Let $p \in E$. The following hold:*

(a) *If $p$ is principal, then $p$ is sharp.*
(b) *If $e \wedge p$ exists for every $e \in E$ (in particular if $E$ is a pseudo-D-lattice) and $p$ is sharp, then $p$ is principal.*

PROOF. – (a) Consider any $d \in E$ with $d \leq p$ and $d \leq p^{\perp}$. We have $p \perp d$ by Proposition 2.2(i), and $p + d \leq p$ because $p$ is principal. Hence $p + d = p$, i.e. $d = 0$.

(b) Let $a, b \in E$, with $a \leq p$, $b \leq p$ and $a \perp b$. We have $a/p \leq p$ and also, by Proposition 2.2(v), $a/p \leq a/1 = a^\perp$. Hence $a/p \leq a^\perp \wedge p$ (which exists by assumption). Moreover, applying Proposition 2.2(xi), (v) and (vii), we have

$$\begin{aligned}(a/p)/(a^\perp \wedge p) &= \big((a/p)/a^\perp\big) \wedge \big((a/p)/p\big) \\ &\leq \big((a/p)/(a/1)\big) \wedge p = (p/1) \wedge p = p^\perp \wedge p = 0,\end{aligned}$$

and hence $a/p = a^\perp \wedge p$. Since $a \perp b$, by Proposition 2.2(i) $b \leq a^\perp$, so that $b \leq a^\perp \wedge p = a/p$. It follows, by Proposition 2.2(iv), that $a + b \leq a + (a/p) = p$. □

Note that principal elements also are defined exactly as in effect algebras.

In the same way we generalize the definition of central element of effect algebras. What we obtain is temporarily called pseudo-central element.

DEFINITION 3.6. – *We say that $p \in E$ is pseudo-central if, for every $a \in E$, both $a \wedge p$ and $a \wedge p^\perp$ exist and we have*

$$a = (a \wedge p) \vee (a \wedge p^\perp). \tag{1}$$

One would expect that, as in the commutative case, the above should turn out to be an alternative equivalent definition of central elements. Indeed, this is just the case, as we will see in Theorem 3.17.

PROPOSITION 3.7. – *If $p \in E$ is pseudo-central, then, for every $a \in E$, both $a \vee p$ and $a \vee p^\perp$ exist*.

PROOF. – Let $a \in E$. Since $p$ is pseudo-central, both ${}^\perp a \wedge p$ and $a^\perp \wedge p^\perp$ exist. Now, by Proposition 2.3 and Proposition 2.2(ii), we have

$${}^\perp(a^\perp \wedge p^\perp) = {}^\perp(a^\perp) \vee {}^\perp(p^\perp) = a \vee p$$

and, similarly,

$$({}^\perp a \wedge p)^\perp = ({}^\perp a)^\perp \vee p^\perp = a \vee p^\perp.$$ □

Now we prove some basic properties of pseudo-central elements.

PROPOSITION 3.8. – *If $p \in E$ is pseudo-central, then $p$ is principal (and hence sharp).*

PROOF. – In view of Proposition 3.5(b), it suffices to show that $p$ is sharp. Now, putting $a = 1$ into (1), we get $1 = (1 \wedge p) \vee (1 \wedge p^\perp) = p \vee p^\perp$, and the conclusion follows from Proposition 3.2. □

PROPOSITION 3.9. – *If $p \in E$ is pseudo-central, then ${}^\perp p = p^\perp$.*

PROOF. – Putting $a = {}^{\perp}p$ into (1), recalling that $p$ is sharp and applying Proposition 3.2, we get

$${}^{\perp}p = ({}^{\perp}p \wedge p) \vee ({}^{\perp}p \wedge p^{\perp}) = 0 \vee ({}^{\perp}p \wedge p^{\perp}) = {}^{\perp}p \wedge p^{\perp},$$

so that ${}^{\perp}p \leq p^{\perp}$, whence $p \perp {}^{\perp}p$. Now (again by Proposition 3.2) we have $p + {}^{\perp}p \geq p \vee {}^{\perp}p = 1$, i.e. $p + {}^{\perp}p = 1$, and therefore ${}^{\perp}p = p^{\perp}$. □

COROLLARY 3.10. – *If $p \in E$ is pseudo-central, then $p^{\perp}$ also is pseudo-central.*

PROOF. – Let $q = p^{\perp}$. From the previous result and Proposition 2.2(ii) it follows that $q^{\perp} = (p^{\perp})^{\perp} = ({}^{\perp}p)^{\perp} = p$. Hence, for every $a \in E$, both $a \wedge q$ and $a \wedge q^{\perp}$ exist and $(a \wedge q) \vee (a \wedge q^{\perp}) = (a \wedge p^{\perp}) \vee (a \wedge p) = a$. □

In order to prove Theorem 3.17, which is the main result of this section, a number of preliminary facts on pseudo-central elements are needed.

LEMMA 3.11. – *If $p \in E$ is pseudo-central, then, for every $a \in E$, we have $a \leq (a \wedge p) + (a \wedge p^{\perp})$ and $a \leq (a \wedge {}^{\perp}p) + (a \wedge p)$.*

PROOF. – Let $a \in E$, and note that by Proposition 2.2(i) the sums $(a \wedge p) + (a \wedge p^{\perp})$ and $(a \wedge {}^{\perp}p) + (a \wedge p)$ are both defined.

Since $p$ is pseudo-central, by Proposition 3.7 and Proposition 2.2(ix), we have

$$a = (a \wedge p) \vee (a \wedge p^{\perp}) \leq (a \wedge p) + (a \wedge p^{\perp}).$$

The second inequality is proved similarly, taking into account also that, by Proposition 3.9, we have ${}^{\perp}p = p^{\perp}$. □

LEMMA 3.12. – *If $p \in E$ is pseudo-central, then, for every $a \in E$, we have $(a \vee p)\backslash a \leq {}^{\perp}a \wedge p$ and $a/(a \vee p) \leq a^{\perp} \wedge p$.*

PROOF. – Let $a \in E$ and note that, as seen in Proposition 3.7, $a \vee p$ exist. Now, applying Proposition 2.2(viii), Proposition 2.4, the previous lemma and Proposition 2.2(v), we get

$$\begin{aligned}(a \vee p)\backslash a &= {}^{\perp}(a \vee p)/{}^{\perp}a = ({}^{\perp}a \wedge {}^{\perp}p)/{}^{\perp}a \\ &\leq ({}^{\perp}a \wedge {}^{\perp}p)/\big(({}^{\perp}a \wedge {}^{\perp}p) + ({}^{\perp}a \wedge p)\big) = {}^{\perp}a \wedge p.\end{aligned}$$

The other inequality goes similarly. □

LEMMA 3.13. – *If $p \in E$ is pseudo-central, then, for every $a \in E$, we have $p + (a \wedge p^{\perp}) = (a \wedge {}^{\perp}p) + p = p \vee (a \wedge p^{\perp})$.*

PROOF. – First observe that, by Proposition 2.2(ix), we have $p + (a \wedge p^{\perp}) \geq p \vee (a \wedge p^{\perp})$; similarly, taking also Proposition 3.9 into account, we have $(a \wedge {}^{\perp}p) + p \geq p \vee (a \wedge p^{\perp})$.

Now applying Proposition 2.4, Proposition 2.2(v), Proposition 3.8 and Proposition 3.2, we get

$$\begin{aligned}&\big(p + (a \wedge p^{\perp})\big) \backslash \big(p \vee (a \wedge p^{\perp})\big)\\ &\quad = \big((p + (a \wedge p^{\perp}))\backslash p\big) \wedge \big((p + (a \wedge p^{\perp}))\backslash (a \wedge p^{\perp})\big)\\ &\quad = \big((p + (a \wedge p^{\perp}))\backslash p\big) \wedge p \leq (1\backslash p) \wedge p = {}^{\perp}p \wedge p = 0.\end{aligned}$$

The other equality goes similarly, taking Proposition 3.9 into account. □

The supremum in the definition of pseudo-central element is in fact a sum.

PROPOSITION 3.14. – *If $p \in E$ is pseudo-central, then, for every $a \in E$, we have $a = (a \wedge p) + (a \wedge p^{\perp}) = (a \wedge {}^{\perp}p) + (a \wedge p)$.*

PROOF. – Let $a \in E$. By Lemma 3.11, we already have $a \leq (a \wedge p) + (a \wedge p^{\perp})$ and $a \leq (a \wedge {}^{\perp}p) + (a \wedge p)$.

First, we claim that

$$a/\big((a \wedge p) + (a \wedge p^{\perp})\big) \leq p^{\perp}. \tag{2}$$

Indeed, applying Proposition 2.2(vi), we get

$$a/(a \wedge p) + (a \wedge p^{\perp}) \leq (a \wedge p)/(a \wedge p) + (a \wedge p^{\perp}) = a \wedge p^{\perp} \leq p^{\perp}.$$

Now we claim that:

$$a/\big((a \wedge p) + (a \wedge p^{\perp})\big) \leq p. \tag{3}$$

Indeed, observe that, by Proposition 2.2(iv) and Lemma 3.13, we have

$$(a \wedge p) + (a \wedge p^{\perp}) \leq p + (a \wedge p^{\perp}) = p \vee (a \wedge p^{\perp}) \leq a \vee p.$$

Therefore, applying Proposition 2.2(v) and Lemma 3.12, we obtain

$$a/\big((a \wedge p) + (a \wedge p^{\perp})\big) \leq a/(a \vee p) \leq a^{\perp} \wedge p \leq p,$$

so that (3) is proved.

Finally, putting (2) and (3) together and recalling Proposition 3.8, we have

$$a/\big((a \wedge p) + (a \wedge p^{\perp})\big) \leq p \wedge p^{\perp} = 0,$$

whence $a = (a \wedge p) + (a \wedge p^{\perp})$.

The arguments to prove that $a = (a \wedge {}^{\perp}p) + (a \wedge p)$ are analogous. □

COROLLARY 3.15. – *If* $p \in E$ *is pseudo-central, then, for every* $a \leq p$ *and* $b \leq p^{\perp}$*, the sums* $a + b$ *and* $b + a$ *both exist and are equal.*

PROOF. – Since $a \leq p$ and $b \leq p^{\perp}$, we have $a \perp b$; moreover, since ${}^{\perp}p = p^{\perp}$ by Proposition 3.9, we also have $b \perp a$.

Let $c = a + b$ and $d = b + a$. We clearly have $c \wedge p \geq a$ and $c \wedge p^{\perp} \geq b$, as well as $d \wedge p \geq a$ and $d \wedge p^{\perp} \geq b$. The previous proposition implies that

$$c = (c \wedge {}^{\perp}p) + (c \wedge p) = (c \wedge p^{\perp}) + (c \wedge p) \geq b + a = d,$$

and also that

$$d = (d \wedge p) + (d \wedge p^{\perp}) \geq a + b = c;$$

hence $c = d$. □

The following observation will be useful in the proof of the next theorem.

LEMMA 3.16. – *Let* $a, b \in E$ *with* $a \perp b$*. If* $c \leq a$*,* $d \leq b$ *and* $c + d = a + b$*, then* $c = a$ *and* $d = b$*.*

PROOF. – Since $c \leq a$, we have $c \perp b$ by Proposition 2.2(i). Hence, applying Proposition 2.2(iv), we get

$$a + b = c + d \leq c + b \leq a + b$$

so that $c + b = a + b$, whence $c = a$ by Proposition 2.2(iii). Similarly one shows that $d = b$. □

We are now ready to show that pseudo-central and central are equivalent concepts.

THEOREM 3.17. – *Any* $p \in E$ *is central if and only if it is pseudo-central.*

PROOF. – Suppose that $p$ is central. We have seen in Proposition 2.6(i) that, for every $a \in E$, $a \wedge p$ and $a \wedge p^{\perp}$ exist; moreover $a = (a \wedge p) \vee (a \wedge p^{\perp})$ by [9, Prop. 2.2(vi) and (vii)]: thus $p$ is pseudo-central.

Conversely, suppose that $p$ is pseudo-central. Define $f: E \to [0, p] \times [0, p^{\perp}]$ as follows:

$$\forall a \in E \quad f(a) = (a \wedge p, a \wedge p^{\perp}). \tag{4}$$

By Proposition 3.8, we have $f(p) = (p \wedge p, p \wedge p^{\perp}) = (p, 0)$, i.e. (C1) is satisfied; furthermore (C2) follows immediately from Proposition 3.14. It remains to show that $f$ is an isomorphism.

To see that $f$ is one-to-one, note that if $f(a) = f(b)$ then, by Proposition 3.14,

$$b = (a \wedge p) + (a \wedge p^{\perp}) = a.$$

Too see that $f$ is onto, consider any $(h,k) \in [0,p] \times [0,p^\perp]$. Observe that $h \perp k$ so we can define $a = h + k$. Since $a \wedge p$ exists and $a \geq h$, we have $a \wedge p \geq h$, too. Similarly $a \wedge p^\perp \geq k$. Being $(a \wedge p) + (a \wedge p^\perp) = a = h + k$, we may apply Lemma 3.16 to get that $h = a \wedge p$ and $k = a \wedge p^\perp$. Therefore $(h,k) = f(a)$.

Now let $a, b \in E$: we complete the proof by showing that $a \perp b$ in $E$ if and only if $f(a) \perp f(b)$ in $[0,p] \times [0,p^\perp]$ and, in this case, $f(a) + f(b) = f(a+b)$.

If $a \perp b$ in $E$, then clearly $(a \wedge p) \perp (b \wedge p)$ and $(a \wedge p^\perp) \perp (b \wedge p^\perp)$, hence we can consider the sum

$$\big((a \wedge p) + (b \wedge p), (a \wedge p^\perp) + (b \wedge p^\perp)\big) = f(a) + f(b),$$

which means that $f(a) \perp f(b)$ in $[0,p] \times [0,p^\perp]$. Conversely, suppose that $f(a) \perp f(b)$ in $[0,p] \times [0,p^\perp]$. Then the sums $c = (a \wedge p) + (b \wedge p)$ and $d = (a \wedge p^\perp) + (b \wedge p^\perp)$ exist in $E$; by Proposition 3.10 and Proposition 3.8, we have $c \leq p$ and $d \leq p^\perp$. Hence it is well defined in $E$ the sum

$$c + d = (a \wedge p) + (b \wedge p) + (a \wedge p^\perp) + (b \wedge p^\perp).$$

Since by Corollary 3.15 $(b \wedge p) + (a \wedge p^\perp) = (a \wedge p^\perp) + (b \wedge p)$, we have, applying Proposition 3.14,

$$c + d = (a \wedge p) + (a \wedge p^\perp) + (b \wedge p) + (b \wedge p^\perp) = a + b, \tag{5}$$

so that $a \perp b$. Finally, by Proposition 2.2(iv) we have $c \leq a + b$ and $d \leq a + b$, hence $c \leq (a+b) \wedge p$ and $d \leq (a+b) \wedge p^\perp$; moreover, by Proposition 3.14,

$$a + b = \big((a+b) \wedge p\big) + \big((a+b) \wedge p^\perp\big)$$

and this, together with (5), implies by Lemma 3.16 that $c = (a+b) \wedge p$ and $d = (a+b) \wedge p^\perp$. Therefore

$$\begin{aligned} f(a+b) &= \big((a+b) \wedge p, (a+b) \wedge p^\perp\big) = (c,d) \\ &= \big((a \wedge p) + (b \wedge p), (a \wedge p^\perp) + (b \wedge p^\perp)\big) = f(a) + f(b). \end{aligned}$$ □

Theorem 3.17 above allows us to characterize central elements in a pseudo-D-lattice $L$.

PROPOSITION 3.18. – *Let $p \in L$. The following are equivalent:*

(1) *$p$ is central.*

(2) *For every $a \in L$, we have $a = (a \wedge p) + (a \wedge p^\perp) = (a \wedge p^\perp) + (a \wedge p)$.*

(3) *$p$ is sharp and, for every $a \in L$, we have $a \backslash (a \wedge p^\perp) \leq p$ and $a \backslash (a \wedge p) \leq p^\perp$.*

(4) *$p$ is sharp and, for every $a \in L$, we have $(a \wedge p)/a \leq p^\perp$ and $(a \wedge p^\perp)/a \leq p$.*

PROOF. – (1) $\Rightarrow$ (2) It follows from the definition and [9, Prop. 2.2(v), (vi) and (vii)]. (Alternatively, apply Theorem 3.17, Proposition 3.9 and Lemma 3.11.)

(2) $\Rightarrow$ (3) Setting $a = p$, we obtain $p = (p \wedge p) + (p \wedge p^{\perp})$, and hence $p \wedge p^{\perp} = 0$. Now, for every $a \in L$, we have $a \backslash (a \wedge p^{\perp}) = \big((a \wedge p) + (a \wedge p^{\perp})\big) \backslash (a \wedge p^{\perp}) = a \wedge p \leq p$ and, similarly, $a \backslash (a \wedge p) = \big((a \wedge p^{\perp}) + (a \wedge p)\big) \backslash (a \wedge p) = a \wedge p^{\perp} \leq p^{\perp}$.

(2) $\Rightarrow$ (4) Analogous to (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) Let $a \in L$. Applying Proposition 2.4, we get

$$a \backslash \big((a \wedge p^{\perp}) \vee (a \wedge p)\big) = \big(a \backslash (a \wedge p^{\perp})\big) \wedge \big(a \backslash (a \wedge p)\big) \leq p \wedge p^{\perp} = 0,$$

and, consequently, $a = (a \wedge p^{\perp}) \vee (a \wedge p)$. Hence $p$ is central by Theorem 3.17.

(4) $\Rightarrow$ (1) Analogous to (3) $\Rightarrow$ (1). □

In the sequel we will also use the following properties of the centre of $L$.

PROPOSITION 3.19. – (a) *A sequence* $(p_1, \ldots, p_n)$ *in* $C(L)$ *is orthogonal if and only if* $p_h \wedge p_k = 0$ *whenever* $h \neq k$. *In this case,* $\sum_{k=1}^{n} p_k = \bigvee_{k=1}^{n} p_k \in C(L)$.

(b) $C(L)$ *is a Boolean algebra as a subalgebra of* $L$.

(c) $C(L)$ *is contained in the lattice-theoretical centre of* $L$.

(d) *If* $\{a_j : j \in J\}$ *is a subset of* $L$ *with* $a = \bigvee_{j \in J} a_j$ *and* $p \in C(L)$, *then* $\bigvee_{j \in J} (a_j \wedge p) = a \wedge p$.

(e) *If* $(a_1, \ldots, a_n)$ *is an orthogonal sequence in* $L$ *and* $p \in C(L)$, *then* $\left(\sum_{k=1}^{n} a_k\right) \wedge p = \sum_{k=1}^{n} (a_k \wedge p)$.

(f) *If* $a \in L$ *and* $(p_1, \ldots, p_n)$ *is an orthogonal sequence in* $C(L)$, *then* $a \wedge \sum_{k=1}^{n} p_k = \sum_{k=1}^{n} (a \wedge p_k)$.

PROOF. – (a) Suppose that $p_h \wedge p_k = 0$ whenever $h \neq k$. Then the sequence is orthogonal and $\sum_{k=1}^{n} p_k = \bigvee_{k=1}^{n} p_k \in C(L)$ by [9, Prop. 2.7(i)].

Conversely, we assume that $(p_1, \ldots, p_n)$ is an orthogonal sequence and prove that $p_h \wedge p_k = 0$ whenever $h \neq k$ $(h, k \in \{1, \ldots, n\})$. We proceed by induction on $n$.

If $n = 1$ the assertion is trivial. So suppose $n > 1$ and let the assertion be true for $n - 1$.

Set $p = \sum_{k=1}^{n-1} p_k$; we have $p_k \leq p$ for every $k \in \{1, \ldots, n-1\}$, hence it suffices to show that $p \wedge p_n = 0$. Now, since $p \perp p_n$, we have $p_n \leq p^{\perp}$ and therefore $p \wedge p_n \leq p \wedge p^{\perp} = 0$ by [9, Prop. 2.2(iii)] (or, alternatively, by Theorem 3.17 and Proposition 3.8).

(b) By Proposition 2.7(i), $C(L)$ is a Boolean algebra and a sublattice of $L$. Now, by (a), if $p, q \in C(L)$ and $p \perp q$, we have $p + q \in C(L)$. Hence $C(L)$ is a subalgebra of $L$.

(c) Let $C_0(L)$ denote the lattice-theoretical centre of $L$. Since $p \in C_0(L)$ if and only if there exists $q \in L$ with $p \wedge q = 0$ and such that, for every $a \in L$, $a = (a \wedge p) \vee (a \wedge q)$, it follows from Theorem 3.17 that $C(L) \subseteq C_0(L)$.

(d) follows from (c).

(e) Let $f_p(a) = a \wedge p$ for every $a \in L$. By Proposition 2.6(iii), we have $\left(\sum_{k=1}^{n} a_k\right) \wedge p = f_p\left(\sum_{k=1}^{n} a_k\right) = \sum_{k=1}^{n} f_p(a_k) = \sum_{k=1}^{n} (a_k \wedge p)$.

(f) follows from (a) and Proposition 2.7(ii). □

## 4. – Hahn decomposition theorem.

*In this section, let $\mu: L \to \mathbb{R}$ be a modular measure.*

We need the following result from [3, Theor. 2.12].

THEOREM 4.1. – *Let $\mathcal{U}$ be a D-uniformity on L. Then:*

(a) *$N(\mathcal{U}) = \cap\{U : U \in \mathcal{U}\}$ is a DV-congruence and a lattice congruence.*

(b) *The quotient $\hat{L} = L/N(\mathcal{U})$ is a pseudo-D-lattice and the quotient uniformity $\hat{\mathcal{U}}$ is a Hausdorff D-uniformity.*

(c) *If $(\tilde{L},\tilde{\mathcal{U}})$ is the uniform completion of $(\hat{L},\hat{\mathcal{U}})$, then $\tilde{L}$ is a pseudo-D-lattice and $\tilde{\mathcal{U}}$ is a D-uniformity on $\tilde{L}$. Moreover, if $\mathcal{U}$ is exhaustive, then $\tilde{\mathcal{U}}$ is o.c. and $(\tilde{L}, \leq)$ is complete.*

(d) *If $\mu$ is $\mathcal{U}$-continuous, then the map $\hat{\mu}$ defined as $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ is a well-defined modular measure on $\hat{L}$ and $\tilde{\mu}$ is the unique $\tilde{\mathcal{U}}$-continuous extension of $\hat{\mu}$ on $\tilde{L}$. Moreover, $\mu(L)$ is dense in $\tilde{\mu}(\tilde{L})$.*

We will denote $N(\mathcal{U}(\mu))$ by $N(\mu)$. We have

$$N(\mu) = \{(a, b) \in L \times L : \mu(c) = 0 \text{ for every } c \leq (a \vee b)\backslash(a \wedge b)\}.$$

Recall that, since $\mu$ is a modular function, the quotient $L/N(\mu)$ is modular [19, Prop. 2.5 and Prop. 3.1].

Moreover, the closure of 0 in $\mathcal{U}(\mu)$ is

$$I(\mu) = \{a \in L : \mu(b) = 0 \text{ for every } b \leq a\}$$

and $(a, b) \in N(\mu)$ if and only if $(a \vee b)\backslash(a \wedge b) \in I(\mu)$.

PROPOSITION 4.2. – *The following statements are equivalent:*

(1) *$\mu$ is bounded.*

(2) *$\mu$ is exhaustive.*

(3) *$\mu$ is of bounded variation.*

PROOF. – By [19, Prop. 2.7] the equivalence holds for all modular measures defined on lattices $L'$ which verify the following condition:

For each finite chain $a_0 \leq \ldots \leq a_n$ in $L'$ and each $I \subseteq \{1, \ldots, n\}$, one has

$$\sum_{i \in I} [\mu(a_i) - \mu(a_{i-1})] \in \mu(L') - \mu(L').$$

We now prove that $L$ satisfies the above condition.

Let $a_0 \leq \ldots \leq a_n$ be a finite chain in $L$. Set $b_i = a_{i-1}/a_i$ for every $i \leq n$. By [3, Lemma 3.3] $(b_1, \ldots, b_n)$ is an orthogonal family in $L$ and, for $I = \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, n\}$ with $h_1 < \ldots < h_k$, we have

$$\sum_{j=1}^{k} [\mu(a_{h_j}) - \mu(a_{h_j - 1})] = \sum_{j=1}^{k} \mu(b_{h_j}) = \mu\left(\sum_{j=1}^{k} b_{h_j}\right) \in \mu(L) \subseteq \mu(L) - \mu(L). \qquad \square$$

DEFINITION 4.3. – *We say that an ideal $I$ is normal if*

(I3) *For every $a, r, s \in E$ with $r \perp a$, $a \perp s$ and $r + a = a + s$, we have $r \in I$ if and only if $s \in I$.*

*We say that an ideal $I$ is a Riesz ideal if it satisfies the following Riesz decomposition property:*

(I4) *Given $a$, $b \in L$ such that there exists the sum $a + b$ and given $c \in I$ with $c \leq a + b$, then there exist $a_1$, $b_1 \in I$ such that $a_1 \leq a$, $b_1 \leq b$ and $c \leq a_1 + b_1$.*

PROPOSITION 4.4 [14, Theor. 4.9]. – *Let $I$ be a normal Riesz ideal of $L$. Suppose that $r = \sup I$ exists. Then $r$ is a central element.*

PROPOSITION 4.5. – *Let $L$ be complete and let $\mu$ be positive. Set $r = \sup I(\mu)$. Then $r$ is a central element.*

PROOF. – By Proposition 4.4 it is sufficient to prove that $I$ is a normal Riesz ideal.

Obviously, $I(\mu)$ is an ideal.

It is normal since, from $r + a = a + s$ it follows that $\mu(r) = \mu(s)$ and hence, $r \in I(\mu)$ if and only if $s \in I(\mu)$.

We now prove that $I(\mu)$ is a Riesz ideal.

Let $a$, $b \in L$ such that there exists $a + b$ and $r \in I(\mu)$ with $r \leq a + b$. Set $i = (r \vee b) \backslash b$ and $j = i/(i \vee r)$.

Since $i \perp b$ and $i + b = r \vee b \leq a + b$, we have $i \leq a$. Moreover, the sum $i + j$ exists and

$$i + j = i \vee r \leq (r \vee b) \vee r = r \vee b = i + b,$$

whence $j \leq b$. As $r \in I$, we have

$$\mu(i) = \mu(r \vee b) - \mu(b) = \mu(r) - \mu(r \wedge b) = 0.$$

Therefore $i \in I$. Since $j \leq i$, we get $j \in I$. So $i + j = i \vee r \geq r$. □

PROPOSITION 4.6. – (a) *Let $p \in L$ with $\mu(p) = \sup \mu(L)$. If $a \leq p$, then $\mu(a) \geq 0$; if $a \leq p^\perp$ or $a \leq {}^\perp p$, then $\mu(a) \leq 0$.*

(b) *If $N(\mu) = \Delta$ and $\mu(p) = \mu(r) = \sup \mu(L)$, then $p = r$.*

(c) *If $N(\mu) = \Delta$ and $\mu(p) = \sup \mu(L)$, then $p^\perp = {}^\perp p$ and $p^\perp$ is the only complement of $p$.*

PROOF. – (a) Let $a \leq p$ and $c = p \backslash a$. Then $\mu(a) + \mu(c) = \mu(p) \geq \mu(c)$, whence $\mu(a) \geq 0$.

If $a \leq p^\perp$, by Proposition 2.2(i) there exists $p + a$ and $\mu(p) \geq \mu(p + a) = \mu(p) + \mu(a)$, so $\mu(a) \leq 0$. Analogously, if $a \leq {}^\perp p$, we obtain that $\mu(a) \leq 0$.

(b) Set $m = \sup \mu(L)$. Since $\mu(p \wedge r) \leq m$, $\mu(p \vee r) \leq m$ and $\mu(p \wedge r) + \mu(p \vee r) = \mu(p) + \mu(r) = 2m$, we have $\mu(p \vee r) = \mu(p \wedge r) = m$. Set $a = (p \vee r) \backslash (p \wedge r)$. We will show that $a = 0$.

Let $b \leq a$. We get

$$0 = \mu(a) = \mu(b) + \mu(a \backslash b). \tag{6}$$

Since $b \leq p \vee r$ and $a \backslash b \leq a \leq p \vee r$, it follows from (a) that $\mu(b) \geq 0$ and $\mu(a \backslash b) \geq 0$. From (6) it follows that $\mu(b) = \mu(a \backslash b) = 0$. Since $N(\mu) = \Delta$, we get $a = 0$.

(c) By (a), we obtain $\mu(c) = 0$ whenever $c \leq p \wedge p^\perp$ or $c \leq {}^\perp p \wedge p$. Since $N(\mu) = \Delta$, we get $p \wedge p^\perp = {}^\perp p \wedge p = 0$. By Proposition 2.3 we have ${}^\perp p \vee p = p^\perp \vee p = 1$.

Let $q$ be a complement of $p$. We have

$$\mu(q^\perp) = \mu(1) - \mu(q) = \mu(p \vee q) + \mu(p \wedge q) - \mu(q) = \mu(p).$$

From (b) it follows that $q^\perp = p$, whence $q = {}^\perp p$. Since ${}^\perp p$ and $p^\perp$ are two complements of $p$, we get ${}^\perp p = p^\perp$. □

THEOREM 4.7. – *If $N(\mu) = \Delta$ and $\mu(p) = \sup \mu(L)$, then $p$ is a central element.*

PROOF. – We first establish a couple of claims.

CLAIM 1. – For every $a \in L$ with $a \wedge p = 0$, we have $a \leq p^\perp$.

By way of contradiction suppose that $a \not\leq p^\perp$. Then $w = a \backslash (a \wedge p^\perp) \neq 0$. Moreover, we get ${}^\perp a \wedge p \neq p$, otherwise we would have $p \leq {}^\perp a$, from which $p^\perp \geq a$.

By Proposition 4.6(b) it follows that $\mu({}^{\perp}a \wedge p) < \mu(p)$. Then

$$\mu({}^{\perp}a \vee p) = \mu({}^{\perp}a) + \mu(p) - \mu(p \wedge {}^{\perp}a) > \mu({}^{\perp}a).$$

It follows that

$$\begin{aligned} 0 < \mu({}^{\perp}a \vee p) - \mu({}^{\perp}a) &= \mu(1) - \mu(a \wedge p^{\perp}) - \mu(1) + \mu(a) \\ &= \mu(a) - \mu(a \wedge p^{\perp}) = \mu(w). \end{aligned}$$

On the other hand, as $a \wedge p = 0$ and $w \leq a$, we get $w \wedge p = 0$.
Hence $\mu(w) = \mu(w \vee p) - \mu(p) \leq 0$, a contradiction.

CLAIM 2. – For every $b \in L$, we have $b \backslash (b \wedge p) \leq p^{\perp}$.

Let $b \in L$. Put $a = b \backslash (b \wedge p)$, $c = a \vee p$, $d = b \vee p$, $e = c \wedge p^{\perp}$ and $f = d \wedge p^{\perp}$. Notice that $e \leq f$, since $c \leq d$.
We will show that $\mu(a \wedge p) = \mu(f \backslash e)$.
We have

$$\begin{aligned} \mu(a \wedge p) &= \mu(a) + \mu(p) - \mu(a \vee p) = \mu(b) - \mu(b \wedge p) + \mu(p) - \mu(a \vee p) \\ &= \mu(b \vee p) - \mu(p) + \mu(p) - \mu(a \vee p) = \mu(d) - \mu(c). \end{aligned}$$

Since $c \leq d$ and $p \vee p^{\perp} = 1$ by Proposition 4.6(c), we have by modularity of $L$

$$c \vee (p^{\perp} \wedge d) = (c \vee p^{\perp}) \wedge d = (a \vee p \vee p^{\perp}) \wedge d = d.$$

Moreover, we get

$$\mu(d) = \mu(c) + \mu(p^{\perp} \wedge d) - \mu(c \wedge p^{\perp} \wedge d) = \mu(c) + \mu(p^{\perp} \wedge d) - \mu(c \wedge p^{\perp}).$$

It follows that

$$\mu(a \wedge p) = \mu(d) - \mu(c) = \mu(p^{\perp} \wedge d) - \mu(c \wedge p^{\perp}) = \mu(f) - \mu(e) = \mu(f \backslash e).$$

As $f \backslash e \leq f \leq p^{\perp}$, by Proposition 4.6(a) we have $\mu(f \backslash e) \leq 0$ and, hence, $\mu(a \wedge p) \leq 0$. On the other hand, since $a \wedge p \leq p$, by Proposition 4.6(a) we have $\mu(a \wedge p) \geq 0$. It follows that $\mu(a \wedge p) = 0$, whence $\mu(p \backslash (a \wedge p)) = \mu(p)$.

By Proposition 4.6(b) it follows that $p \backslash (a \wedge p) = p$, from which $a \wedge p = 0$. From Claim 1 it follows that $a \leq p^{\perp}$, and Claim 2 is proved.

Now, as $-\mu$ is a modular measure with $N(-\mu) = N(\mu)$ and $-\mu(p^{\perp}) = \sup(-\mu(L))$, we apply Claim 2 and we obtain that

$$\forall b \in L \quad b \backslash (b \wedge p^{\perp}) \leq p. \tag{7}$$

Finally, since by Proposition 4.6(c) $p$ is sharp, from Claim 2 and (7) we deduce, applying Proposition 3.18(3), that $p$ is central. □

In what follows, we denote by $\mu_a$ the map defined by $\mu_a(b) = \mu(a \wedge b)$ for

$b \in L$. As in [4] it is easy to see that, if $a$ is a central element, then $\mu_a$ is a modular measure and $\mu = \mu_a + \mu_{a^\perp} = \mu_a + \mu_{^\perp a}$. Moreover, if $\mu$ is o.c., then $\mu_a$ and $\mu_{a^\perp} = \mu_{^\perp a}$ are o.c., too.

THEOREM 4.8. – *Let $p \in L$, $q = p^\perp$ and $r = {}^\perp p$. Then*

(a) *The following conditions are equivalent*

(1) $\mu(p) = \sup \mu(L)$.

(2) $\mu(q) = \mu(r) = \inf \mu(L)$.

(3) $\mu_p$, $\mu_q$ *and* $\mu_r$ *are modular measures with* $\mu_q = \mu_r$, $\mu_p \geq 0$, $\mu_q \leq 0$ *and* $\mu = \mu_p + \mu_q$.

(b) *If $\mu(p) = \sup \mu(L)$, the equivalence classes $\hat{p}, \hat{q}$ and $\hat{r}$ in $\hat{L} = L/N(\mu)$ are central elements, $\hat{q} = \hat{r}$, $\hat{q}$ is the unique complement of $\hat{p}$ and $\mu(w) = \sup \mu(L)$ implies $\hat{w} = \hat{p}$.*

(c) *If $L$ is $\sigma$-complete and $\mu$ is $\sigma$-additive, there exists an element $p \in L$ such that $\mu(p) = \sup \mu(L)$.*

PROOF. – (a) $\Leftrightarrow$ (2) is obvious since $\mu(q) = \mu(r) = \mu(1) - \mu(p)$.

(1) $\Rightarrow$ (3): We may suppose $N(\mu) = \Delta$ since we may substitute $L$ by $\hat{L}$ and $\mu$ by $\hat{\mu}$. By Proposition 4.5 $p$ is a central element. By Proposition 3.9, Corollary 3.10 and Theorem 3.17 we get $q = r$ and $q$ is central element, too. By Proposition 4.6 $\mu_p \geq 0$ and $\mu_q \leq 0$.

(3) $\Leftrightarrow$ (1): Let $a \in L$. We have $\mu(a) = \mu_p(a) + \mu_q(a) \leq \mu_p(a) \leq \mu_p(1) = \mu(p)$. Therefore $\mu(p) = \sup \mu(L)$.

(b) follows from Proposition 4.6 and Proposition 4.5.

(c) follows from [19, Prop. 1.2.2]. □

COROLLARY 4.9. – *If $L$ is complete and $\mu$ is o.c., then there exists a central element $p \in L$ such that $\mu(p) = \sup \mu(L)$.*

PROOF. – We can suppose $\mu \neq 0$.

Observe that $\mu$ is exhaustive. Indeed, by [19, Prop. 3.5 and Prop. 3.6] $\mu$ is o.c. ($\sigma$-o.c., exhaustive) if and only if the $\mu$-uniformity is so and, by [18, Prop. 8.1.2], every $\sigma$-o.c. lattice uniformity on a $\sigma$-complete lattice is exhaustive.

Then by Proposition 4.2 the total variation $v$ of $\mu$ is bounded, and therefore, by [20, Theor. 1.3.11], $\mu$ and $v$ generate the same topology. Then we have

$$I(\mu) = \{a \in L : v(a) = 0\}.$$

By Proposition 4.5 there exists a central element $c \in L$ such that $I(\mu) = [0, c]$. Then, for every $a \in L$, $a = (a \wedge c) + (a \wedge c^\perp)$, from which

$$\mu(a) = \mu(a \wedge c) + \mu(a \wedge c^\perp) = \mu(a \wedge c^\perp).$$

and hence $\sup \mu(L) = \sup \mu([0, c^\perp])$.

Now observe that $[0, c^{\perp}]$ is a complete pseudo-D-lattice and the restriction of $\mu$ to $[0, c^{\perp}]$ is an o.c. modular measure. Moreover, since

$$I(\mu') = \{a \in [0, c^{\perp}] : \mu'(b) = 0 \text{ for all } b \leq a\} \subseteq I(\mu) \cap [0, c^{\perp}] = \{0\},$$

we have $N(\mu') = \Delta$.

Then by Theorem 4.8 there exists a central element $p \in [0, c^{\perp}]$ such that $\mu'(p) = \sup \mu'(L) = \sup \mu(L)$. Since $c^{\perp}$ is central in $L$, by [9, Prop. 2.8] $p$ is central in $L$, too. □

**5. – Uhl's and Kadets' theorems.**

*In this section $X$ is a Banach space and $\mu: L \to X$ is a modular measure.*

We will show a Uhl type theorem and a Kadets type theorem concerning the convexity and compactness of the closure of the range.

We need some definitions.

DEFINITION 5.1. – *We say that $\mu$ is nonatomic (or strongly continuous) if for every $\varepsilon > 0$ there exists an orthogonal family $(a_1, \ldots, a_n)$ in $L$ such that $a_1 + \ldots + a_n = 1$ and $\|\mu(b)\| < \varepsilon$ for $b \leq a_i$ with $i \leq n$.*

*We say that $\mu$ is atomless if for every $a \in L$ with $\mu(a) \neq 0$ there exists $b < a$ with $\mu(b) \neq 0$ and $\mu(b) \neq \mu(a)$.*

PROPOSITION 5.2. – *The modular measure $\mu$ is nonatomic if and only if for every 0-neighbourhood $U$ in $\mathcal{U}(\mu)$, there exists an orthogonal family $(a_1, \ldots, a_n)$ in $L$ such that $a_1 + \ldots + a_n = 1$ and $a_i \in U$ for $i \leq n$.*

PROOF. – By [3, Theor. 2.9] a basis of $\mathcal{U}(\mu)$ is the family formed by the sets

$$\{(a, b) \in L \times L : \ \|\mu(c)\| < \varepsilon \text{ for every } c \leq (a \vee b) \backslash (a \wedge b)\}$$

with $\varepsilon > 0$.

Hence a basis of 0-neighbourhoods in $\mathcal{U}(\mu)$ is the family formed by the sets

$$\{a \in L : \ \|\mu(b)\| < \varepsilon \text{ for } b \leq a\}$$

with $\varepsilon > 0$. The proof is now straightforward. □

PROPOSITION 5.3. – *For every $a \in L$, $|\mu|(a) = \sup\left\{\sum_{i=1}^{n} \|\mu(b_i)\| : (b_1, \ldots, b_n) \text{ is an orthogonal family in } L \text{ with } b_1 + \ldots + b_n = a\right\}$.*

PROOF. – Let $a \in L$ and $\bar{\mu}(a) = \sup\left\{\sum_{i=1}^{n} \|\mu(b_i)\| : (b_1, \dots, b_n) \text{ is an orthogonal family in } L \text{ with } b_1 + \cdots + b_n = a\right\}$.

Let $a_0, \dots, a_n$ be elements in $L$ such that $0 = a_0 \leq a_1 \leq a_n = a$. For every $i \leq n$, put $b_i = a_{i-1}/a_i$.

By [3, Lemma 3.3] we have that $(b_1, \dots, b_n)$ is an orthogonal family in $L$ and $b_1 + \ldots + b_n = 0/a = a$. It follows that

$$\sum_{i=1}^{n} \|\mu(a_i) - \mu(a_{i-1})\| = \sum_{i=1}^{n} \|\mu(b_i)\| \leq \bar{\mu}(a),$$

whence $|\mu|(a) \leq \bar{\mu}(a)$.

Now let $(b_1, \dots, b_n)$ be an orthogonal family in $L$ with $b_1 + \ldots + b_n = a$. Put $a_0 = 0$ and $a_i = b_1 + \cdots + b_i$ for $i \in \{1, \dots, n\}$.

We get $0 = a_0 \leq a_1 \leq \cdots \leq a_n = a$ and $b_i = a_{i-1}/a_i$ for every $i \in \{1, \dots, n\}$. It follows that

$$\sum_{i=1}^{n} \|\mu(b_i)\| = \sum_{i=1}^{n} \|\mu(a_i) - \mu(a_{i-1})\| \leq |\mu|(a),$$

whence $\bar{\mu}(a) \leq |\mu|(a)$. □

DEFINITION 5.4. – *We say that a poset $E$ is a $B$-poset if it is endowed with* 0 *and* 1, *a binary relation $\perp$ on $L$ and a partially defined binary operation $\oplus$ which satisfy the following conditions:*

(B1) *The sum $a \oplus b$ is defined if and only if $a \perp b$.*

(B2) *For every $a \in E$ we have $a \oplus 0 = 0 \oplus a = a$.*

(B3) *If $a \leq b$, there exists $c \in E$ with $a \perp c$ and $a \oplus c = b$.*

(B4) *If $d \leq c$, $b \leq a$ and $c \perp a$, then $d \perp b$ and $d \oplus b \leq c \oplus a$.*

(B5) *If $a \leq c \leq a \oplus b$, then there exists $d \leq b$ with $a \oplus d = c$.*

Arguing as in the proof of [3, Theor. 3.7], one sees that every pseudo-effect algebra is a $B$-poset if one defines $a \oplus b = a + b$ and the sum $a + b$ is defined if and only if $a \perp b$.

Consequently we obtain

THEOREM 5.5. – (a) *$|\mu|$ is a modular measure.*

(b) *If $L$ is $\sigma$-complete and $\mu$ is $\sigma$-additive, then $\mu$ is nonatomic if and only if $\mu$ is atomless.*

(c) *If $L$ is $\sigma$-complete, $X = \mathbb{R}^n$ and $\mu$ is nonatomic, then $\mu([0, a])$ is convex for every $a \in L$.*

PROOF. – (a) $|\mu|$ is modular by [20, Prop. 1.3.10], and is a measure by [5, Prop. 4.6].

(b) is observed in [5, § 4].

(c) is proved in [3, Theor. 3.7] applying [5, Theor. 3.8]. □

In the sequel we will use the following notation:

NOTATION 5.6. – Let $\nu : L \to [0, \infty[$ be a measure, and $\mathcal{A}$ be a Boolean subalgebra of $C(L)$. Put

$$S_\nu(\mathcal{A}) := \left\{ \sum_{i=1}^{n} x_i \nu_{a_i} : n \in \mathbb{N},\, x_i \in X,\, a_i \in \mathcal{A} \right\}.$$

The following results go similarly to Lemma 3.1 and Theorem 3.2 of [4], making use of Proposition 3.19(a), (d), (e) and (f).

LEMMA 5.7. – *Let $\nu : L \to [0, \infty[$ be a measure and $\mathcal{A}$ be a Boolean subalgebra of $C(L)$. Then:*

(a) *Every element $\gamma \in S_\nu(\mathcal{A})$ is a measure on $L$ with values in $X$. If $\nu$ is modular, $\gamma$ is modular, too. Moreover, if $\nu$ is o.c. or nonatomic, $\gamma$ is so.*

(b) *For every element $\gamma \in S_\nu(\mathcal{A})$ there exists $s \in \mathbb{N}$, $y_1, \ldots, y_s$ in $X$ and a disjoint family $(b_1, \ldots, b_s)$ in $\mathcal{A}$ with $\bigvee_{i=1}^{s} b_i = 1$, $\gamma = \sum_{i=1}^{s} y_i \nu_{b_i}$ and $|\gamma|_{\rceil \mathcal{A}}(1) = \sum_{i=1}^{s} \|y_i\| \nu(b_i)$, where $|\gamma|_{\rceil \mathcal{A}}$ denotes the restriction of $|\gamma|$ to $\mathcal{A}$.*

We recall that $X$ has the *Radon-Nikodym property* if, for every $\sigma$-algebra $\Sigma$ of subsets of $X$, for every real $\sigma$-additive measure $\nu$ on $\Sigma$ and for every $X$-valued $\nu$-continuous measure $\mu$ of bounded variation, there exists a $\nu$-integrable function $f$ with $\mu(A) = \int_A f d\mu$ for all $A \in \Sigma$.

THEOREM 5.8. – *Suppose that $X$ has the Radon-Nikodym property. Let $\nu : L \to [0, \infty[$ be a measure, $\mathcal{A}$ be a Boolean subalgebra of $C(L)$ and $\lambda : \mathcal{A} \to X$ be a $\nu$-continuous measure of bounded variation.*

*Then there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ in $S_\nu(\mathcal{A})$ (see Notation 5.6) such that the map $\mu$ defined as $\mu(a) = \sum_{n=1}^{\infty} \nu_n(a)$ for $a \in L$ is an $X$-valued measure which extends $\lambda$.*

LEMMA 5.9. – *Suppose that for every $\varepsilon > 0$ and for every $a \in L$, there exists $b \le a$ with $\left\| \mu(b) - \frac{1}{2}\mu(a) \right\| < \varepsilon$. Then the closure of $\mu(L)$ is convex.*

PROOF. – It suffices to show that for every $\varepsilon > 0$ and for every $a,\ b \in L$, there exists $s \in L$ with $\left\| \mu(s) - \frac{\mu(a)+\mu(b)}{2} \right\| < \varepsilon$.

Let $\varepsilon > 0$ and $a,\ b \in L$. Put $c = a \backslash (a \wedge b)$ and $d = b \backslash (a \wedge b)$. By assumptions there exists $c_\varepsilon \leq c$ and $d_\varepsilon \leq d$ such that

$$\left\| \mu(c_\varepsilon) - \frac{\mu(c)}{2} \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \mu(d_\varepsilon) - \frac{\mu(d)}{2} \right\| < \frac{\varepsilon}{2}.$$

From Proposition 2.2(x) it follows that $c \wedge d = 0$, so that $c_\varepsilon \wedge d_\varepsilon = 0$, too. Hence $\mu(c_\varepsilon \vee d_\varepsilon) = \mu(c_\varepsilon) + \mu(d_\varepsilon)$.

Now, as $c \leq^{\perp} (a \wedge b)$, we have $a \wedge b \leq c^\perp \leq c_\varepsilon^\perp$ and analogously $a \wedge b \leq d^\perp \leq d_\varepsilon^\perp$, whence

$$a \wedge b \leq c_\varepsilon^\perp \wedge d_\varepsilon^\perp = (c_\varepsilon \vee d_\varepsilon)^\perp.$$

Put $s = (c_\varepsilon \vee d_\varepsilon) + (a \wedge b)$. Since

$$\mu(s) = \mu(a \wedge b) + \mu(c_\varepsilon \vee d_\varepsilon) = \mu(a \wedge b) + \mu(c_\varepsilon) + \mu(d_\varepsilon),$$

we have

$$\begin{aligned} &\left\| \mu(s) - \frac{\mu(a)+\mu(b)}{2} \right\| \\ &= \left\| \mu(a \wedge b) + \mu(c_\varepsilon) + \mu(d_\varepsilon) - \frac{\mu(c)}{2} - \frac{\mu(a \wedge b)}{2} - \frac{\mu(d)}{2} - \frac{\mu(a \wedge b)}{2} \right\| \\ &\leq \left\| \mu(c_\varepsilon) - \frac{\mu(c)}{2} \right\| + \left\| \mu(d_\varepsilon) - \frac{\mu(d)}{2} \right\| < \varepsilon. \end{aligned}$$

□

Now we can prove a Uhl type theorem:

THEOREM 5.10. – *Suppose that X has the Radon-Nikodym property and $\mu$ of bounded variation. Then $\mu(L)$ is relatively compact. Moreover, if $\mu$ is nonatomic, then $\overline{\mu(L)}$ is convex.*

PROOF. – We denote by $\nu$ the total variation of $\mu$.

The proof goes into two steps:

STEP 1. – First suppose that $(L, \leq)$ is complete and $\mu$ is o.c.. As $\nu$ is bounded, by [20, Theor. 1.3.11] $\mu$ and $\nu$ generate the same topology. So $\nu$ is o.c., too (see Proposition 5.2). Moreover, if $\mu$ is nonatomic, by Proposition 5.2 $\nu$ is nonatomic, too.

Applying Theorem 5.8 we find a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $S_\nu(C(L))$ (see Notation 5.6) such that $\mu(a) = \sum_{n=1}^{\infty} \nu_n(a)$.

Now the argument proceeds in a similar way as in [4, Theor. 5.2], applying Corollary 4.9.

STEP 2. – We now drop the assumption $(L, \leq)$ being complete and $\mu$ being o.c. We use the same notation as in Theorem 4.1.

Let $\mathcal{U}$ be the D-uniformity generated by $v$, $\hat{L} = L/N(\mathcal{U})$, $\hat{\mathcal{U}}$ be the quotient uniformity and $(\tilde{L},\tilde{\mathcal{U}})$ the uniform completion of $(\hat{L},\hat{\mathcal{U}})$. As $v$ is bounded,by Proposition 4.2 $v$ is exhaustive, too; so is $\mathcal{U}$ by [19, Prop. 3.6].

From Theorem 4.1 it follows that $\tilde{\mathcal{U}}$ is o.c. and $(\tilde{L}, \leq)$ is a complete D-lattice. We recall that by [3, Theor. 2.9] a basis of the D-uniformity generated by a modular measure $\lambda$ is the family formed by the sets

$$\{(a, b) \in L \times L : \|\lambda(c)\| < \varepsilon, \text{ for every } c \leq (a \vee b) \backslash (a \wedge b)\}$$

with $\varepsilon > 0$.

Since $\|\mu(a)\| \leq v(a)$ for every $a \in L$, we have $\mathcal{U}(\mu) \leq \mathcal{U}$, so $\mu$ and $v$ are $\mathcal{U}$-continuous. By Theorem 4.1 the maps $\hat{\mu}(a) := \mu(a)$ and $\hat{v}(a) := v(a)$ for $a \in \hat{a} \in \hat{L}$ are well-defined $\hat{\mathcal{U}}$-continuous modular measures.

Observe that $\tilde{\mu}$ and $\tilde{v}$ are the unique o.c. $\tilde{\mathcal{U}}$-continuous extensions of $\mu$ and $v$, respectively, with $\tilde{\mu}(\tilde{L}) = \overline{\mu(L)}$.

Moreover, $\tilde{\mu}$ is of bounded variation. In fact we show that $\|\tilde{\mu}(a)\| \leq \tilde{v}(a)$ for all $a \in \tilde{L}$.

Indeed, if $a \in \tilde{L}$ and $\varepsilon > 0$, there exists $b \in \hat{L}$ such that $\|\hat{\mu}(b) - \tilde{\mu}(a)\| < \frac{\varepsilon}{2}$ and $|\hat{v}(b) - \tilde{v}(a)| < \frac{\varepsilon}{2}$, since $\tilde{\mu}$ and $\tilde{v}$ are $\tilde{\mathcal{U}}$-continuous. Then

$$\|\tilde{\mu}(a)\| < \frac{\varepsilon}{2} + \|\hat{\mu}(b)\| < \frac{\varepsilon}{2} + \hat{v}(b) < \varepsilon + \tilde{v}(a).$$

In a similar way we see that if $\mu$ is nonatomic, $\hat{\mu}$ and $\tilde{\mu}$ are nonatomic, too. Therefore we can apply Step 1. □

COROLLARY 5.11. – *If $X = \mathbb{R}^n$ and $\mu$ is nonatomic, then $\overline{\mu(L)}$ is convex and compact.*

PROOF. – Apply Theorem 5.10, since $\mu$ being nonatomic implies $\mu$ of bounded variation. Indeed, if $\mu = (\mu_1, \ldots, \mu_n)$ is nonatomic, then, for every $i \leq n$, $\mu_i$ is nonatomic and hence bounded. By Proposition 4.2 every $\mu_i$ is of bounded variation and so is $\mu$. □

REMARK 5.12. – If $L$ is $\sigma$-complete and $\mu$ is $\sigma$-additive, in 5.10 the assumption $\mu$ being nonatomic can be substituted by $\mu$ being atomless like in [17], since these two notions are equivalent (see Theorem 5.5(b)).

The last result has been proved by Kadets in [15] for $\sigma$-additive measures on $\sigma$-algebras. We need a lemma.

LEMMA 5.13. – *Let $\nu : L \to [0, +\infty[$ be a measure. Suppose that, for every $a \in L$ and $\alpha \in \mathbb{R}$ with $0 < \alpha < \nu(a)$, there exists $b \leq a$ such that $\nu(b) = \alpha$.*

*Then, for every $a \in L$ and $m \in \mathbb{N}$, there exists an orthogonal family $(a_1, \ldots, a_m)$ in $L$ with $\sum_{i=1}^{m} a_i = a$ and $\nu(a_i) = \frac{1}{m}\nu(a)$ for each $i \leq m$.*

PROOF. – Let $\nu(a) \in L$ with $\nu(a) > 0$.

We proceed by induction on $m \in \mathbb{N}$. The beginning is obvious.

By assumption there exists $b \leq a$ with $\nu(b) = \frac{m-1}{m}\nu(a)$. By induction there exists an orthogonal family $(a_1, \ldots, a_{m-1})$ whose sum equals $b$ such that $\nu(a_i) = \frac{1}{m-1}\nu(b) = \frac{1}{m}\nu(a)$.

Set $a_m := b/a$. Then, as $a = b + a_m = a_1 + \ldots + a_{m-1} + a_m$, we have that $(a_1, \ldots, a_m)$ is an orthogonal family with $\nu(a_m) = \nu(a) - \nu(b) = \frac{1}{m}\nu(a)$. $\square$

We recall that $X$ is said to be *B-convex* if there exist an integer $n \geq 2$ and a real number $0 < k < 1$ such that, for every $x_1, \ldots, x_n$ in $X$, $\min_{a_i = \pm 1} \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq kn \sup_{i \leq n} \|x_i\|$.

The concept of $B$-convexity is independent from the Radon-Nikodym property, as observed in [4, p. 164].

THEOREM 5.14. – *Suppose that $X$ is B-convex and $\mu$ is nonatomic with bounded variation. Then $\overline{\mu(L)}$ is convex.*

PROOF. – As in Theorem 5.10 we can suppose that $L$ is complete.

Let $\varepsilon > 0$ and $a \in L$. By Lemma 5.9 it is sufficient to prove that there exists $b \leq a$ such that $\left\| \mu(b) - \frac{1}{2}\mu(a) \right\| < \varepsilon$, and we may clearly assume $\mu(a) \neq 0$.

Choose $n$ and $k$ as in the definition of $B$-convex spaces and $s \in \mathbb{N}$ such that $k^s \mu(a) < \varepsilon$. Set $r = n^s$. Since $\mu$ is bounded, by [20, Theor. 1.3.11] $\mu$ and $|\mu|$ generate the same topology. Hence by Proposition 5.2 $|\mu|$ is nonatomic, too.

Since $L$ is $\sigma$-complete, by Theorem 5.5 $|\mu|([0, a])$ is convex. Therefore, by Lemma 5.13 we can find an orthogonal family $(a_1, \ldots, a_r)$ in $L$ such that $\sum_{i=1}^{r} a_i = a$ and $|\mu|(a_i) = \frac{1}{r}|\mu|(a)$ for each $i \leq r$.

Set

$$I := \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, r\}, \text{ with } h_1 < \cdots < h_k$$

and

$$\{1, \ldots, r\} \backslash I := \{t_1, \ldots, t_w\}, \text{ with } t_1 < \cdots < t_w.$$

Let $a_I = \sum_{j=1}^{k} a_{h_j}$. We have $\mu(a_I) = \sum_{j=1}^{k} \mu(a_{h_j})$.

Then

$$\left\|\mu(a_I) - \frac{1}{2}\mu(a)\right\| = \frac{1}{2}\left\| \sum_{j=1}^{k} \mu(a_{h_j}) - \sum_{j=1}^{w} \mu(a_{t_j})\right\|$$

from which we obtain

$$\min_I \left\|\mu(a_I) - \frac{1}{2}\mu(a)\right\| = \frac{1}{2} \min_{a_i=\pm 1} \left\| \sum_{i=1}^{r} a_i \mu(a_i)\right\|$$

Choose $b \le a$ such that $\|\mu(b) - \frac{1}{2}\mu(a)\| = \min_I \|\mu(a_I) - \frac{1}{2}\mu(a)\|$. Then we get

$$\left\|\mu(b) - \frac{1}{2}\mu(a)\right\| \le \frac{1}{2} k^s r \sup_{i \le r} \|\mu(a_i)\| \le \frac{1}{2} k^s r \sup_{i \le r} |\mu|(a_i) \le \frac{1}{2} k^s |\mu|(a) < \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

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