Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **3** (2010), n.3, p. 421–446.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2010_9_3_3_421_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2010.

Bollettino U. M. I.
(9) **III** (2010), 421-446

Classical Free Energies of a Heat Conductor with Memory and the Minimum Free Energy for its Discrete Spectrum Model

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Abstract. – *Free energies, originally proposed for viscoelastic solids, together with their corresponding internal dissipations, are here considered under forms adapted to the case of rigid heat conductors with memory. The results related to the minimum free energy of the discrete spectrum model are then compared with some of the classical free energies of such conductors.*

1. – Introduction.

The free energy functional, in the case of materials with memory, is not necessarily uniquely defined, hence the importance of its explicit expressions is well known. The non uniqueness of the free energy has been shown in the linear case [7] as well as in the general one by means of abstract formulations of thermodynamics.

This problem was firstly studied in the case of viscoelastic solids (in alphabetical order, see [7], [10]-[12], [15]-[17], [20]), for which some interesting functionals have been proposed as free energies. In particular, the proof of existence of a bounded and convex set, with a minimum and a maximum element, when free energies of materials with memory are considered, is comprised, for instance, in [17].

Later on, this study has been extended to the case of linear rigid heat conductors with memory within the framework previously proposed by Gurtin and Pipkin in [18]. The linearization of such a theory, when isotropic materials are examined, yields a constitutive equation involving the heat flux expressed by a linear functional of the history of the temperature gradient [18]. It is well known that such a relation represents a generalization of the Cattaneo-Maxwell equation [8]. Thus, in such a context, some expressions for the free energy of these materials have been given in [4] and [1], by adapting those ones proposed for viscoelastic solids, under the hypothesis that the material states of these conductors are characterized by the present value of the temperature $\vartheta(t)$, its past history ${}_r\vartheta^t$ and the integrated history of the temperature gradient $\bar{\boldsymbol{g}}^t$ (see [3] too).

In [2] the analogous problem is considered in the case when rigid heat conductors are characterized by states expressed in terms of the temperature $\vartheta(t)$, its past history ${}_r\vartheta^t$ and the past history of the temperature gradient ${}_r\boldsymbol{g}^t$. Any free energy is given by the sum of two parts, one of which is due to the internal energy, while the other one is related to the heat flux. These two contributes are derived in [4] and in [2], for the classical free energies related to viscoelastic solids. Only for the *Graffi-Volterra* functional it has been necessary to derive the contribute due to the heat flux by integrating by parts the form given in terms of $\bar{\boldsymbol{g}}^t$ in [4]; thus, the proposed expression is not formally similar to the one introduced for viscoelastic solids.

Here, after recalling these expressions, we further develop the results related to the minimum free energy of the discrete model of a heat conductor with memory, characterized by relaxation functions given by the sum of exponentials, (see [5], [6]). Thus, we are able to show that the minimum free energy of such a model is very interesting since it can be compared with some classical free energies. From its form we can derive the kernels related to the general form of free energy. Their expressions show the equivalence of the minimum free energy to the *Breur-Onat* functional and, in addition, the kernels which characterize the related rate of dissipation are obtained. Finally, the *Day* functional is shown to be a special case of the minimum free energy which corresponds to the case when only one exponential, instead of a sum of exponentials, characterizes the two kernels for the discrete model.

The material is organized as follows. Section 2 concerns the basic equations of the linear theory of rigid heat conductors with memory, the thermodynamic restrictions on the constitutive equations further to some useful relations. In Section 3, states and processes are introduced. In Section 4, the linearized local form of the Second Law of Thermodynamics and the internal dissipation are considered; moreover, the notion of the thermal work and its expression are given together with the definition of the minimum free energy. In Section 5, the classical free energies introduced for viscoelastic solids and already adapted to rigid heat conductors with memory are recalled. In Section 6, the discrete spectrum model together with the related minimum free energy is considered. In particular, the expression for the minimum free energy is modified in a suitable way to be compared with few classical free energies, which are recognized to represent its special cases.

2. – Basic equations.

The linearized constitutive equations in the case of a homogeneous and isotropic rigid heat conductor with memory $\mathcal{B}$, read

$$e(\boldsymbol{x},t) = a_0 \vartheta(\boldsymbol{x},t) + \int_0^{+\infty} a'(s)\,{}_r\vartheta^t(\boldsymbol{x},s)ds, \tag{2.1}$$

$$\boldsymbol{q}(\boldsymbol{x},t) = -\int_0^{+\infty} k(s)\,{}_r\boldsymbol{g}^t(\boldsymbol{x},s)ds \tag{2.2}$$

where, in turn, e represents the internal energy, $\boldsymbol{q}$ the heat flux, ϑ the linear part of the temperature variation, with respect to a uniform absolute temperature Θ_0, and $\boldsymbol{g} = \nabla\vartheta$ the temperature gradient, $\boldsymbol{x} \in \Omega$ denotes the vector position within the fixed bounded three-dimensional domain occupied by $\mathcal{B}$, while ${}_r\vartheta^t(\boldsymbol{x}, s) := \vartheta(\boldsymbol{x}, t-s)$ and ${}_r\boldsymbol{g}^t(\boldsymbol{x}, s) := \boldsymbol{g}(\boldsymbol{x}, t-s)$, $\forall s \in \mathbb{R}^{++} \equiv (0, +\infty)$, the past histories, respectively, of ϑ and $\boldsymbol{g}$ related to time t.

According to such a notation, the couple $(\vartheta(\boldsymbol{x}, t), {}_r\vartheta^t(\boldsymbol{x}, s))$ expresses the whole history, up to time t, of the temperature, $\vartheta(\boldsymbol{x}, t-s)$, $\forall s \in \mathbb{R}^+ \equiv [0, +\infty)$. Therefore, the internal energy depends on this couple, while the heat flux depends only on the past history ${}_r\boldsymbol{g}^t$.

The relative history up to time t of the temperature, denoted by ϑ_r^t, is defined by

$$\vartheta_r^t(\boldsymbol{x}, s) = {}_r\vartheta^t(\boldsymbol{x}, s) - \vartheta(\boldsymbol{x}, t). \tag{2.3}$$

Henceforth, since our attention is focussed on a specific point of $\mathcal{B}$, the dependence on $\boldsymbol{x}$ will be omitted.

We assume that the two kernels in (2.1)-(2.2), $a' : \mathbb{R}^+ \to \mathbb{R}$ and $k : \mathbb{R}^+ \to \mathbb{R}$, are such that $a', a'', a''' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ and $k, k', k'' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. The thermal conductivity k and the heat capacity

$$a(t) = a_0 + \int_0^t a'(s)ds \tag{2.4}$$

are such that $a_0 \equiv a(0) > 0$, and, in addition, $\lim_{t \to +\infty} k(t) = 0$, as suggested by physical considerations. Recall the definition of Fourier transform of any function $f : \mathbb{R} \to \mathbb{R}$,

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(s)e^{-i\omega s}ds = f_-(\omega) + f_+(\omega) \qquad \forall \omega \in \mathbb{R}, \tag{2.5}$$

where

$$f_-(\omega) = \int_{-\infty}^{0} f(s)e^{-i\omega s}ds, \qquad f_+(\omega) = \int_0^{+\infty} f(s)e^{-i\omega s}ds, \tag{2.6}$$

and of half-range Fourier cosine and sine transforms,

$$f_c(\omega) = \int_0^{+\infty} f(s)\cos(\omega s)ds, \qquad f_s(\omega) = \int_0^{+\infty} f(s)\sin(\omega s)ds. \tag{2.7}$$

Note that definitions $(2.6)_2$ and (2.7) are unchanged if f is defined only on $\mathbb{R}^+$. If f is a function defined on $\mathbb{R}^+$, its even and odd prolongations can, in turn, be defined via $f(\xi) = f(-\xi)$, $f(\xi) = -f(-\xi)$ or $f(\xi) = 0$, for any $\xi < 0$, the last of which is the causal extension; the corresponding Fourier transforms, respectively, read

$$f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -2if_s(\omega), \quad f_F(\omega) = f_c(\omega) - if_s(\omega). \tag{2.8}$$

If $f, f' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, then

$$f'_s(\omega) = -\omega f_c(\omega); \tag{2.9}$$

furthermore, if $f'' \in L^1(\mathbb{R}^+)$, also [13]

$$\omega f'_s(\omega) = f'(0) + f''_c(\omega). \tag{2.10}$$

It is well known that the thermodynamic laws impose restrictions on the constitutive equations; in our case, these constraints on (2.1)-(2.2) are [14]

$$\omega a'_s(\omega) > 0, \qquad k_c(\omega) > 0 \qquad \omega \neq 0. \tag{2.11}$$

Formulae (2.9)-(2.11) together with the inverse half-range Fourier transforms, induce

$$a''_c(\omega) = \omega a'_s(\omega) - a'(0), \quad a(t) - a_0 = \frac{2}{\pi} \int_0^{+\infty} \frac{a'_s(\omega)}{\omega} [1 - \cos(\omega t)] d\omega > 0, \tag{2.12}$$

if $a'' \in L^1(\mathbb{R}^+)$, and

$$k_c(\omega) = -\frac{1}{\omega} k'_s(\omega) > 0 \ \forall \omega \neq 0, \quad k_0 \equiv k(0) = -\frac{2}{\pi} \int_0^{+\infty} \frac{k'_s(\omega)}{\omega} d\omega > 0. \tag{2.13}$$

Hence, it follows

$$a_\infty - a_0 = \frac{2}{\pi} \int_0^{+\infty} \frac{a'_s(\omega)}{\omega} d\omega > 0, \qquad \lim_{\omega \to +\infty} \omega a'_s(\omega) = a'(0) \geq 0 \tag{2.14}$$

and

$$\lim_{\omega \to +\infty} \omega k'_s(\omega) = -\lim_{\omega \to +\infty} \omega^2 k_c(\omega) = k'(0) \leq 0. \tag{2.15}$$

We assume

$$k_c(0) > 0, \qquad k'(0) < 0, \qquad a'(0) > 0. \tag{2.16}$$

Finally, the Fourier transforms $f_\pm(\omega)$, defined in (2.6), can be extended on the complex z-plane $\mathbb{C}$ and are analytic, respectively, in

$$\mathbb{C}^{(-)} = \{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{--}\}, \quad \mathbb{C}^{(+)} = \{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{++}\}, \tag{2.17}$$

where $\mathbb{R}^{--} \equiv (-\infty, 0)$. Moreover, when f is analytic on $\mathbb{R}$, the corresponding

Fourier transforms [15], $f_{\pm}(z)$ are analytic, respectively, in

$$\mathbb{C}^{-} = \{z \in \mathbb{C};\ \operatorname{Im} z \in \mathbb{R}^{-}\}, \quad \mathbb{C}^{+} = \{z \in \mathbb{C};\ \operatorname{Im} z \in \mathbb{R}^{+}\}. \tag{2.18}$$

The notation $f_{(\pm)}(z)$ indicates that the zeros and the singularities of f are in $\mathbb{C}^{\pm}$.

3. – States and processes.

The body $\mathcal{B}$, characterized by the constitutive equations (2.1)-(2.2), can be considered as a simple material; therefore, its behaviour can be described by means of states and processes.

Let

$$\sigma(t) = (\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t) \tag{3.1}$$

represent the thermodynamic state of $\mathcal{B}$ and the piecewise continuous map $P : [0, d) \to \mathbb{R} \times \mathbb{R}^3$ defined by

$$P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau)) \qquad \forall \tau \in [0, d), \tag{3.2}$$

the related thermodynamic process with duration $d \in \mathbb{R}^{++}$, where $\dot{\vartheta}_P(\tau)$ denotes the derivative of ϑ with respect to τ. Such a process can be applied to the body at any time t.

We denote by Σ and Π the sets of states and of processes, which are admissible for the material. For any process $P \in \Pi$ with duration $d \in \mathbb{R}^{++}$ we can consider its restriction $P_{[\tau_1,\tau_2)} \in \Pi$ to any interval $[\tau_1, \tau_2) \subset [0, d)$.

The state transition function $\rho : \Sigma \times \Pi \to \Sigma$ maps any initial state $\sigma^i \in \Sigma$ and any $P \in \Pi$ into the final state $\sigma^f = \rho(\sigma^i, P) \in \Sigma$. If we apply the restriction $P_{[0,\tau)}$ to $\sigma^i = \sigma(0)$, the final state is $\sigma(\tau) = \rho(\sigma(0), P_{[0,\tau)})$; in particular, if $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$, the pair (σ,P) is termed *cycle*.

Let the process $P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau))$, $\forall \tau \in [0, d)$ be applied at the instant $t > 0$, when $\sigma(t) = (\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t)$, then the subsequent states $\sigma(t + \tau)$, $\forall \tau \in (0, d]$ are characterized by

$$\vartheta_P : (0, d] \to \mathbb{R}, \quad \vartheta_P(\tau) \equiv \vartheta(t + \tau) = \vartheta(t) + \int_0^{\tau} \dot{\vartheta}_P(\eta) d\eta, \tag{3.3}$$

the continuation of the past history of the temperature

$${}_r\vartheta^{t+\tau}(s) = (\vartheta_P *_r \vartheta)^{t+\tau}(s) = \begin{cases} \vartheta_P(\tau - s) & \forall s \in (0, \tau], \\ \vartheta(t + \tau - s) & \forall s > \tau \end{cases} \tag{3.4}$$

and the past history of the temperature gradient

$${}_r\boldsymbol{g}^{t+\tau}(s) = (\boldsymbol{g}_P * {}_r\boldsymbol{g})^{t+\tau}(s) = \begin{cases} \boldsymbol{g}_P(\tau - s) & \forall s \in (0, \tau], \\ \boldsymbol{g}(t + \tau - s), & \forall s > \tau. \end{cases} \tag{3.5}$$

Recalling (2.1)-(2.2), the internal energy and the heat flux are given by

$$e(t+\tau) = a_0 \vartheta_P(\tau) + \int_0^\tau a'(s)\vartheta_P^\tau(s)ds + \int_\tau^{+\infty} a'(s)\, {}_r\vartheta^{t+\tau}(s)ds, \tag{3.6}$$

$$\boldsymbol{q}(t+\tau) = -\int_0^\tau k(s)\boldsymbol{g}_P^\tau(s)ds - \int_\tau^{+\infty} k(s)\, {}_r\boldsymbol{g}^{t+\tau}(s)ds\,. \tag{3.7}$$

Consider the static continuation of two given histories $(\vartheta(t), {}_r\vartheta^t)$ and ${}_r\boldsymbol{g}^t$, with duration $a \in \mathbb{R}^{++}$, defined by

$$\vartheta^{t_{(a)}} = \begin{cases} \vartheta(t) & \forall s \in [0,a], \\ {}_r\vartheta^t(s-a) & \forall s > a, \end{cases} \qquad \boldsymbol{g}^{t_{(a)}} = \begin{cases} \boldsymbol{g}(t) & \forall s \in [0,a], \\ {}_r\boldsymbol{g}^t(s-a) & \forall s > a\,, \end{cases} \tag{3.8}$$

then, on use of (3.6)-(3.7), or directly from (2.1)-(2.2), it follows

$$e(t+a) = a(a)\vartheta(t) + \int_0^{+\infty} a'(\xi+a)\, {}_r\vartheta^t(\xi)d\xi, \tag{3.9}$$

$$\boldsymbol{q}(t+a) = -\,\nu^{(k)}(a)\boldsymbol{g}(t) - \int_0^{+\infty} k(\xi+a)\, {}_r\boldsymbol{g}^t(\xi)d\xi, \tag{3.10}$$

where

$$\nu^{(k)}(t) = \int_0^t k(\eta)d\eta. \tag{3.11}$$

The constitutive equations (2.1)-(2.2) allow to consider two functionals $\tilde{e} : \mathbb{R} \times \Gamma_a \to \mathbb{R}$ and $\tilde{\boldsymbol{q}} : \Gamma_k \to \mathbb{R}^3$ defined by

$$e(\sigma(t)) = \tilde{e}(\vartheta(t), {}_r\vartheta^t), \qquad \boldsymbol{q}(\sigma(t)) = \tilde{\boldsymbol{q}}({}_r\boldsymbol{g}^t), \tag{3.12}$$

where, by virtue of (3.9)-(3.10), Γ_a and Γ_k denote, respectively, the following function spaces

$$\Gamma_a = \left\{ {}_r\vartheta^t : (0,+\infty) \to \mathbb{R};\ \left| \int_0^{+\infty} a'(\xi+\tau)\, {}_r\vartheta^t(\xi)d\xi \right| < +\infty\ \forall\tau \in \mathbb{R}^+ \right\}, \tag{3.13}$$

$$\Gamma_k = \left\{ {}_r\boldsymbol{g}^t : (0,+\infty) \to \mathbb{R}^3;\ \left| \int_0^{+\infty} k(\xi+\tau)\, {}_r\boldsymbol{g}^t(\xi)d\xi \right| < +\infty\ \forall\tau \in \mathbb{R}^+ \right\}, \tag{3.14}$$

where t is a parameter.

4. – Thermodynamics and thermal work.

Thermodynamics plays a central role in the study of any physical problem. In particular, its laws impose constraints on the constitutive equations and state the existence of the free energy, which, in the linear theory of materials with memory, suggests the choice of the most suitable norms. It is well known (see [17]) that, in the case of materials with memory, there is no unique expression for the free energy; in fact, there is a convex set of these functions, with a minimum and a maximum element, here denoted by ψ_m and ψ_M, respectively. Thus, the interest in finding expressions for the free energy for materials with memory follows.

In the linear theory, the thermodynamical constraints imposed on the constitutive equations of $\mathcal{B}$ are given in (2.11) and, moreover, the linearized local form of the Clausius-Duhem classical inequality reads

$$\dot{\psi}(\boldsymbol{x},t) \leq \dot{e}(\boldsymbol{x},t)\vartheta(\boldsymbol{x},t) - \boldsymbol{q}(\boldsymbol{x},t)\cdot\boldsymbol{g}(\boldsymbol{x},t), \tag{4.1}$$

where ψ enjoys the same properties of a canonical free energy and, accordingly, is called free energy, for short.

Inequality (4.1) implies the following equality

$$\dot{\psi}(\boldsymbol{x},t) + D(\boldsymbol{x},t) = \dot{e}(\boldsymbol{x},t)\vartheta(\boldsymbol{x},t) - \boldsymbol{q}(\boldsymbol{x},t)\cdot\boldsymbol{g}(\boldsymbol{x},t), \tag{4.2}$$

where, according to the Second Law, $D(\boldsymbol{x},t)$, termed *the internal dissipation*, is a non-negative quantity.

The thermal power, taking into account (4.1), can be written in the form

$$w(t) = \dot{e}(t)\vartheta(t) - \boldsymbol{q}(t)\cdot\boldsymbol{g}(t). \tag{4.3}$$

Therefore, the thermal work, produced by the application of a process $P(\tau) = (\dot{\vartheta}_P(\tau), \boldsymbol{g}_P(\tau))$, $\forall\tau \in [0,d)$, at the initial time t, when the material state is $\sigma(t) = (\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t)$, is expressed by

$$W(\sigma,P) = \widetilde{W}(\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t; \dot{\vartheta}_P, \boldsymbol{g}_P) = \int_0^d [\dot{e}(t+\tau)\vartheta_P(\tau) - \boldsymbol{q}(t+\tau)\cdot\boldsymbol{g}_P(\tau)]d\tau. \tag{4.4}$$

Here, $\boldsymbol{q}(t+\tau)$ is given by (3.7) and $\dot{e}(t+\tau)$, the derivative with respect to τ of (3.6), after two integrations by parts, takes the form

$$\dot{e}(t+\tau) = a_0\dot{\vartheta}_P(\tau) + a'(0)\vartheta_P(\tau) + \int_0^\tau a''(s)\vartheta_P(\tau-s)ds + \int_\tau^{+\infty} a''(s)\vartheta(t+\tau-s)ds. \tag{4.5}$$

The thermal work on the rigid heat conductor, only during the application of a given process, can be evaluated by considering the zero state $\sigma_0(0) = (0, {}_r0^\dagger, {}_r\boldsymbol{0}^\dagger)$

as the initial state at time $t=0$, when the process, $P(t)=(\dot{\vartheta}_P(t), \boldsymbol{g}_P(t))$, $\forall t \in [0,d)$ with a duration $d<+\infty$, is applied. When $(\vartheta_0, {}_r\vartheta_0^t, {}_r\boldsymbol{g}_0^t)$ denotes the ensuing fields, whose form can be derived by (3.3)-(3.5), in [6] it is shown that the thermal work is

$$
\begin{aligned}
W(\sigma_0(0),P) = \widetilde{W}(0, {}_r0^\dagger, {}_r\boldsymbol{0}^\dagger; \dot{\vartheta}_P, \boldsymbol{g}_P) &= \frac{1}{2} a_0 \vartheta_0^2(d) + a'(0)\int_0^d \vartheta_0^2(t)dt \\
+\int_0^d\int_0^t a''(s)\, {}_r\vartheta_0^t(s)ds\vartheta_0(t)dt &+ \int_0^d\int_0^t k(s)\, {}_r\boldsymbol{g}_0^t(t)ds \cdot \boldsymbol{g}_0(t)dt > 0,
\end{aligned} \tag{4.6}
$$

and the process is termed a finite thermal work process.

Also in [6], (4.6), modified by means of the prolongation of P on $\mathbb{R}^+$, defined letting $P(\tau)=(0,\boldsymbol{0})$, $\forall\tau \geq d$ and $\vartheta_P(\tau)=0$, for any $\tau > d$, combined with the application of Plancherel's theorem, has suggested the introduction of the function spaces

$$
\widetilde{H}_a(\mathbb{R}^+,\mathbb{R}) = \left\{ \vartheta : \mathbb{R}^+ \to \mathbb{R};\ \int_{-\infty}^{+\infty} \omega a_s'(\omega)\vartheta_{P+}(\omega)[\vartheta_{P+}(\omega)]^* d\omega < +\infty \right\}, \tag{4.7}
$$

$$
\widetilde{H}_k(\mathbb{R}^+,\mathbb{R}^3) = \left\{ \boldsymbol{g} : \mathbb{R}^+ \to \mathbb{R}^3;\ \int_{-\infty}^{+\infty} k_c(\omega)\boldsymbol{g}_+(\omega)\cdot[\boldsymbol{g}_{P+}(\omega)]^* d\omega < +\infty \right\}. \tag{4.8}
$$

to characterize all finite work processes.

In general, apply the process $P=(\dot{\vartheta}_P, \boldsymbol{g}_P)$, with duration $d<+\infty$, to the body $\mathcal{B}$ at time $t>0$ when $\sigma(t)=(\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t)$. Here ${}_r\vartheta^t$ and ${}_r\boldsymbol{g}^t$ are possible past histories, which belong to Γ_a and Γ_k, defined in (3.13)-(3.14), so that they give a finite work for any process in (4.7)-(4.8). When the trivial prolongation on $\mathbb{R}^+$, via $P(\tau)=(0,\boldsymbol{0})$, $\forall\tau\geq d$, of such a process, assuming $\vartheta_P(\tau)=0$, $\forall\tau>d$, the work (4.4), on substitution of expressions (4.5) and (3.7) of $\dot{e}(t+\tau)$ and $\boldsymbol{q}(t+\tau)$, after some integrations by parts, can be written

$$
\begin{aligned}
W(\sigma(t),P) &= \widetilde{W}(\vartheta(t), {}_r\vartheta^t, {}_r\boldsymbol{g}^t; \dot{\vartheta}_P, \boldsymbol{g}_P) \\
&= \frac{1}{2} a_0 \left[\vartheta_P^2(d) - \vartheta_P^2(0)\right] + a'(0)\int_0^{+\infty} \vartheta_P^2(\tau)d\tau \\
&+ \int_0^{+\infty} \left[\frac{1}{2}\int_0^{+\infty} a''(|\tau-\eta|)\vartheta_P(\eta)d\eta - \tilde{I}_{(a)}^t(\tau, {}_r\vartheta^t)\right]\vartheta_P(\tau)d\tau \\
&+ \int_0^{+\infty} \left[\frac{1}{2}\int_0^{+\infty} k(|\tau-\eta|)\boldsymbol{g}_P(\eta)d\eta - \boldsymbol{I}_{(k)}^t(\tau, {}_r\boldsymbol{g}^t)\right]\cdot \boldsymbol{g}_P(\tau)d\tau,
\end{aligned} \tag{4.9}
$$

where, for any $\tau \in \mathbb{R}^+$,

$$\tilde{I}^t_{(a)}(\tau, {}_r\vartheta^t) = -\int_0^{+\infty} a''(\tau+\xi)\,{}_r\vartheta^t(\xi)d\xi,\ \boldsymbol{I}^t_{(k)}(\tau, {}_r\boldsymbol{g}^t) = -\int_0^{+\infty} k(\tau+\xi)\,{}_r\boldsymbol{g}^t(\xi)d\xi. \tag{4.10}$$

Note that, the heat flux $\boldsymbol{q}(t+\tau)$, given by (3.10), depends on $\boldsymbol{I}^t_{(k)}$, conversely the energy $e(t+\tau)$, which represents the final value of the internal energy after a static continuation with a finite duration, does not depend on $\tilde{I}^t_{(a)}$. Indeed, this follows from expression (4.4) for the work which depends on $\dot{e}$ and not on e.

Since the minimum free energy coincides with the maximum recoverable work obtained from a given state of the material, it follows

$$\psi_m(t) = W_R(\sigma) = \sup\{-W(\sigma, P) : \ P \in \Pi\}, \tag{4.11}$$

where Π denotes the set of finite work processes.

Thermodynamic considerations yield $W_R(\sigma) < +\infty$; moreover, $W_R(\sigma)$ is a non-negative function of the state σ, since the null process, which also belongs to Π, gives a null work.

5. – Free energies for heat conductors.

This Section is devoted to analyze various free energies which have been introduced in viscoelasticity. They are all amenable to be adopted in the case of a rigid heat conductor with memory according to results here presented, based also on previous works [2]-[6] (see [9] too).

For this purpose, some requirements on the kernels in (2.1)-(2.2) and on their derivatives need to be imposed. Specifically, the following hypotheses

$$a'(s) > 0, \qquad a''(s) \le 0 \quad \forall s \in \mathbb{R}^+ \tag{5.1}$$

are adopted together with the further one

$$a'''(s) \ge 0, \tag{5.2}$$

imposed on the kernel related to e, while the restrictions for the kernel k are given by

$$k'(s) < 0, \qquad k''(s) \ge 0 \quad \forall s \in \mathbb{R}^+. \tag{5.3}$$

In particular, as a consequence, $(2.16)_{2,3}$, previously assumed, are satisfied.

The separate contributes $\psi^{(e)}$ and $\psi^{(\boldsymbol{q})}$ due to e and $\boldsymbol{q}$, in any free energy of $\mathcal{B}$, can be emphasized via the following sum

$$\psi(t) = \psi^{(e)}(t) + \psi^{(\boldsymbol{q})}(t). \tag{5.4}$$

The Day free energy. In [2] the free energy functional proposed by Day [10] in the case of a viscoelastic solids, has been adapted to heat conduction with memory; it reads

$$\psi_{Day}(t) = \psi^{(e)}_{Day}(t) + \psi^{(\boldsymbol{q})}_{Day}(t) = \frac{1}{2} a_0 \vartheta^2(t) + \frac{1}{2}\left[(a_\infty - a_0)^{-\frac{1}{2}} \int_0^{+\infty} a'(s)\, {}_r\vartheta^t(s) ds\right]^2 + \frac{1}{2}\left[k_0^{-\frac{1}{2}} \int_0^{+\infty} k(s)\, {}_r\boldsymbol{g}^t(s) ds\right]^2 . \tag{5.5}$$

Here, the kernels must have the exponential form, required by Day's functional for viscoelastic solids, given by

$$a'(s) = a'(0) e^{-\delta s} \quad \delta > 0, \qquad k(s) = k(0) e^{-\gamma s} \quad \gamma > 0, \tag{5.6}$$

whence it follows that

$$a''(s) = -\delta a'(s), \qquad k'(s) = -\gamma k(s), \tag{5.7}$$

which satisfy conditions (5.1) and (5.3), because of $(2.16)_3$ and $(2.13)_2$. Thus, the corresponding internal dissipation is given by

$$D_{Day}(t) \equiv D^{(e)}_{Day}(t) + D^{(\boldsymbol{q})}_{Day}(t) = \frac{\delta}{a_\infty - a_0}\left[\int_0^{+\infty} a'(s) \vartheta^t_r(s) ds\right]^2 + \frac{\gamma}{k_0}\left[\int_0^{+\infty} k(s)\, {}_r\boldsymbol{g}^t(s) ds\right]^2 \geq 0. \tag{5.8}$$

The Dill free energy. The analogous, in thermodynamics with memory, of the free energy introduced by Dill [12] for viscoelastic solids, reads

$$\psi_{Dill}(t) = \psi^{(e)}_{Dill}(t) + \psi^{(\boldsymbol{q})}_{Dill}(t) = \frac{1}{2} a_0 \vartheta^2(t) - \frac{1}{2} \int_0^{+\infty}\int_0^{+\infty} a''(\xi_1 + \xi_2)\, {}_r\vartheta^t(\xi_1)\, {}_r\vartheta^t(\xi_2) d\xi_1 d\xi_2 + \frac{1}{2} \int_0^{+\infty}\int_0^{+\infty} k(\xi_1 + \xi_2)\, {}_r\boldsymbol{g}^t(\xi_1) \cdot {}_r\boldsymbol{g}^t(\xi_2) d\xi_1 d\xi_2, \tag{5.9}$$

where the kernels k and a'' are required to be non-negative.

The related internal dissipation is given by

$$D_{Dill}(t) = D^{(e)}_{Dill}(t) + D^{(\boldsymbol{q})}_{Dill}(t) = \int_0^{+\infty}\int_0^{+\infty} a'''(\xi_1 + \xi_2) \vartheta^t_r(\xi_1) \vartheta^t_r(\xi_2) d\xi_1 d\xi_2 - \int_0^{+\infty}\int_0^{+\infty} k'(\xi_1 + \xi_2)\, {}_r\boldsymbol{g}^t(\xi_1) \cdot {}_r\boldsymbol{g}^t(\xi_2) d\xi_1 d\xi_2, \tag{5.10}$$

which, by virtue of (5.2) and $(5.3)_1$, is a non-negative quantity.

The Fabrizio free energy. A new free energy in terms of the minimal state has been recently introduced by Fabrizio for viscoelastic solids (see, for example, [11]). The corresponding free energy functional is obtained in [2], where also the functionals (4.10) as well as the following ones

$$I^t_{(a)}(\tau, {}_r\vartheta^t) = -\int_0^{+\infty} a'(\tau+\xi)\,{}_r\vartheta^t(\xi)d\xi,\; \breve{I}^t_{(a)}(\tau, \vartheta^t_r) = -\int_0^{+\infty} a'(\tau+\xi)\vartheta^t_r(\xi)d\xi, \tag{5.11}$$

are introduced together with some useful relations they satisfy. Thus, the free energy of the heat conductor with memory $\mathcal{B}$ is written as

$$\begin{aligned}\psi_F(t) = \psi_F^{(e)}(t) + \psi_F^{(\boldsymbol{q})}(t) &= \frac{1}{2}a_0\vartheta^2(t) + \frac{1}{2}\int_0^{+\infty} \frac{1}{a'(\tau)}\left[I^t_{(a)(1)}(\tau, {}_r\vartheta^t)\right]^2 d\tau \\ &-\frac{1}{2}\int_0^{+\infty} \frac{1}{k'(\tau)}\left[\boldsymbol{I}^t_{(k)(1)}(\tau, {}_r\boldsymbol{g}^t)\right]^2 d\tau,\end{aligned} \tag{5.12}$$

where $I^t_{(a)(1)}(\tau, {}_r\vartheta^t)$ and $\boldsymbol{I}^t_{(k)(1)}(\tau, {}_r\boldsymbol{g}^t)$ denote, respectively, the derivatives with respect to τ of $(5.11)_1$ and $(4.10)_2$. They are both positive by virtue of $(5.1)_1$ and $(5.3)_1$.

The corresponding internal dissipation is given by

$$\begin{aligned}D_F(t) \equiv D_F^{(e)}(t) + D_F^{(\boldsymbol{q})}(t) &= \frac{1}{2a'(0)}\left[\breve{I}^t_{(a)(1)}(0, \vartheta^t_r)\right]^2 \\ &+\frac{1}{2}\int_0^{+\infty} \frac{d}{d\tau}\left[\frac{1}{a'(\tau)}\right]\left[\breve{I}^t_{(a)(1)}(\tau, \vartheta^t_r)\right]^2 d\tau - \frac{1}{2}\frac{1}{k'(0)}\left[\boldsymbol{I}^t_{(k)(1)}(0, {}_r\boldsymbol{g}^t)\right]^2 \\ &-\frac{1}{2}\int_0^{+\infty} \frac{d}{d\tau}\left[\frac{1}{k'(\tau)}\right]\left[\boldsymbol{I}^t_{(k)(1)}(\tau, {}_r\boldsymbol{g}^t)\right]^2 \geq 0.\end{aligned} \tag{5.13}$$

The Breuer and Onat functional. An alternative form of the most general quadratic form, proposed by Breuer and Onat [7] for viscoelastic solids, has been given in the scalar case by *Golden* [15]; this alternative functional, adapted to $\mathcal{B}$ ([2]), reads

$$\begin{aligned}\psi_g(t) &= \psi_g^{(e)}(t) + \psi_g^{(\boldsymbol{q})}(t) \\ &= \frac{1}{2}a_0\vartheta^2(t) - \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} a_{12}(s,u)\,{}_r\vartheta^t(s)\,{}_r\vartheta^t(u)dsdu \\ &+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k(s,u)\,{}_r\boldsymbol{g}^t(s)\cdot{}_r\boldsymbol{g}^t(u)dsdu.\end{aligned} \tag{5.14}$$

Here, we distinguish the new kernels from the previous ones by means of their two variables. Obviously, $k(s,u)$ and $a_{12}(s,u)=\dfrac{\partial^2}{\partial s\partial u}a(s,u)$ are required to be positive operators. For such kernels some conditions are assumed; that is

$$a_{12}(+\infty,u)=a_{12}(s,+\infty)=0, \tag{5.15}$$

$$a_1(s,0)=a_2(0,s)=a'(s),\quad a_1(u,+\infty)=a_2(+\infty,s)=0 \tag{5.16}$$

and

$$k(s,0)=k(0,s)=k(s) \tag{5.17}$$

and, in addition, the derivatives $k_1(s,u)$ and $k_2(s,u)$ satisfy $(5.3)_1$.

The corresponding internal dissipation is

$$\begin{aligned} D_g(t)\equiv D_g^{(e)}(t)+D_g^{(\boldsymbol{q})}(t)=&\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}K_{12}^{(a)}(s,u)\vartheta_r^t(s)\vartheta_r^t(u)dsdu\\ &-\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}K^{(k)}(s,u)\,{}_r\boldsymbol{g}^t(s)\cdot{}_r\boldsymbol{g}^t(u)dsdu\geq 0,\end{aligned} \tag{5.18}$$

where, in order to ensure the non-negativity of D_g, the derivative of

$$K^{(a)}(s,u)=a_1(s,u)+a_2(s,u) \tag{5.19}$$

is required to be a non-negative, that is

$$K_{12}^{(a)}(s,u)=a_{112}(s,u)+a_{212}(s,u)\geq 0 \tag{5.20}$$

in accordance with (5.2),

$$K^{(k)}(s,u)=k_1(s,u)+k_2(s,u)<0, \tag{5.21}$$

by virtue of the hypotheses on the first derivatives of $k(s,u)$.

The Graffi-Volterra free energy. The Graffi-Volterra functional proposed for viscoelastic solids ([16], [20]), adapted to $\mathcal{B}$ in [2], reads

$$\begin{aligned}\psi_G(t)\equiv\psi_G^{(e)}(t)+\psi_G^{(\boldsymbol{q})}(t)=&\frac{1}{2}a_0\vartheta^2(t)+\frac{1}{2}\int_0^{+\infty}a'(s)[{}_r\vartheta^t(s)]^2ds\\ &+\int_0^{+\infty}\left[\int_0^{+\infty}k(s+\xi)\,{}_r\boldsymbol{g}^t(s+\xi)d\xi\right]\cdot{}_r\boldsymbol{g}^t(s)ds,\end{aligned} \tag{5.22}$$

under the hypotheses (5.1) and (5.3).

In such an expression, as observed in [2], the functional $\psi_G^{(e)}$ due to the internal energy is deduced in a natural way; while the contribute $\psi_G^{(\boldsymbol{q})}$ due to the heat flux is derived in terms of ${}_r\boldsymbol{g}^t$, via integration by parts, the expression for the free energy in terms of $\bar{\boldsymbol{g}}^t$. Thus, the expression of $\psi_G^{(\boldsymbol{q})}$ in (5.22), even if is not formally similar to the corresponding one in viscoelasticity, yields a free energy since obtained from that one in terms of $\bar{\boldsymbol{g}}^t$.

The related internal dissipation is given by

$$
\begin{aligned}
D_G(t) \equiv D_G^{(e)}(t) + D_G^{(\boldsymbol{q})}(t) = -\frac{1}{2}\int_0^{+\infty} a''(s)[\vartheta_r^t(s)]^2 ds \\
- \int_0^{+\infty}\int_0^{+\infty} k'(s+\xi)\, {}_r\boldsymbol{g}^t(s+\xi)\cdot {}_r\boldsymbol{g}^t(s)\, ds\, d\xi \geq 0.
\end{aligned}
\tag{5.23}
$$

The maximum free energy. A particular expression of the general representation (5.14) for $\mathcal{B}$, as it occurs for viscoelastic solids, has the form

$$
\begin{aligned}
\psi_M(t) \equiv \psi_M^{(e)}(t) + \psi_M^{(\boldsymbol{q})}(t) = \frac{1}{2} a_0 \vartheta^2(t) \\
-\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} a_{12}(|s_1 - s_2|)\, {}_r\vartheta^t(s_1)\, {}_r\vartheta^t(s_2)\, ds_1 ds_2 \\
+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k(|s_1 - s_2|)\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)\, ds_1 ds_2,
\end{aligned}
\tag{5.24}
$$

where each of the kernels $a_{12}(|s_1 - s_2|)$ and $k(|s_1 - s_2|)$ is required to be non-negative.

Such a functional is called the maximum free energy of $\mathcal{B}$; in fact, the corresponding internal dissipation vanishes [2], that is

$$
D_M(t) \equiv D_M^{(e)}(t) + D_M^{(\boldsymbol{q})}(t) = 0. \tag{5.25}
$$

The Golden free energy. The analog of the minimum free energy [15] in viscoelasticity, derived in the frequency domain by *Golden*, is proposed in [5] and [6] referring to heat conduction with memory. According to the viscoelastic case, two equivalent expressions for the minimum free energy of $\mathcal{B}$, which represent the parts related to the heat flux and to the internal energy, respectively, are further given. A first expression for the minimum free energy.

The thermodynamic constraint $(2.11)_1$ imposed on a' allows us to introduce the even function

$$
H(\omega) = \omega a_s'(\omega) = H(-\omega) > 0 \quad \forall \omega \in \mathbb{R}\backslash\{0\}, \tag{5.26}
$$

which, on use of $(2.12)_1$ and $(2.16)_3$, satisfies

$$
H_\infty = \lim_{\omega\to+\infty} \omega a_s'(\omega) = a'(0) > 0 \tag{5.27}
$$

and goes to zero at least quadratically at the origin. Hence, its behaviour is assumed to be not stronger then quadratic.

According to the assumptions in [15], the functions $H(\omega)$ and of $k_c(\omega)$, both positive valued by virtue of (2.11), are required to factorize as follows

$$H(\omega) = H_{(+)}(\omega)H_{(-)}(\omega), \quad k_c(\omega) = k_{(+)}(\omega)k_{(-)}(\omega). \tag{5.28}$$

Furthermore, the hypothesis of the analyticity of the Fourier transforms on $\mathbb{R}$, implies that $H_{(+)}(\omega)$ which exhibits no singularities and zeros in $\mathbb{C}^{(-)}$, is analytic in $\mathbb{C}^-$, while $H_{(-)}(\omega)$ has no zeros and singularities in $\mathbb{C}^{(+)}$ so that is analytic in $\mathbb{C}^+$. Similar results hold for $k_{(\pm)}(\omega)$.

Let us introduce

$$p^t_{(a)}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{I^t_{(a)+}(\omega, {}_r\vartheta^t)/[2H_{(-)}(\omega)]}{\omega - z} d\omega, \tag{5.29}$$

$$\boldsymbol{p}^t_{(k)}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\boldsymbol{I}^t_{(k)+}(\omega, {}_r\boldsymbol{g}^t)/[2k_{(-)}(\omega)]}{\omega - z} d\omega \tag{5.30}$$

and let

$$p^t_{(a)(\pm)}(\omega) = \lim_{\beta\to 0^\mp} p^t_{(a)}(\omega + i\beta), \quad \boldsymbol{p}^t_{(k)(\pm)}(\omega) = \lim_{\beta\to 0^\mp} \boldsymbol{p}^t_{(k)}(\omega + i\beta). \tag{5.31}$$

Note that the quantity $\boldsymbol{p}^t_{(k)}$ is indicated as $\boldsymbol{P}^t$ in [5].

The use of Plemelj formulae [19] yields

$$\frac{1}{2}\frac{I^t_{(a)+}(\omega, {}_r\vartheta^t)}{H_{(-)}(\omega)} = p^t_{(a)(-)}(\omega) - p^t_{(a)(+)}(\omega), \tag{5.32}$$

$$\frac{1}{2}\frac{\boldsymbol{I}^t_{(k)+}(\omega, {}_r\boldsymbol{g}^t)}{k_{(-)}(\omega)} = \boldsymbol{p}^t_{(k)(-)}(\omega) - \boldsymbol{p}^t_{(k)(+)}(\omega), \tag{5.33}$$

where $p^t_{(a)(\pm)}(\omega)$ and $\boldsymbol{p}^t_{(k)(\pm)}(\omega)$ can be extended on the complex plane to give two functions $p^t_{(a)(\pm)}(z)$ and $\boldsymbol{p}^t_{(k)(\pm)}(z)$ with zeros and singularities for $z \in \mathbb{C}^\pm$ and hence analytic not only in $\mathbb{C}^{(\mp)}$, but also in $\mathbb{R}$ by virtue of the hypothesis on the Fourier transforms.

Thus, the minimum free energy (4.11) is given by

$$\psi_m(t) = \frac{1}{2}a_0\vartheta^2(t) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\, p^t_{(a)(+)}(\omega)\,|^2\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\, \boldsymbol{p}^t_{(k)(+)}(\omega)\,|^2\, d\omega. \tag{5.34}$$

Another expression for the minimum free energy.

A second different but equivalent form of $\psi_m(t)$ has been derived by writing (5.34) in terms of ${}_r\vartheta^t_+(\omega)$ and ${}_r\boldsymbol{g}^t_+(\omega)$, related to $p^t_{(a)(+)}(\omega)$ and $\boldsymbol{p}^t_{(k)(+)}(\omega)$. Thus, (5.29)-(5.30) can be written in the form

$$p^t_{(a)(+)}(\omega) = -\lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(+)}(\omega')\left[{}_r\vartheta^t_+(\omega')\right]^*}{\omega'-z}d\omega' + \lim_{z\to\omega^-}\frac{a'(0)}{2\pi i}\int_{-\infty}^{+\infty}\frac{\left[{}_r\vartheta^t_+(\omega')\right]^*/H_{(-)}(\omega')}{\omega'-z}d\omega', \tag{5.35}$$

$$\boldsymbol{p}^t_{(k)(+)}(\omega) = -\lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{k_{(+)}(\omega')\left[{}_r\boldsymbol{g}^t_+(\omega')\right]^*}{\omega'-z}d\omega', \tag{5.36}$$

whence, denoting by * the complex conjugate,

$$\left[p^t_{(a)(+)}(\omega)\right]^* = \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(-)}(\omega')\,{}_r\vartheta^t_+(\omega')}{\omega'-w}d\omega' - \lim_{w\to\omega^+}\frac{a'(0)}{2\pi i}\int_{-\infty}^{+\infty}\frac{{}_r\vartheta^t_+(\omega')/H_{(+)}(\omega')}{\omega'-w}d\omega', \tag{5.37}$$

$$\left[\boldsymbol{p}^t_{(k)(+)}(\omega)\right]^* = \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{k_{(-)}(\omega')\,{}_r\boldsymbol{g}^t_+(\omega')}{\omega'-w}d\omega' \tag{5.38}$$

and, in particular, (5.37) becomes

$$\left[p^t_{(a)(+)}(\omega)\right]^* = \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(-)}(\omega')\,{}_r\vartheta^t_+(\omega')}{\omega'-w}d\omega', \tag{5.39}$$

where the second integral of (5.37) vanishes, since its integrand functions ${}_r\vartheta^t_+(\omega)$ and $H_{(+)}(\omega)$, without singularities and zeros in $\mathbb{C}^{(-)}$, are analytic in $\mathbb{C}^-$.

The application of the Plemelj formulae gives

$$H_{(-)}(\omega)\,{}_r\vartheta^t_+(\omega) = q^t_{(a)(-)}(\omega) - q^t_{(a)(+)}(\omega), \tag{5.40}$$

where

$$q^t_{(a)(\pm)}(\omega) = \lim_{z\to\omega^\mp}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{H_{(-)}(\omega')\,{}_r\vartheta^t_+(\omega')}{\omega'-z}d\omega'; \tag{5.41}$$

moreover,

$$k_{(-)}(\omega)\,{}_r\boldsymbol{g}^t_+(\omega) = \boldsymbol{q}^t_{(k)(-)}(\omega) - \boldsymbol{q}^t_{(k)(+)}(\omega) \tag{5.42}$$

wherein

$$\boldsymbol{q}^t_{(k)(\pm)}(\omega) = \lim_{z\to\omega^\mp} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{k_{(-)}(\omega')\,{}_r\boldsymbol{g}^t_+(\omega')}{\omega' - z} d\omega'. \tag{5.43}$$

Note that the quantity $\boldsymbol{Q}^t_{(\pm)}(\omega)$ of [5] is replaced by $\boldsymbol{q}^t_{(k)(\pm)}(\omega)$, where the subscript $_{(k)}$ is added to distinguish this quantity from the heat flux $\boldsymbol{q}$.

Formulae (5.38), (5.39), (5.41) together with the last relation imply

$$\left[p^t_{(a)(+)}(\omega)\right]^* = q^t_{(a)(-)}(\omega), \quad \left[\boldsymbol{p}^t_{(k)(+)}(\omega)\right]^* = \boldsymbol{q}^t_{(k)(-)}(\omega), \tag{5.44}$$

and hence the following further equivalent form of the free energy

$$\psi_m(t) = \frac{1}{2} a_0 \vartheta^2(t) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\, q^t_{(a)(-)}(\omega)\,|^2\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\, \boldsymbol{q}^t_{(k)(-)}(\omega)\,|^2\, d\omega \tag{5.45}$$

is obtained in terms of ${}_r\vartheta^t$ and ${}_r\boldsymbol{g}^t$, besides to $\vartheta(t)$.

6. – The minimum free energy for the discrete spectrum model and its relation with classical free energies.

This Section is concerned about generalizations for the minimum free energy in linear heat conduction with memory in the special case in which the relaxation function is expressed as a sum of exponentials in order to show that some classical free energies can be derived from the results obtained for this discrete spectrum model. This model represents the analogous of that one, previously studied in [15] by *Golden* in viscoelasticity. At first, we recall the fundamental formulae derived for this model, in rigid heat thermodynamics, in [5] and [6], where the minimum free energy is derived by assuming different states. Since here the states of $\mathcal{B}$ are defined in (3.1) in terms of $\vartheta(t)$, ${}_r\vartheta^t$ and ${}_r\boldsymbol{g}^t$, we refer to [6] only as far as the contribute to the free energy due to e is concerned. Indeed, while in [6] the states are expressed by means of $\bar{\boldsymbol{g}}^t$, we must consider the results comprised in [5], where $\boldsymbol{g}^t$ appears in the states, for the contribute related to $\boldsymbol{q}$.

Let the relaxation functions a and k be expressed by

$$a(t) = \begin{cases} a_\infty - \sum_{i=1}^{n} h_i e^{-a_i t} & \forall t \geq 0, \\ 0 & \forall t < 0, \end{cases} \qquad k(t) = \begin{cases} \sum_{i=1}^{n} g_i e^{-k_i t} & \forall t \geq 0, \\ 0 & \forall t < 0, \end{cases} \tag{6.1}$$

where n is a positive integer, the inverse decay times a_i and k_i $(i = 1, 2, \ldots, n)$ as well as the coefficients h_i and g_i $(i = 1, 2, \ldots, n)$ are positive; moreover, a_i and k_i

are ordered so that $a_j < a_{j+1}$ and $k_j < k_{j+1}$ $(j = 1, 2, \ldots, n-1)$. The form now assumed assures that the following conditions $a_\infty - a(0) = \sum_{i=1}^{n} h_i > 0$ and $k(0) = \sum_{i=1}^{n} g_i > 0$, assumed in $(2.14)_1$ and $(2.13)_2$, are satisfied.

The Fourier transforms of $a'(t)$ and $k(t)$, by virtue of $(2.8)_3$, give

$$a'_s(\omega) = \omega \sum_{i=1}^{n} \frac{a_i h_i}{\omega^2 + a_i^2}, \quad k_c(\omega) = \sum_{i=1}^{n} \frac{k_i g_i}{\omega^2 + k_i^2}. \tag{6.2}$$

Let us consider the kernel a. Formulae $(6.2)_1$, (5.26) and (5.27) imply

$$H(\omega) = \omega^2 \sum_{i=1}^{n} \frac{a_i h_i}{\omega^2 + a_i^2} \geq 0 \quad \forall \omega \in \mathbb{R}, \quad H_\infty = \sum_{i=1}^{n} a_i h_i = a'(0) > 0. \tag{6.3}$$

Then, we rewrite $H(\omega)$ as

$$H(\omega) = H_\infty \prod_{i=1}^{n} \left\{ \frac{\omega^2 + \gamma_i^2}{\omega^2 + a_i^2} \right\}, \tag{6.4}$$

where $\gamma_1^2 = 0$ and γ_j^2 $(j = 2, 3, \ldots, n)$ are simple zeros of $f(z) = H(\omega)$ with $z = -\omega^2$, such that

$$a_1^2 < \gamma_2^2 < a_2^2 < \ldots < a_{n-1}^2 < \gamma_n^2 < a_n^2. \tag{6.5}$$

Hence, the factorization $(5.28)_1$, in particular, yields

$$H_{(-)}(\omega) = h_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + ia_i} \right\} = h_\infty \left(1 + i \sum_{i=1}^{n} \frac{R_i}{\omega + ia_i} \right), \ h_\infty = H_\infty^{1/2}, \tag{6.6}$$

where

$$R_i = (\gamma_i - a_i) \prod_{j=1, j \neq i}^{n} \left\{ \frac{\gamma_j - a_i}{a_j - a_i} \right\}. \tag{6.7}$$

The expression (5.41) of $q^t_{(a)(-)}(\omega)$ becomes, on use of (6.6), choosing to close the integration path in $\mathbb{C}^{(-)}$, becomes

$$q^t_{(a)(-)}(\omega) = ih_\infty \sum_{i=1}^{n} \frac{R_i}{\omega + ia_i} {}_r\vartheta^t_+(-ia_i), \tag{6.8}$$

where, by virtue of $(2.6)_2$,

$${}_r\vartheta^t_+(-ia_i) = \int_0^{+\infty} {}_r\vartheta^t(s) e^{-a_i s} ds = \left[{}_r\vartheta^t_+(-ia_i) \right]^*. \tag{6.9}$$

Thus, the contribute due to e in (5.45), on evaluation of $\mid q^t_{(a)(-)}(\omega) \mid^2$, closing the integration path in $\mathbb{C}^{(+)}$, is given by

$$\psi^{(e)}_m(t)=\frac{1}{2}a_0\vartheta^2(t)+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2H_\infty\sum_{i,j=1}^{n}\frac{R_iR_j}{a_i+a_j}e^{-(a_is_1+a_js_2)}\,{}_r\vartheta^t(s_1)\,{}_r\vartheta^t(s_2)ds_1ds_2. \tag{6.10}$$

Let us consider the kernel k. Firstly, taking into account $(6.2)_2$, we introduce

$$K(\omega)=(1+\omega^2)k_c(\omega), \qquad K_\infty=\lim_{\omega\to\infty}K(\omega)=\sum_{i=1}^{n}k_ig_i>0, \tag{6.11}$$

which can be written in the form

$$K(\omega)=K_\infty\prod_{i=1}^{n}\left\{\frac{\omega^2+\delta_i^2}{\omega^2+k_i^2}\right\}, \tag{6.12}$$

where $\delta_1^2=1$ and δ_j^2 $(j=2,3,\dots,n)$ are the simple zeros of $g(z)=K(\omega)$ with $z=-\omega^2$, ordered as the analogous quantities for a in (6.5). Its factorization gives

$$K_{(-)}(\omega)=k_\infty\prod_{i=1}^{n}\left\{\frac{\omega+i\delta_i}{\omega+ik_i}\right\}, \quad k_\infty=(K_\infty)^{\frac{1}{2}}, \tag{6.13}$$

whence, by virtue of $(6.11)_1$,

$$k_{(-)}(\omega)=ik_\infty\frac{\prod_{j=2}^{n}(\omega+i\delta_j)}{\prod_{i=1}^{n}(\omega+ik_i)}=ik_\infty\sum_{r=1}^{n}\frac{U_r}{\omega+ik_r}. \tag{6.14}$$

where

$$U_r=\frac{\prod_{j=2}^{n}(\delta_j-k_r)}{\prod_{i=1,i\neq r}^{n}(k_i-k_r)} \quad (r=1,2,\dots,n). \tag{6.15}$$

Taking into account (6.14), $\boldsymbol{q}^t_{(k)(-)}(\omega)$, expressed by (5.43), can be evaluated on closing the integration contour in $\mathbb{C}^{(-)}$; it follows

$$\boldsymbol{q}^t_{(k)(-)}(\omega)=ik_\infty\sum_{r=1}^{n}\frac{U_r}{\omega+ik_r}\,{}_r\boldsymbol{g}^t_+(-ik_r), \tag{6.16}$$

where

$${}_r\boldsymbol{g}^t_+(-ik_r)=\int_0^{+\infty}{}_r\boldsymbol{g}^t(s)e^{-k_rs}ds=\left[{}_r\boldsymbol{g}^t_+(-ik_r)\right]^*. \tag{6.17}$$

Thus, the contribute due to $\boldsymbol{q}$ in (5.45), on evaluation of $|\boldsymbol{q}^t_{(k)(-)}(\omega)|^2$ and, then, closing the integration contour in $\mathbb{C}^{(+)}$, is given by

$$\psi_m^{(\boldsymbol{q})}(t) = \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} e^{-(k_i s_1 + k_j s_2)}\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)\, ds_1 ds_2. \tag{6.18}$$

Therefore, we have

$$\begin{aligned}\psi_m(t) &= \psi_m^{(e)}(t) + \psi_m^{(\boldsymbol{q})}(t) = \frac{1}{2}a_0 \vartheta^2(t)\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{a_i + a_j} e^{-(a_i s_1 + a_j s_2)}\, {}_r\vartheta^t(s_1)\, {}_r\vartheta^t(s_2)\, ds_1 ds_2\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} e^{-(k_i s_1 + k_j s_2)}\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)\, ds_1 ds_2.\end{aligned} \tag{6.19}$$

Such an expression of the minimum free energy of the discrete model can be compared with other free energies.

To show this, it turns out to be convenient to rewrite (6.19) as follows

$$\begin{aligned}\psi_m(t) &= \frac{1}{2}a_0 \vartheta^2(t)\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \mathcal{N}_{(a)}(s_1, s_2)\, {}_r\vartheta^t(s_1)\, {}_r\vartheta^t(s_2)\, ds_1 ds_2\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \mathcal{N}_{(k)}\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)\, ds_1 ds_2,\end{aligned} \tag{6.20}$$

where

$$\mathcal{N}_{(a)}(s_1, s_2) \equiv -\, a_{12}(s_1, s_2) = 2H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{a_i + a_j} e^{-(a_i s_1 + a_j s_2)}, \tag{6.21}$$

$$\mathcal{N}_{(k)}(s_1, s_2) \equiv k(s_1, s_2) \equiv 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} e^{-(k_i s_1 + k_j s_2)}. \tag{6.22}$$

Consequently, the form (6.20), together with the kernels $(6.21)_2$-$(6.22)_2$, is cast in *the general form* of the free energy (5.14).

Firstly, the part of $\psi_m(t)$ in (6.19), or equivalently in (6.20), which expresses the contribute due to e, is examined.

A double integration by parts on $(s_2, +\infty)$ and, then, on $(s_1, +\infty)$, solves the differential equation $(6.21)_2$ and gives the solution

$$a(s_1, s_2) = -2H_\infty \sum_{i,j=1}^{n} \frac{1}{a_i + a_j}\frac{R_i}{a_i}\frac{R_j}{a_j} e^{-(a_i s_1 + a_j s_2)}. \tag{6.23}$$

Since this solution is subject to (5.16), some interesting identities, which can be used to verify both the positiveness of $\psi_m(t)$ as well as of its corresponding internal dissipation $D_m(t)$ can be derived. These results are analogous to those obtained by Golden in [15] for viscoelastic solids; however, some of them are only similar to the ones derived in [15]; indeed, the behaviour of a with respect to the one of the kernel of viscoelastic solids, is different.

Thus, note that the solution (6.23), using $(6.1)_1$, needs to satisfy

$$\begin{aligned} a(0,s_2)|_{s_2=s} \equiv a(s_1,0)|_{s_1=s} &= -2H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{(a_i+a_j)a_i a_j} e^{-a_i s} \\ &\equiv a(s) - a_\infty = \sum_{i=1}^{n} h_i e^{-a_i s}, \end{aligned} \tag{6.24}$$

$$a(0,0) = -2H_\infty \sum_{i,j=1}^{n} \frac{1}{a_i+a_j} \frac{R_i}{a_i} \frac{R_j}{a_j} \equiv a_0 - a_\infty = -\sum_{i=1}^{n} h_i. \tag{6.25}$$

We can easily obtain a first relation by evaluating the two expressions for $H_{(-)}(\omega)$, given in (6.6) where $\gamma_1 = 0$, for $\omega = 0$

$$\sum_{i=1}^{n} \frac{R_i}{a_i} = -1. \tag{6.26}$$

Then, by equating the two expression in (6.24), we obtain a relation

$$\sum_{i=1}^{n} \left[2H_\infty \frac{R_i}{a_i} \sum_{j=1}^{n} \frac{R_j}{(a_i+a_j)a_j} - h_i \right] e^{-a_i s} = 0, \tag{6.27}$$

which is identically satisfied by means of

$$\sum_{j=1}^{n} \frac{R_j}{(a_i+a_j)a_j} = \frac{a_i h_i}{2H_\infty R_i}. \tag{6.28}$$

Let us now consider the second expression for $H_{(-)}(\omega)$ in (6.6) and evaluate its complex conjugate; thus, $H(\omega)$ can be written

$$\begin{aligned} H(\omega) = &H_{(-)}(\omega)[H_{(-)}(\omega)]^* = H_\infty \left[1 - \sum_{i=1}^{n} R_i \frac{2a_i}{\omega^2 + a_i^2} \right. \\ &\left. + \sum_{i=1}^{n} \frac{R_i}{\omega + i a_i} \sum_{j=1}^{n} \frac{R_j}{\omega - i a_j} \right] \equiv \omega^2 \sum_{i=1}^{n} \frac{a_i h_i}{\omega^2 + a_i^2}, \end{aligned} \tag{6.29}$$

the last equality follows by virtue of $(6.3)_1$. This identity, after factorization of the term $(\omega - i a_p)^{-1}$, where $\omega = i a_p$ is a fixed pole of $H_{(+)}(\omega)$, gives the following condition, necessarily satisfied for any ω,

$$\frac{1}{\omega - ia_p}\left[R_p\left(\frac{2a_p}{\omega + ia_p} - \sum_{i=1}^{n}\frac{R_i}{\omega + ia_i}\right) + \frac{\omega^2}{H_\infty}\frac{a_p h_p}{\omega + ia_p}\right]$$

$$= 1 - \sum_{j=1, j\neq p}^{n}\frac{2R_j a_j}{(\omega - ia_j)(\omega + ia_j)} + \sum_{i=1}^{n}\frac{R_i}{\omega + ia_i}\sum_{j=1,j\neq p}^{n}\frac{R_j}{\omega - ia_j} \tag{6.30}$$

$$- \frac{\omega^2}{H_\infty}\sum_{j=1,j\neq p}^{n}\frac{a_j h_j}{(\omega - ia_j)(\omega + ia_j)}\ .$$

Comparison between the two members of this equality, near the pole $\omega = ia_p$, that is, evaluating the limit as $\omega \to ia_p$, the right-hand side admits a finite limit and, consequently, also the limit of the left-hand side is finite. Thus, the quantity in the square brackets in the left-hand side must vanish for $\omega = ia_p$. The same way of reasoning holds true for any a_p $(p = 1, 2, \ldots, n)$; thus, it follows

$$\sum_{j=1}^{n}\frac{R_j}{a_i + a_j} = 1 - \frac{a_i^2 h_i}{2H_\infty R_i} \qquad (i = 1, 2, \ldots, n). \tag{6.31}$$

Finally, equating (6.3) with (6.4) and multiplying this identity by $\omega^2 + a_i^2$ we obtain a relation, which, in the limit $\omega \to ia_i$, yields

$$-\frac{a_i h_i^3}{H_\infty} = (\gamma_i^2 - a_i^2)\prod_{j=1,j\neq i}^{n}\left\{\frac{\gamma_j^2 - a_i^2}{a_j^2 - a_i^2}\right\} = R_i B_i < 0, \tag{6.32}$$

where R_i is given by (6.7) and, by virtue of (6.5),

$$B_i = (\gamma_i + a_i)\prod_{j=1,j\neq i}^{n}\left\{\frac{\gamma_j + a_i}{a_j + a_i}\right\} > 0 \quad (i = 1, 2, \ldots, n). \tag{6.33}$$

In particular, as already observed, it is easy to see that (6.23), differentiating and using (6.28), satisfies (5.16).

Thus, on substitution of the expression (6.32) of R_i in terms of B_i, into (6.23), we obtain

$$a(s_1, s_2) = -\frac{2}{H_\infty}\sum_{i,j=1}^{n}\frac{a_i^2 a_j^2 h_i h_j}{(a_i + a_j)B_i B_j}e^{-(a_i s_1 + a_j s_2)} < 0. \tag{6.34}$$

The latter allows us to state the equivalence of the part of the free energy due to e, that is the first two terms of (5.14), here considered with that one in the *Breur-Onat formula* and hence to *Golden's alternative form*.

Therefore, such a contribute can be rewritten as

$$\psi_m^{(e)}(t) = \frac{1}{2} a_0 \vartheta^2(t) + \frac{1}{H_\infty} \int_0^{+\infty} \int_0^{+\infty} \sum_{i,j=1}^{n} \frac{a_i^3 a_j^3 h_i h_j}{(a_i + a_j) B_i B_j} e^{-(a_i s_1 + a_j s_2)} \, {}_r\vartheta^t(s_1) \, {}_r\vartheta^t(s_2) ds_1 ds_2. \tag{6.35}$$

Moreover, on use of (6.23), we can evaluate (5.19),

$$K^{(a)}(s_1, s_2) = 2H_\infty \sum_{i,j=1}^{n} \frac{R_i}{a_i} \frac{R_j}{a_j} e^{-(a_i s_1 + a_j s_2)}, \tag{6.36}$$

and hence, differentiation and, then, again, use of (6.32), give

$$K_{12}^{(a)}(s_1, s_2) = \frac{2}{H_\infty} \sum_{i,j=1}^{n} \frac{a_i^3 a_j^3}{B_i B_j} e^{-(a_i s_1 + a_j s_2)} > 0, \tag{6.37}$$

which appears in (5.18) for $D_m^{(e)}(t)$.

Thus, the corresponding internal dissipation can be derived

$$D_m^{(e)}(t) = \frac{1}{H_\infty} \int_0^{+\infty} \int_0^{+\infty} \sum_{i,j=1}^{n} \frac{a_i^3 a_j^3 h_i h_j}{B_i B_j} e^{-(a_i s_1 + a_j s_2)} \, {}_r\vartheta^t(s_1) \, {}_r\vartheta^t(s_2) ds_1 ds_2, \tag{6.38}$$

where the coefficients are all positive, by virtue of (6.33). Moreover, this expression can be written in a form, which is clearly non-negative, that is

$$\begin{aligned} D_m^{(e)}(t) &= \frac{1}{H_\infty} \left[\sum_{i=1}^{n} \frac{a_i^3 h_i}{B_i} \int_0^{+\infty} e^{-a_i s_1} \, {}_r\vartheta^t(s_1) ds_1 \right] \left[\sum_{j=1}^{n} \frac{a_j^3 h_j}{B_j} \int_0^{+\infty} e^{-a_j s_2} \, {}_r\vartheta^t(s_2) ds_2 \right] \\ &= \frac{1}{H_\infty} \left[\sum_{i=1}^{n} \frac{a_i^3 h_i}{B_i} \int_0^{+\infty} e^{-a_i s} \, {}_r\vartheta^t(s) ds \right]^2 > 0. \end{aligned} \tag{6.39}$$

Let us, now, consider the part of $\psi_m(t)$ in (6.19), or in (6.20), due to $\boldsymbol{q}$.

The assumption $k(s_1, s_2)$ is given by the expression $(6.22)_2$, that is

$$k(s_1, s_2) = 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} e^{-(k_i s_1 + k_j s_2)}, \tag{6.40}$$

subject to (5.17), implies

$$k(0, s_2)|_{s_2 = s} \equiv k(s_1, 0)|_{s_1 = s} = 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} e^{-k_j s} \equiv k(s) = \sum_{i=1}^{n} g_i e^{-k_i s}, \tag{6.41}$$

$$k(0,0) = 2K_\infty \sum_{i,j=1}^{n} \frac{U_i U_j}{k_i + k_j} \equiv k_0 = \sum_{i=1}^{n} g_i. \tag{6.42}$$

Now, some identities analogous to the ones related to e are needed.

As before, evaluation of the two expressions (6.14) for $k_{(-)}(\omega)$ in $\omega = 0$ gives

$$\sum_{i=1}^{n} \frac{U_i}{k_i} = \frac{\prod_{j=2}^{n} (\delta_i)}{\prod_{i=1}^{n} (k_i)} > 0, \tag{6.43}$$

by virtue of the order in (6.5), which also δ_j ($j = 2, 3, \ldots, n$) have. Moreover, from (6.41) it follows that

$$\sum_{i=1}^{n} \left[2K_\infty U_i \sum_{j=1}^{n} \frac{U_j}{k_i + k_j} - g_i \right] e^{-k_i s} = 0, \tag{6.44}$$

which is identically satisfied with

$$\sum_{j=1}^{n} \frac{U_j}{k_i + k_j} = \frac{g_i}{2K_\infty U_i}. \tag{6.45}$$

This result can be derived via the same way of reasoning already followed to prove also the analogous relation (6.31) valid in the case of a. Accordingly, it can be derived on evaluation of $k(\omega)$ together with the use of $(6.14)_2$ and its conjugate combined with the comparison of the result with $(6.2)_2$; then, as before, in the case of a to derive (6.31), by isolating in this identity the quantities with the factor $(\omega - ik_p)^{-1}$, where $\omega = ik_p$ is a fixed pole of $K_{(+)}(\omega)$, we obtain a relation similar to (6.30), whence the limit as $\omega \to ik_p$ allows us to derive (6.45).

Furthermore, equating $(6.2)_2$ with the expression $k_c(\omega)$, deduced by the product $k_{(-)}(\omega)[k_{(-)}(\omega)]^*$, on use of $(6.14)_1$, and multiplying this identity by $\omega^2 + k_r^2$, it follows

$$\sum_{i=1, i\neq r}^{n} \frac{k_i g_i \left(\omega^2 + k_r^2\right)}{\omega^2 + k_i^2} + k_r g_r = K_\infty \frac{\prod_{j=2}^{n} \left(\omega^2 + \delta_j^2\right)}{\prod_{i=1, i\neq r}^{n} \left(\omega^2 + k_i^2\right)}, \tag{6.46}$$

whence, as $\omega \to ik_r$,

$$\frac{k_r g_r}{K_\infty} = \frac{\prod_{j=2}^{n} \left(\delta_j^2 - k_r^2\right)}{\prod_{i=1, i\neq r}^{n} \left(k_i^2 - k_r^2\right)} = U_r C_r, \tag{6.47}$$

where U_r is given by (6.15) and

$$C_r = \frac{\prod_{j=2}^{n}(\delta_j + k_r)}{\prod_{i=1, i\neq r}^{n}(k_i + k_r)} > 0 \quad (r = 1, 2, \ldots, n). \tag{6.48}$$

Now, U_r given by the expression (6.47), can be written in terms of C_r so that, when substituted into (6.40), gives

$$k(s_1, s_2) = \frac{2}{K_\infty}\sum_{i,j=1}^{n}\frac{k_i k_j g_i g_j}{(k_i + k_j)C_i C_j}e^{-(k_i s_1 + k_j s_2)} > 0; \tag{6.49}$$

thus, the equivalence of *the Breur-Onat formula* is also stated as far as the part of the free energy due to $\boldsymbol{q}$ is concerned and, hence, also the one of *Golden's alternative form*.

In fact, we have

$$\psi_m^{(\boldsymbol{q})}(t) = \frac{1}{K_\infty}\int_0^{+\infty}\int_0^{+\infty}\sum_{i,j=1}^{n}\frac{k_i k_j g_i g_j}{(k_i + k_j)C_i C_j}e^{-(k_i s_1 + k_j s_2)}\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)ds_1 ds_2. \tag{6.50}$$

The corresponding internal dissipation can be derived by its expression given (5.18). To do this, using (6.40) and (6.47), we firstly derive the expression for $K^{(k)}$, defined by (5.21),

$$\begin{aligned} K^{(k)}(s_1, s_2) &= -\,2K_\infty \sum_{i,j=1}^{n} U_i U_j e^{-(k_i s_1 + k_j s_2)} \\ &= -\frac{2}{K_\infty}\sum_{i,j=1}^{n}\frac{k_i k_j g_i g_j}{C_i C_j}e^{-(k_i s_1 + k_j s_2)} < 0; \end{aligned} \tag{6.51}$$

hence,

$$\begin{aligned} D_m^{(\boldsymbol{q})}(t) &= \frac{1}{K_\infty}\int_0^{+\infty}\int_0^{+\infty}\sum_{i,j=1}^{n}\frac{k_i k_j g_i g_j}{C_i C_j}e^{-(k_i s_1 + k_j s_2)}\, {}_r\boldsymbol{g}^t(s_1)\cdot {}_r\boldsymbol{g}^t(s_2)ds_1 ds_2 \\ &= \frac{1}{K_\infty}\left[\sum_{i=1}^{n}\frac{k_i g_i}{C_i}\int_0^{+\infty} e^{-k_i s_1}\, {}_r\boldsymbol{g}^t(s_1)ds_1\right]\cdot\left[\sum_{j=1}^{n}\frac{k_j g_j}{C_j}\int_0^{+\infty} e^{-k_j s_2}\, {}_r\boldsymbol{g}^t(s_2)ds_2\right] \\ &= \frac{1}{K_\infty}\left[\sum_{i=1}^{n}\frac{k_i g_i}{C_i}\int_0^{+\infty} e^{-k_i s}\, {}_r\boldsymbol{g}^t(s)ds\right]^2 > 0. \end{aligned} \tag{6.52}$$

Obviously, the sum of (6.35) and (6.50) gives the whole free energy, as well as the sum of (6.39) and (6.52) gives the corresponding internal dissipation.

An interesting case is the one when $n = 1$ in (6.1). In this case, for a, since $\gamma_1 = 0$, (6.6) becomes

$$H_{(-)}(\omega) = h_\infty \frac{\omega}{\omega + ia_1} = h_\infty \left(1 + i \frac{R_1}{\omega + ia_1}\right), \; R_1 = -\,a_1, \; H_\infty = a_1 h_1, \tag{6.53}$$

while, for k, (6.14) yields

$$k_{(-)}(\omega) = ik_\infty \frac{1}{\omega + ik_1}, \quad U_1 = 1, \quad K_\infty = k_1 g_1. \tag{6.54}$$

Thus, the expression (6.10) of $\psi_m^{(e)}(t)$ reduces to

$$\psi_m^{(e)}(t) = \frac{1}{2} a_0 \vartheta^2(t) + \frac{1}{2} a_1^2 h_1 \left[\int_0^{+\infty} e^{-a_1 s} \, {}_r\vartheta^t(s) ds \right]^2, \tag{6.55}$$

which is in agreement with *the Day functional* $\psi_{Day}^{(e)}(t)$ in (5.5), by virtue of (6.53) and of the relation $a_\infty - a_0 = h_1$.

Analogously, the part of $\psi_m^{(\boldsymbol{q})}(t)$, given in (6.18), on use of (6.54), becomes

$$\psi_m^{(\boldsymbol{q})}(t) = \frac{1}{2} g_1 \left[\int_0^{+\infty} e^{-k_1 s} \, {}_r\boldsymbol{g}^t(s) ds \right]^2; \tag{6.56}$$

it agrees with *the Day functional* $\psi_{Day}^{(\boldsymbol{q})}(t)$ in (5.5), since $k_0 \equiv k_1$ and in $(5.7)_2$ $\gamma = k_1$.

Finally, evaluating the sum of (6.39) and (6.52) with $n = 1$, the internal dissipation reads

$$D_m(t) = a_1^3 h_1 \left[\int_0^{+\infty} e^{-a_1 s} \, {}_r\vartheta^t(s) ds \right]^2 + k_1 g_1 \left[\int_0^{+\infty} e^{-k_1 s} \, {}_r\boldsymbol{g}^t(s) ds \right]^2, \tag{6.57}$$

since from (6.32) it follows that $B_1 = a_1$ and, by virtue of $(6.53)_{2,3}$ and $(6.54)_{2,3}$, (6.47) yields $C_1 = 1$.

Acknowledgements. The support of G.N.F.M.-I.N.D.A.M. and of M.I.U.R. are gratefully acknowledged, S.C. thanks also SAPIENZA Università di Roma.

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Received December 31, 2009 and in revised form May 10, 2010