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L^2- Singular Dichotomy for Orbital Measures on Complex Groups

S. K. Gupta - K. E. Hare (*)

Abstract. – *It is known that all continuous orbital measures, μ, on a compact, connected, classical simple Lie group G or its Lie algebra satisfy a dichotomy: either $\mu^k \in L^2$ or μ^k is purely singular to Haar measure. In this note we prove that the same dichotomy holds for the dual situation, continuous orbital measures on the complex group $G^{\mathbb{C}}$. We also determine the sharp exponent k such that any k-fold convolution product of continuous G-bi-invariant measures on $G^{\mathbb{C}}$ is absolute continuous with respect to Haar measure.*

1. – Introduction.

Let G be a compact, classical, connected, simple Lie group and $\mathfrak{g}$ its Lie algebra. The orbital measure, μ_H, is the G-invariant, probability measure supported on the adjoint orbit generated by $H \in \mathfrak{g}$. It is a continuous measure whenever $H \neq 0$ and is always singular to Haar measure on $\mathfrak{g}$. In [5] it was shown that all continuous orbital measures satisfy the following L^2- *singular dichotomy*: If $k \in \mathbb{N}$, then the k-fold convolution product, μ_H^k, is either singular to Haar measure or belongs to $L^2(\mathfrak{g}) \cap L^1(\mathfrak{g})$, meaning μ_H^k is absolutely continuous to Haar measure and has an $L^1 \cap L^2$ density function. Moreover, for every $H \neq 0$ there is always an exponent k with $\mu_H^k \in L^2$ and the minimal choice of $k = k(H)$ can be specified. A similar result was also established in [4] for the continuous orbital measures on the group G, i.e., the G-invariant, probability measures supported on non-trivial conjugacy classes in G. In both the Lie algebra and Lie group case, the (known) overall minimal exponent, $k_0 = k_0(\mathfrak{g})$ (respectively, $k_0(G)$) is at most $rank\, G + 1$ (resp., $2\, rank\, G$). A consequence of the L^2 result is that $\mu_1 * \cdots * \mu_{k_0} \in L^1$ whenever $\mu_1, \ldots, \mu_{k_0}$ are continuous, G-invariant measures on G.

These results improved upon earlier work of Ragozin [7] who showed, in

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particular, that $\mu_1 * \cdots * \mu_{\dim G} \in L^1(G)$ whenever $\mu_1, \ldots, \mu_{\dim G}$ are continuous, G-invariant measures on G. More generally, Ragozin in [8] studied zonal measures on the symmetric space G/K, where G is a connected Lie group, K is a connected compact subgroup and G/K is isotropy irreducible. He showed that if $\mu_1, \ldots, \mu_{\dim G/K}$ are any continuous K-invariant measures, (these are called zonal measures) then $\mu_1 * \cdots * \mu_{\dim G/K} \in L^1(G/K)$. A compact, connected, simple Lie group G is a special instance of this setting as it can be viewed as the compact symmetric space $(G \times G)/D$ where $D = \{(x, x) : x \in G\}$. Under this identification the D-invariant measures on the symmetric space correspond to the G-invariant measures on G.

In this article, we consider the non-compact symmetric space, $G^{\mathbb{C}}/G$, which is dual to the compact symmetric space $(G \times G)/D$. The zonal measures on $G^{\mathbb{C}}/G$ are the G-invariant measures and so can also be considered as G-bi-invariant measures on the complex group $G^{\mathbb{C}}$. We show that when G is one of the classical Lie groups the L^2- singular dichotomy continues to hold for the analogue of the orbital measures on $G^{\mathbb{C}}$. Moreover, $\mu_1 * \cdots * \mu_{k_0(\mathfrak{g})} \in L^1$ for all G-bi-invariant measures on $G^{\mathbb{C}}$ and the number $k_0(\mathfrak{g})$ is sharp with this property. It is an improvement of Ragozin's result for compact symmetric spaces since $\dim G^{\mathbb{C}}/G = \dim G > k_0(\mathfrak{g})$.

2. – Basic Facts and Notation.

Let G be a compact, connected, simple Lie group and $\mathfrak{g}$ its Lie algebra. Following Ragozin [8] we will consider elements of $\mathfrak{g}$ as right invariant vector fields on G. The Lie group acts on the algebra by the adjoint action $Ad(\cdot)$. We will also write $Ad(\cdot)$ for the conjugation action of G on G. The orbital measures on G or $\mathfrak{g}$ are the G-invariant, probability measures, μ_a, for $a \in G$ or $\mathfrak{g}$ respectively, defined by:

$$\int_{\mathcal{G}} f d\mu_a = \int_G f(Ad(g)a)dm(g) \text{ for } f \in C_0(\mathcal{G})$$

where $\mathcal{G} = G$ or $\mathfrak{g}$ and m denotes Haar measure on G. These measures are supported on the conjugacy class, $C_a \subseteq G$, or adjoint orbit, $O_a \subseteq \mathfrak{g}$, generated by the element a and are continuous (non-atomic) if a is not in the centre of G in the group case, and $a \neq 0$ in the algebra case. They are the extreme points of the convex set which is the intersection of the unit sphere with the cone of positive measures in the space of continuous, G-invariant measures.

Let $G^{\mathbb{C}}$ be the complexification of G; that is, $G^{\mathbb{C}}$ is the Lie group corresponding to the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ and thus has dimension over $\mathbb{R}$ equal to $\dim \mathfrak{g}^{\mathbb{C}}$. The compact group G can be identified with the compact symmetric space $(G \times G)/D$ where $D = \{(g, g) : g \in G\}$. The group $G^{\mathbb{C}}$ is non-compact and

$G^{\mathbb{C}}/G$ is the non-compact symmetric space dual to $(G \times G)/D$. A good reference for basic facts about analysis on symmetric spaces is [6].

In this paper we consider the zonal measures on the symmetric space $G^{\mathbb{C}}/G$ as the G-bi-invariant measures on the complex group $G^{\mathbb{C}}$. When $\mu(xG) = 0$ for all $x \in G^{\mathbb{C}}$ the measure μ is called continuous; these are the non-atomic measures on G^C/G. The extreme points of the the convex set which is the intersection of the unit sphere with the cone of positive measures in the space of continuous G-bi-invariant measures are the continuous measures on $G^{\mathbb{C}}$ given by $\mu_a = m_G * \delta_a * m_G$, where $a \in G^{\mathbb{C}}$, $a \neq$ identity element e. They satisfy the integration formula

$$\int_{G^{\mathbb{C}}} f d\mu_a = \int_G \int_G f(xay) dm_G(x) dm_G(y) \text{ for } f \in C_0(G^{\mathbb{C}})$$

and are supported on the double coset GaG. By analogy, we call these the orbital measures on $G^{\mathbb{C}}$.

Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{g}$ and $\mathcal{A} = exp(i\mathfrak{t}) \subseteq G^{\mathbb{C}}$. Every orbital measure is equal to $\mu_a = \mu_{\exp iH}$ for some $a \in \mathcal{A}$, $H \in \mathfrak{t}$ (see page 458, [1]).

Let Φ (resp. Φ^+) denote the (positive) roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the complexified torus $\mathfrak{t}^{\mathbb{C}}$. The root space decomposition of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \bigoplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

For every root $\alpha \in \Phi$, let $E_\alpha \in \mathfrak{g}^{\mathbb{C}}$ denote a root vector and choose $RE_\alpha, IE_\alpha \in \mathfrak{g}$ such that $E_\alpha = RE_\alpha + iIE_\alpha$. The theory of roots and root vectors is important in our arguments and the following easy fact, in particular, will be very useful.

LEMMA 2.1. – *Let $H \in \mathfrak{t}$ and $a = exp(iH)$. Then*

(i) $Ad(a)RE_\alpha = RE_\alpha \cosh \alpha(H) - iIE_\alpha \sinh \alpha(H)$,
(ii) $Ad(a)IE_\alpha = IE_\alpha \cosh \alpha(H) + iRE_\alpha \sinh \alpha(H)$.

PROOF. – (i) For $\alpha \in \Phi$, $[H, E_\alpha] = i\alpha(H)E_\alpha$, where $\alpha(H) \in \mathbb{R}$. By equating real and imaginary parts of the previous equality we get that $[H, RE_\alpha] = -\alpha(H)IE_\alpha$, and $[H, IE_\alpha] = \alpha(H)RE_\alpha$. An easy calculation gives

$$\begin{aligned} Ad(a)RE_\alpha &= Ad(exp(iH))RE_\alpha = exp(ad(iH))RE_\alpha \\ &= \sum_{n=0}^{\infty} \frac{(ad(iH))^n}{n!} RE_\alpha \\ &= \sum_{n \text{ even}} \frac{\alpha(H)^n}{n!} RE_\alpha - i \sum_{n \text{ odd}} \frac{\alpha(H)^n}{n!} IE_\alpha \\ &= \cosh \alpha(H) RE_\alpha - i \sinh \alpha(H) IE_\alpha. \end{aligned}$$

The proof of (ii) is similar. □

3. – L^2 Results.

We can obtain L^2 results for convolution powers of orbital measures on the complex group $G^{\mathbb{C}}$ by comparing with the L^2 results for the corresponding orbital measures on the Lie algebra $\mathfrak{g}$.

THEOREM 3.1. – *Suppose $H \in \mathfrak{t}$, μ_H is the orbital measure supported on the adjoint orbit in $\mathfrak{g}$ generated by H and $\mu_{\exp iH}$ is the orbital measure on $G^{\mathbb{C}}$ supported on the double coset $G(\exp iH)G$. There is a positive constant $B = B(H)$ such that*

$$\|\mu_H^k\|_{L^2(\mathfrak{g})} = B\|\mu_{\exp iH}^k\|_{L^2(G^{\mathbb{C}})}.$$

PROOF. – It is convenient to introduce some additional notation. Let W be the Weyl group and let ρ denote half the sum of the positive roots. For $\lambda \in \mathfrak{t}^*$ we will put

$$\Pi(\lambda) = \prod_{\alpha\in\Phi^+} (\alpha, \lambda)$$

and let $c(\lambda)$ denote the Harish-Chandra c function

$$c(\lambda) = \frac{\Pi(\rho)}{\Pi(i\lambda)}.$$

For a complex group, Plancherel's formula ([6, p. 454]) states that

$$\|f\|^2_{L^2(G^{\mathbb{C}})} = \int_{\mathfrak{t}^*} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda.$$

Put $a = \exp iH$. The Fourier transform of an orbital measure on $G^{\mathbb{C}}$ is a continuous function on $\mathfrak{t}^*$ and it equals ([6, p. 432]),

$$\widehat{\mu_a}(\lambda) = \phi_\lambda(a) = c(\lambda)\frac{\sum_{s\in W} \det s\; e^{is(\lambda,H)}}{\sum_{s\in W} \det s\; e^{s(\rho,H)}},$$

when the denominator is not zero. If the denominator is zero then numerator is also zero and $\phi_\lambda(a)$ is defined as a limit which exists.

Given $Z \in \mathfrak{t}$, put

$$A(\lambda)(Z) = \sum_{s\in W} \det s\; e^{is(\lambda,Z)}.$$

It is known (c.f. [10, 4.14.4]) that

$$\begin{aligned}\sum_{s\in W} \det s\; e^{s(\rho,H)} &= e^{-(\rho,H)} \prod_{\alpha\in\Phi^+} \left(e^{(\alpha,H)} - 1\right)\\ &= e^{-(\rho,H)} \prod_{\alpha\in\Phi^+} e^{\alpha(H)/2} \prod_{\alpha\in\Phi^+} 2\sinh \alpha(H).\end{aligned}$$

Thus $\phi_\lambda(a)$ is an indeterminate form if and only if there are positive roots α such that $\alpha(H)=0$. (We call such H singular.) In this case

$$\phi_\lambda(a) = c(\lambda)\lim_{Z\to H}\frac{A(\lambda)(Z)}{e^{-(\rho,Z)}\prod\limits_{\alpha\in\Phi^+}e^{\alpha(Z)/2}\prod\limits_{\alpha\in\Phi^+}2\sinh\,\alpha(Z)}$$

where the limit is taken over non-singular Z. Hence

$$\left|\phi_\lambda(a)\right| = |c(\lambda)|b_1\lim_{Z\to H}\frac{|A(\lambda)(Z)|}{\left|\prod\limits_{\alpha(H)=0}2\sinh\,\alpha(Z)\right|}$$

where

$$b_1 = b_1(H) = \left|\frac{e^{(\rho,H)}\prod\limits_{\alpha\in\Phi^+}e^{-\alpha(H)/2}}{\prod\limits_{\alpha(H)\neq 0}2\sinh\,\alpha(H)}\right| > 0.$$

Of course, $\lim\limits_{\theta\to 0}\sinh\,(\theta)/\theta = 1$, thus

$$\lim_{Z\to H}\frac{|A(\lambda)(Z)|}{\left|\prod\limits_{\alpha(H)=0}\sinh\,\alpha(Z)/2\right|} = b_2\lim_{Z\to H}\frac{|A(\lambda)(Z)|}{\left|\prod\limits_{\alpha(H)=0}\alpha(Z)\right|}$$

where $b_2 = 2^{-card\{\alpha:\alpha(H)=0\}}$. These observations imply that

$$\begin{aligned}\left\|\mu_a^k\right\|^2_{L^2(G^{\mathbb{C}})} &= \int_{\mathfrak{t}^*}\left|\phi_\lambda(a)\right|^{2k}|c(\lambda)|^{-2}d\lambda\\ &= (b_1b_2)^{2k}\int_{\mathfrak{t}^*}|c(\lambda)|^{2k-2}\left(\lim_{Z\to H}\frac{|A(\lambda)(Z)|}{\left|\prod\limits_{\alpha(H)=0}\alpha(Z)\right|}\right)^{2k}d\lambda \qquad (1)\\ &= b_3\int_{\mathfrak{t}^*}\left(\lim_{Z\to H}\frac{|A(\lambda)(Z)|}{\left|\prod\limits_{\alpha(H)=0}\alpha(Z)\right|}\right)^{2k}\frac{d\lambda}{|\Pi(\lambda)|^{2k-2}}\end{aligned}$$

where $b_3 = (b_1b_2)^{2k}|\Pi(\rho)|^{2k-2} > 0$.

On the other hand, the Fourier transform for the orbital measure μ_H is a continuous function on $\mathfrak{g}$ given by the formula

$$\widehat{\mu_H}(\lambda) = \frac{\Pi(\rho)}{\Pi(\lambda)\prod\limits_{\alpha(H)\neq 0}\alpha(H)}\lim_{Z\to H}\frac{A(\lambda)(Z)}{\prod\limits_{\alpha(H)=0}\alpha(Z)}$$

where, again, the limit is taken over Z non-singular and it exists. (This can be deduced, for example, from the formulas developed in [2], noting that their orbital measures are normalized so that the measure of the adjoint orbit is $\Pi_{\alpha\in\Phi^+}\alpha(H)^2$, rather than 1, as in our case.) Together with the Weyl integration formula this shows that

$$(2)\qquad \begin{aligned} \|\mu_H^k\|^2_{L^2(\mathfrak{g})} &= \int_{\mathfrak{t}^*} |\Pi(\lambda)|^2 |\widehat{\mu_H}(\lambda)|^{2k} d\lambda \\ &= b_4 \int_{\mathfrak{t}^*} \left| \lim_{Z\to H} \frac{A(\lambda)(Z)}{\prod_{\alpha(H)=0} \alpha(Z)} \right|^{2k} \frac{d\lambda}{|\Pi(\lambda)|^{2k-2}} \end{aligned}$$

where

$$b_4 = \left| \frac{\Pi(\rho)}{\prod_{\alpha(H)\neq 0} \alpha(H)} \right|^{2k} > 0.$$

Combining (1) and (2) clearly gives the result. □

4. – Singularity Results.

We can also obtain singularity results for measures on $G^{\mathbb{C}}$ by comparing with the singularity properties of the corresponding orbital measures on the Lie algebra. Our arguments depend strongly on the following characterizations of singularity for orbital measures on the complex group $G^{\mathbb{C}}$ and the Lie algebra $\mathfrak{g}$.

PROPOSITION 4.1. – *Let $a_1, \dots, a_k \in \mathcal{A} \subseteq G^{\mathbb{C}}$ and suppose $\mu_{a_1}, \dots, \mu_{a_k}$ are the orbital measures on $G^{\mathbb{C}}$ supported on the double cosets Ga_iG. Then $\mu_{a_1} * \cdots * \mu_{a_k}$ is singular with respect to the Haar measure on $G^{\mathbb{C}}$ if and only if the function $f_k = f_k(a_1, \dots, a_k) : G^{k+1} \to G^{\mathbb{C}}$ given by*

$$f_k(g_1, g_2, \dots, g_{k+1}) = g_1 a_1 g_2 \cdots g_k a_k g_{k+1}$$

has rank less than the dimension of $G^{\mathbb{C}}$ at almost every $(g_1, \dots, g_{k+1}) \in G^{k+1}$.

PROOF. – Ragozin [8, Thm. 2.5] essentially proves that if the rank of f_K is equal to the dimension of $G^{\mathbb{C}}$ at almost every $g_1, \dots, g_{k+1}$, then $\mu_{a_1} * \cdots * \mu_{a_k}$ is absolutely continuous. Conversely, if *rank* f_k is less than the dimension of $G^{\mathbb{C}}$ at almost every $g_1, \dots, g_{k+1}$, then a continuity argument implies *rank* $f_k < \dim G^{\mathbb{C}}$ at every $(g_1, \dots, g_{k+1})$ and thus Sard's theorem says the measure of the image of f_k is zero. But the image is the product of the double cosets, $Ga_1G \cdots Ga_kG$, which supports the measure $\mu_{a_1} * \cdots * \mu_{a_k}$, and hence this measure is singular. □

A similar argument (see [7] and [3]) gives the analogous result for $\mathfrak{g}$.

PROPOSITION 4.2. – *Let $H_1, \dots, H_k \in \mathfrak{t}$ and suppose $\mu_{H_1}, \dots, \mu_{H_k}$ are the orbital measures on $\mathfrak{g}$ supported on the orbits O_{H_i}. Then $\mu_{H_1} * \cdots * \mu_{H_k}$ is singular with respect to the Haar measure on $\mathfrak{g}$ if and only if the function $F_k = F_k(H_1, \dots, H_k) : G^k \to \mathfrak{g}$ given by*

$$F_k(g_1, \dots, g_k) = Ad(g_1)H_1 + \cdots + Ad(g_k)H_k$$

has rank less than the dimension of $\mathfrak{g}$ at almost every $(g_1, \dots, g_k) \in G^k$.

Thus to determine the singularity of μ_a^k, $a \in \mathcal{A}$, we will need to study the differential of $f_k(a, \dots, a)$ at the point $(g_1, g_2, g_3, \dots, g_{k+1})$. This is the map $df_k|_{(g_1,\dots,g_{k+1})}$ from $\mathfrak{g}^{k+1}$ to $\mathfrak{g}^{\mathbb{C}}$ whose value at $(X_1, \dots, X_{k+1}) \subseteq \mathfrak{g}^{k+1}$ is (1)

$$X_1 + Ad(g_1a)X_2 + Ad(g_1ag_2a)X_3 + \cdots + Ad(g_1ag_2a \dots g_ka)X_{k+1}.$$

Thus *rank* f_k at $(g_1, g_2, \dots, g_{k+1})$ is the dimension of the vector subspace

$$\mathfrak{g} + Ad(g_1a)\mathfrak{g} + Ad(g_1ag_2a)\mathfrak{g} + \cdots + Ad(g_1ag_2a \dots g_ka)\mathfrak{g}.$$

As $g_1 \in G$, $Ad(g_1)\mathfrak{g} = \mathfrak{g}$, so there is no loss of generality in assuming $g_1 = e$, the identity element in G. Thus one can deduce the following useful characterization of singularity.

COROLLARY 4.1. – *Let $a \in \mathcal{A}$. Then μ_a^k is singular to the Haar measure on $G^{\mathbb{C}}$ if and only if for almost all $(g_2, g_3, \dots, g_k) \in G^{k-1}$,*

$$\mathfrak{g} + Ad(a)\mathfrak{g} + Ad(ag_2a)\mathfrak{g} + \cdots + Ad(ag_2 \dots ag_ka)\mathfrak{g} \subsetneq \mathfrak{g}^{\mathbb{C}}.$$

To reduce this to a computation of something more tangible, given $a \in \mathcal{A}$, say $a = \exp iH$ with $H \in \mathfrak{t}$, we will set

$$\begin{aligned} \mathcal{N} &= span\{RE_\alpha, IE_\alpha : \sinh \alpha(H) \neq 0\} \\ &= span\{RE_\alpha, IE_\alpha : \alpha(H) \neq 0\}. \end{aligned}$$

It is well known that $\mathcal{N}$ is not trivial if $H \neq 0$ ($a \neq e$). We will also write $\mathcal{N}$ for $\{\alpha \in \Phi : \alpha(H) \neq 0\}$. It should be clear from the context which is meant.

LEMMA 4.1. – *Let $H \in \mathfrak{t}$, $a = exp(iH) \in \mathcal{A}$ and $g_2, g_3, \dots, g_k \in \mathfrak{g}$. Then*

$$\begin{aligned} &\mathfrak{g} + Ad(a)\mathfrak{g} + Ad(ag_2a)\mathfrak{g} + \cdots + Ad(ag_2 \dots ag_ka)\mathfrak{g} \\ &= \mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + \cdots + Ad(ag_2 \dots ag_k)i\mathcal{N}. \end{aligned}$$

(1) This is true because of our convention of using right invariant vector fields.

PROOF. – Lemma 2.1 gives the formula

$$Ad(a)RE_\alpha = \cosh \alpha(H)RE_\alpha - \sinh \alpha(H)iIE_\alpha \tag{3}$$

and a similar formula for $Ad(a)IE_\alpha$. Thus by definition of $\mathcal{N}$, $Ad(a)RE_\alpha$ (or $Ad(a)IE_\alpha$) belongs to $\mathfrak{g}$ if $\alpha \notin \mathcal{N}$ and belongs to $\mathfrak{g} + i\mathcal{N}$ if $\alpha \in \mathcal{N}$. In particular, $Ad(a)\mathfrak{g} \subseteq \mathfrak{g} + i\mathcal{N}$. Thus $\mathfrak{g} + Ad(a)\mathfrak{g} \subseteq \mathfrak{g} + i\mathcal{N}$ and from this one can deduce that

$$Ad(ag_2a)\mathfrak{g} \subseteq Ad(ag_2)(\mathfrak{g} + i\mathcal{N}) \subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N}.$$

Continuing inductively, we see that $LHS \subseteq RHS$.

For the other inclusion, we note that formula (3) also implies that if $\alpha \in \mathcal{N}$, then $iIE_\alpha \in \mathfrak{g} + Ad(a)\mathfrak{g}$. A similar statement is true for iRE_α, hence $\mathfrak{g} + i\mathcal{N} \subseteq \mathfrak{g} + Ad(a)\mathfrak{g}$. More generally,

$$\begin{aligned}\sinh \alpha(H)Ad(ag_2 \ldots ag_j)iIE_\alpha &= -Ad(ag_2 \ldots ag_j)(Ad(a)RE_\alpha - \cosh \alpha(H)RE_\alpha) \\ &\subseteq Ad(ag_2 \ldots ag_ja)\mathfrak{g} + Ad(ag_2 \ldots ag_{j-1}a)\mathfrak{g}\end{aligned}$$

and similarly for iRE_α. Thus $RHS \subseteq LHS$. □

We make one more simplification in the next lemma.

LEMMA 4.2. – *Let $H \in \mathfrak{t}$ and $a = exp(iH)$. Let $g_2, g_3, \ldots, g_k \in \mathfrak{g}$. Then*

$$\begin{aligned}&\mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + Ad(ag_2ag_3)i\mathcal{N} + \cdots + Ad(ag_2 \ldots ag_k)i\mathcal{N} \\ &= \mathfrak{g} + i\mathcal{N} + Ad(g_2)i\mathcal{N} + Ad(g_2g_3)i\mathcal{N} + \cdots + Ad(g_2 \ldots g_k)i\mathcal{N}.\end{aligned}$$

PROOF. – Let β be a root, $g \in G$ and assume

$$Ad(g)RE_\beta = t + \sum_\alpha (c_\alpha RE_\alpha + d_\alpha IE_\alpha) \text{ for some } t \in \mathfrak{t}.$$

By Lemma 2.1,

$$\begin{aligned}Ad(ag)RE_\beta &= Ad(a)\left(t + \sum_\alpha (c_\alpha RE_\alpha + d_\alpha IE_\alpha)\right) \\ &= t + \sum_{\alpha \notin \mathcal{N}} (c_\alpha RE_\alpha + d_\alpha IE_\alpha) + \sum_{\alpha \in \mathcal{N}} (c_\alpha RE_\alpha + d_\alpha IE_\alpha) \cosh \alpha(H) \\ &\quad + \sum_{\alpha \in \mathcal{N}} i(-c_\alpha IE_\alpha + d_\alpha RE_\alpha) \sinh \alpha(H).\end{aligned}$$

Therefore

$$\begin{aligned}Ad(ag)iRE_\beta - Ad(g)iRE_\beta &= \sum_{\alpha \in \mathcal{N}} i(c_\alpha RE_\alpha + d_\alpha IE_\alpha)(\cosh \alpha(H) - 1) \\ &\quad - \sum_{\alpha \in \mathcal{N}} (-c_\alpha IE_\alpha + d_\alpha RE_\alpha) \sinh \alpha(H) \\ &\in \mathfrak{g} + i\mathcal{N}.\end{aligned}$$

A similar result holds for iIE_β. Thus for any $g \in G$,

$$Ad(ag)i\mathcal{N} \subseteq \mathfrak{g} + i\mathcal{N} + Ad(g)i\mathcal{N} \text{ and } Ad(g)i\mathcal{N} \subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag)i\mathcal{N}.$$

Since $Ad(a)\mathfrak{g} = \mathfrak{g} + i\mathcal{N}$, this gives

$$\begin{aligned} Ad(ag_2g_3)i\mathcal{N} &\subseteq Ad(ag_2)(\mathfrak{g} + i\mathcal{N} + Ad(ag_3)i\mathcal{N}) \\ &\subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + Ad(ag_2ag_3)i\mathcal{N}. \end{aligned}$$

Combining these results we see that

$$\begin{aligned} Ad(g_2g_3)i\mathcal{N} &\subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag_2g_3)i\mathcal{N} \\ &\subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + Ad(ag_2ag_3)i\mathcal{N}, \end{aligned}$$

hence

$$\mathfrak{g} + i\mathcal{N} + Ad(g_2)i\mathcal{N} + Ad(g_2g_3)i\mathcal{N} \subseteq \mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + Ad(ag_2ag_3)i\mathcal{N}.$$

One can similarly deduce that

$$\mathfrak{g} + i\mathcal{N} + Ad(ag_2)i\mathcal{N} + Ad(ag_2ag_3)i\mathcal{N} \subseteq \mathfrak{g} + i\mathcal{N} + Ad(g_2)i\mathcal{N} + Ad(g_2g_3)i\mathcal{N}.$$

The proof follows inductively. □

Together the two lemmas give

COROLLARY 4.2. – *Let $H \in \mathfrak{t}$, $a = exp(iH)$ and $g_2, g_3, \ldots, g_k \in \mathfrak{g}$. Then*

$$\begin{aligned} &\mathfrak{g} + Ad(a)\mathfrak{g} + Ad(ag_2a)\mathfrak{g} + \cdots + Ad(ag_2 \ldots ag_ka)\mathfrak{g} \\ &= \mathfrak{g} + i(\mathcal{N} + Ad(g_2)\mathcal{N} + Ad(g_2g_3)\mathcal{N} + \cdots + Ad(g_2 \ldots g_k)\mathcal{N}). \end{aligned}$$

We can now prove our main singularity result.

THEOREM 4.1. – *Let $H \in \mathfrak{t}$, $a = \exp iH$ and $k \in \mathbb{N}$. Then μ_a^k is a singular orbital measure on $G^{\mathbb{C}}$ if and only if μ_H^k is a singular orbital measure on $\mathfrak{g}$.*

PROOF. – Prop. 4.2 says that μ_H^k is singular if and only if $\dim F_k < \dim \mathfrak{g}$ a.e., where $F_k(x_1, \ldots, x_k) = Ad(x_1)H + \cdots Ad(x_k)H$ is the addition map onto the k-fold sum of the orbit O_H. The differential of F_k at $(x_1, \ldots, x_k) \in G^k$ maps onto the sum of the tangent spaces to O_H at $x_iHx_i^{-1}$, $\sum_{i=1}^{k} T_{x_iHx_i^{-1}}(O_H)$. Without loss of generality we can assume $x_1 = e$.

It is known that $T_H(O_H) = \mathcal{N}$ [3, Prop. 2.1], thus $T_{x_iHx_i^{-1}}(O_H) = Ad(x_i)\mathcal{N}$. Therefore μ_H^k is singular if and only if

$$\mathcal{N} + Ad(x_2)\mathcal{N} + Ad(x_3)\mathcal{N} + \cdots + Ad(x_k)\mathcal{N} \subsetneq \mathfrak{g},$$

or, equivalently,

$$\mathfrak{g} + \mathfrak{i}(\mathcal{N} + Ad(x_2)\mathcal{N} + Ad(x_3)\mathcal{N} + \cdots + Ad(x_k)\mathcal{N} \subsetneq \mathfrak{g}^{\mathbb{C}} \tag{4}$$

for almost all $x_1, \ldots, x_k$. Now put $g_2 = x_2$, $g_3 = g_2^{-1}x_2$ and $g_k = g_{k-1}^{-1} \ldots g_2^{-1}x_k$. Combined with Cor. 4.2 this shows that μ_H^k is singular if and only if for almost all $g_2, \ldots, g_k$,

$$\mathfrak{g} + Ad(a)\mathfrak{g} + Ad(ag_2a)\mathfrak{g} + \cdots + Ad(ag_2 \ldots ag_ka)\mathfrak{g} \subsetneq \mathfrak{g}^{\mathbb{C}}.$$

By Corollary 4.1 it follows that μ_H^k is singular if and only if μ_a^k is singular. □

When we write that a measure $\mu \in L^1 \cap L^2$ we mean that it is absolutely continuous with respect to the Haar measure and its density function belongs to $L^1 \cap L^2$. The following dichotomy theorem was proved in [5, Theorem 8.2].

THEOREM 4.2. – *Suppose $H \neq 0$ belongs to the torus of any of the classical, compact, connected, simple Lie algebras $\mathfrak{g}$. There exists an integer $k_0(H)$ such that $\mu_H^k \in L^2(\mathfrak{g}) \bigcap L^1(\mathfrak{g})$ for all $k \geq k_0(H)$ and μ_H^k is singular to the Haar measure on $\mathfrak{g}$ for all $k < k_0(H)$.*

The value of $k_0(H)$ is specified in [5]. The maximum value depends on the Lie algebra, but in all cases is at most $rank\, G + 1$.

This theorem in combination with Theorems 3.1 and 4.1 yields the dichotomy result for complex groups.

COROLLARY 4.3 (Dichotomy Theorem for Complex Groups). – *Let G be a classical, compact, connected, simple Lie group and let $G^{\mathbb{C}}$ be its complexification. Suppose $a \in \mathcal{A}\backslash\{e\}$, say $a = exp(iH)$ with $H \in \mathfrak{t}\backslash\{0\}$. Then $\mu_a^k \in L^2(G^{\mathbb{C}}) \bigcap L^1(G^{\mathbb{C}})$ for all $k \geq k_0(H)$ and μ_a^k is singular to the Haar measure on $G^{\mathbb{C}}$ for all $k < k_0(H)$.*

COROLLARY 4.4. – *Let $\kappa(\mathfrak{g}) = \max\{k_0(H) : H \in \mathfrak{t}\backslash\{0\}\}$. If $\mu_1, \ldots, \mu_\kappa$ are any continuous G-bi-invariant measures on $G^{\mathbb{C}}$, then $\mu_1 * \cdots * \mu_\kappa \in L^1(G^{\mathbb{C}})$.*

PROOF. – Let $a_1, \ldots, a_\kappa \in \mathcal{A}\backslash\{e\}$. By Holder's inequality and the dichotomy theorem for complex groups, $\mu_{a_1} * \cdots * \mu_{a_\kappa} \in L^2$. Being a non-zero measure, its support, $Ga_1G \cdots Ga_\kappa G$, has positive Haar measure. The arguments in Prop. 4.1 show that this implies that the rank of the function $f_\kappa(g_1, \ldots, g_{\kappa+1}) = g_1a_1g_2 \ldots g_\kappa a_\kappa g_{\kappa+1}$ is equal to the dimension of $G^{\mathbb{C}}$ on a set of positive measure. But then an analyticity result implies the same is true almost everywhere.

Since this holds for all $a_1, \ldots, a_\kappa \in \mathcal{A}\backslash\{e\}$ it follows from the proof of Theorem 2.5 in [8] that any κ-fold convolution product of continuous G-bi-invariant measures is absolutely continuous. □

Of course, κ is sharp with this property since there are continuous orbital measures with $\mu_a^{\kappa-1}$ purely singular.

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