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Viscosity and Almost Everywhere Solutions of First-Order Carnot-Carathèodory Hamilton-Jacobi Equations

PIERPAOLO SORAVIA (*)

Abstract. – *We consider viscosity and distributional derivatives of functions in the directions of a family of vector fields, generators of a Carnot-Carathèodory (C-C in brief) metric. In the framework of convex and non coercive Hamilton-Jacobi equations of C-C type we show that viscosity and a.e. subsolutions are equivalent concepts. The latter is a concept related to Lipschitz continuity with respect to the metric generated by the family of vector fields. Under more restrictive assumptions that include Carnot groups, we prove that viscosity solutions of C-C HJ equations are Lipschitz continuous with respect to the corresponding Carnot-Carathèodory metric and satisfy the equation a.e.*

1. – Introduction.

Since the notion of viscosity solutions has been introduced by Crandall-Lions [6], value functions of deterministic optimal control problems, which are known to be non differentiable in general, have been shown to solve the Bellman equation in the viscosity sense, see [14]. It is well known however that coercive Hamiltonians have locally Lipschitz continuous solutions and the consistency between viscosity and a.e. solutions has been long established, being a consequence of the classical Rademacher Theorem, see e.g. the books by Lions [14] and Bardi-Capuzzo Dolcetta [1]. Indeed when we consider a first order HJ equation

$$\tau u(x) + H(x, Du(x)) = \lambda, \quad x \in \Omega(\subset \mathbb{R}^n), \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\tau \geq 0$ and $H(x, \cdot)$ is convex, then it is equivalent for a locally Lipschitz continuous function to satisfy (1.1) as an a.e. or a viscosity subsolution. Moreover if u is a locally Lipschitz viscosity solution of (1.1), then the equation is also satisfied a.e.. Therefore the two concepts of weak solutions are compatible. It has to be noted however, that adding to (1.1) appropriate boundary conditions and structure assumptions, continuous viscosity solutions are unique, while a.e. solutions are not, in general.

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A special subclass of HJ equations with noncoercive Hamiltonian, that we name of C-C type, usually has solutions which are expected to be not merely continuous but Lipschitz continuous with respect to a sub-riemannian or Carnot-Carathèodory distance. We write such equations in the form

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) = \lambda, \quad x \in \Omega \tag{1.2}$$

where $\sigma(x)$ is an $n \times m$ matrix ($m \leq n$), the superscript t means transposition and now $H : \Omega \times \mathbb{R}^m \to \mathbb{R}$. The columns of σ are a family of m vector fields that will be supposed to generate a Carnot-Carathèodory metric d. For (1.2), the concept of viscosity solution still applies, by interpreting the gradient $Du(x)$ in the viscosity sense. However, a more recent theory of Sobolev spaces within this framework has been introduced, see e.g. Franchi-Serapioni-Serra Cassano [10, 9], Garofalo-Nhieu [13], Franchi-Hajlasz-Koskela [11] and the very recent book by Bonfiglioli-Lanconelli-Uguzzoni [4]. Namely, we can denote the vector fields σ_j as differential operators X_j and then, locally d-Lipschitz continuous functions have locally bounded directional derivatives with respect to the family of vector fields in the sense of distributions. We denote them as $X_j u(x)$. Moreover these derivatives coincide a.e. with pointwise directional derivatives in an appropriate sense in the directions of the vector fields, see e.g. Pansu [17], Monti-Serra Cassano [16] and Monti [15]. Thus we can also interpret (1.2) as

$$\tau u(x) + H(x, Xu(x)) = \lambda, \quad \text{a.e. } x \in \Omega, \tag{1.3}$$

where $Xu(x) = (X_j u(x))$.

Our plan in this paper is to create a direct correspondence between viscosity solutions and a.e. solutions in the previous sense, a question that we did not find explained elsewhere, thus extending the classical results holding in the Euclidian setting. Our motivation, besides being quite a natural question in order to bridge the two theories, relies also in the theory of absolutely minimizing functions, in particular the Lipschitz extension problem, the infinity-Laplace equation, and the way we extend the definition of *absolute minimizers* in a subelliptic setting. It turns out that, besides the classical definition, one can equivalently define absolute minimizers in a *viscosity sense*, see [19] and also Bieske [3] and Wang [20].

In the C-C case (1.2) we will prove parallel results to those holding for (1.1). In Section 2 we first show that, when $H(x, \cdot)$ is coercive (but note that coercivity of H does not imply that the equation (1.2) is coercive in the gradient as well), any viscosity subsolution of (1.2) is d-Lipschitz continuous. Then in Section 3, for a general uniformly continuous Carnot-Carathèodory metric, we prove that it is equivalent that a locally d-Lipschitz continuous function u satisfies (1.2) as a subsolution in the viscosity sense or (1.3) as an a.e. subsolution. In Section 4, under structure conditions satisfied by Carnot groups, we show that if u is a locally d-Lipschitz continuous viscosity solution of (1.2), in particular if $H(x, \cdot)$ is coercive, then u also satisfies the equation a.e.. Hence existence results for a.e.

solutions of HJ equations in Carnot groups become a consequence of the well known classical ones for viscosity solutions.

We finally want to recall that the role of the degenerate eikonal equation in pde theory, a special but important example of HJ C-C equations, has been recently studied by Dragoni, see [7, 8] extending some classical results holding in the Euclidian setting.

2. – Preliminaries and notations.

We start recalling some basic facts on families of vector fields and the Carnot-Carathèodory metric that they define. The starting point here is a family of $m \leq n$ Lipschitz continuous vector fields $\sigma_j : \Omega \to \mathbb{R}^n$, $j = 1, \dots, m$, $\Omega \subset \mathbb{R}^n$ bounded, open and connected set, which we put as columns of the matrix valued function $\sigma : \Omega \to \mathbb{R}^{n \times m}$. To such family of vector fields we associate a deterministic optimal control system, namely

$$\begin{cases} \dot{\tilde{y}}(t) = \sigma(\tilde{y}(t))\tilde{a}(t) = \displaystyle\sum_{j=1}^{m} \tilde{a}_j \sigma_j(\tilde{y}(t)), \\ \tilde{y}(0) = x, \end{cases} \tag{2.1}$$

for any given control function $\tilde{a}(\,\cdot\,) \in L^\infty(0, +\infty, B_1(0))$ (here $B_1(0) \subset \mathbb{R}^m$ is the unit ball). In addition, we assume throughout the paper that the family of vector fields is of Carnot-Carathèodory type, namely the following basic assumption holds: for all $x, z \in \Omega$ the set

$$\mathcal{A}_{x,z} = \{\tilde{a}(\,\cdot\,) \in L^\infty(0, +\infty, B_1(0)) : \tilde{y}(0; \tilde{a}) = x,\ \tilde{y}(t_{x,z}; \tilde{a}) = z,\ t_{x,z} \in [0, +\infty)\}$$

is nonempty. This allows to define the following function

$$d(x, z) = \inf_{a(\cdot) \in L^\infty(0, +\infty, B_1(0))} t_{x,z} < +\infty,$$

which is a distance function, also called the Carnot-Carathèodory distance associated to the family $\{\sigma_j\}$. A particular case where the previous property is satisfied is for instance that of Riemannian metrics ($m = n$), when the matrix $A(x) = \sigma(x)\sigma^t(x)$ is positive definite at each point x. In the sub-Riemannian case, the vector fields are smooth and satisfy the Hörmander condition, namely $\{\sigma_j : j = 1, \dots, n\}$ and their Lie brackets generate the space $\mathbb{R}^n$ at each point of Ω. By Chow Theorem this is a sufficient condition for a well defined Carnot-Carathèodory distance.

For given $z \in \Omega$, the function $d(\cdot, z)$ is the value function of an optimal control problem and then it solves the corresponding Hamilton-Jacobi-Bellman equation, the eikonal equation, see [2, 1] and the references therein,

$$\max_{|\tilde{a}| \leq 1} \{-\sigma(x)\tilde{a} \cdot D_x d(x, z)\} = |\sigma^t(x) D_x d(x, z)| = 1, \quad x \in \Omega \backslash \{z\},$$

as a continuous viscosity solution. Such equation is an important example of HJ equation of C-C type where $H(x,p) = |p|$ is the Hamiltonian. Notice that if we denote

$$\sigma = \begin{pmatrix} \sigma_{(1)} \\ \sigma_{(2)} \end{pmatrix},$$

and $\sigma_{(1)}(x)$ is an $m \times m$ invertible matrix for all $x \in \Omega$, then we can also represent the eikonal equation as $|\sigma^t_{(1)}(x)\tilde{\sigma}^t(x)Du(x)| = 1$, where

$$\tilde{\sigma}(x) = \sigma(x)\sigma^{-1}_{(1)}(x) = \begin{pmatrix} I \\ \sigma_{(2)}(x)\sigma^{-1}_{(1)}(x) \end{pmatrix} \tag{2.2}$$

is a new family of vector fields. The equation is still C-C with Hamiltonian $H(x,\tilde{p}) = |\sigma^t_{(1)}(x)\tilde{p}|$ in this case. Vector fields of the form (2.2) appear for instance in the case of Carnot groups, see also Section 4.

We will always consider families of vector fields such that the distance d is uniformly continuous in the following sense: for any open set $D \subset\subset \Omega$ there is a modulus ω^D such that

$$d(x,z) \leq \omega^D(|x-z|), \quad \text{for all } x, z \in \overline{D}.$$

As a special case, it is well known that if the vector fields $\sigma_j \in C^\infty(\Omega)$ and their Lie algebra satisfy the Hörmander finite rank condition of order r, then for any $D \subset\subset \mathbb{R}^n$ the distance d satisfies the estimate

$$d(x,z) \leq L_D|x-z|^{\frac{1}{r}}, \quad \text{for } x, z \in D.$$

We also denote the vector fields as differential operators, i.e.

$$X_j = \sum_{i=1}^n \sigma_{ij}(x)\partial_i.$$

When $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, we define the derivatives of φ along the directions of the vector fields as

$$X_j\varphi(x) = \sum_{i=1}^n \sigma_{ij}(x)\frac{\partial}{\partial x_i}\varphi(x). \tag{2.3}$$

If otherwise $u \in L^1(\Omega)$ we can define $X_j u$ in the sense of distributions when the following equation is satisfied

$$\int_\Omega X_j u(x)\varphi(x)dx = -\int_\Omega u(x)\sum_{i=1}^n \frac{\partial}{\partial x_i}(\sigma_{ij}(x)\varphi(x))dx \equiv -\int_\Omega u(x)X_j^*\varphi(x)dx,$$

for all $\varphi \in C^1_c(\Omega)$. If in particular u is locally Lipschitz continuous in the Euclidean sense, then equation (2.3) is still valid for a.e. $x \in \Omega$.

Notice that we have set

$$X_j^* \varphi(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\sigma_{ij}(x)\varphi(x)) = (\text{div } \sigma_j(x))\ \varphi(x) + X_j\varphi(x),$$

in particular X_j is self-adjoint if and only if div $\sigma_j \equiv 0$. One can henceforth define the corresponding Sobolev spaces taking into account the derivatives with respect to the directions of the vector fields. It is well known, among their properties, see e.g. Franchi-Serapioni-Serra Cassano [10, 9] and Garofalo-Nhieu [13], that $u \in W_X^{1,\infty}(D)$ if and only if the function u is Lipschitz continuous in $D \subset\subset \Omega$ with respect to the Carnot-Carathèodory metric d determined by the family of vector fields, i.e.

$$|u(x) - u(y)| \leq L_u d(x, y), \quad \text{for all } x, y \in D,$$

for an appropriate $L_u \in \mathbb{R}$. In the following we will denote $Xu = (X_1 u, \dots, X_m u)$, when u is locally d-Lipschitz continuous.

We now consider a continuous Hamiltonian $H : \Omega \times \mathbb{R}^m \to \mathbb{R}$ with the following properties: $H(x, \cdot)$ is convex for all x and locally Lipschitz continuous in the form

$$\begin{aligned} H(x, p) - H(y, p) &\leq L_D(1 + |p|^k)|x - y|, \\ H(x, p) - H(y, q) &\leq L_D(1 + (|p| \vee |q|)^k)|p - q|, \end{aligned} \tag{2.4}$$

for all $x, y \in D \subset\subset \Omega$, $p, q \in \mathbb{R}^m$ and some $k \geq 1$. Local Lipschitz continuity of H in x could be relaxed to local uniformly continuous, but we skip these details. We notice that our Hamiltonian $H(x, \cdot)$ may or may not be coercive, where this means that at least $\lim_{|p|\to\infty} \inf_{x\in\Omega} H(x, p) = +\infty$. Nevertheless even though a coercivity condition holds, the Hamiltonian

$$\tilde{H}(x, q) = H(x, \sigma^t(x)q), \quad (x, q) \in \Omega \times \mathbb{R}^n$$

will not be coercive in q if $m < n$.

Our goal in this paper is to study the relationships between the *horizontal* Hamilton-Jacobi equations

$$\begin{aligned} &\tau u(x) + H(x, Xu(x)) = \lambda \in \mathbb{R}, \quad &&\text{a.e. } x \in \Omega, \\ &\tau u(x) + H(x, \sigma^t(x)Du(x)) = \lambda, \quad &&\text{in } \Omega \text{ in the viscosity sense,} \end{aligned}$$

where here and in the following $\tau \geq 0$. Our approach will use convexity of the Hamiltonian as a crucial ingredient to study subsolutions, and refined almost everywhere differentiability statements for d-Lipschitz continuous functions due to Monti [15] (see also Pansu [17]), holding in a Carnot group setting (and more), to study supersolutions.

We end this section by showing that the two previous equations are clearly related by a general regularity result of viscosity subsolutions of C-C HJ equations. We start by considering Hamiltonians H of Bellman type, see however Remark 2.3 below. We suppose that

$$H(x,p) = \max_{a\in A}\{-f(x,a)\cdot p - l(x,a)\}, \quad (x,p)\in \Omega\times\mathbb{R}^m, \tag{2.5}$$

where A is a compact subset of a metric space and $f:\Omega\times A\to\mathbb{R}^m$, $l:\Omega\times A\to\mathbb{R}$ are continuous functions satisfying

$$|f(x,a)-f(z,a)|\le L_D|x-z|,\ |l(x,a)-l(z,a)|\le L_D|x-z|, \tag{2.6}$$

for all $x,z\in D\subset\subset\Omega$, $a\in A$. Since the whole discussion in the paper concerns local properties, we will also assume the data to be bounded. Then we also have the following representation formula

$$\tilde H(x,q) = \max_{a\in A}\{-\sigma(x)f(x,a)\cdot q - l(x,a)\},\quad \text{for all } (x,q)\in\Omega\times\mathbb{R}^n.$$

The latter is the Bellman Hamiltonian of an optimal control problem governed by the following control system

$$\dot y(t) = \sigma(y(t))f(y(t),a(t)),\quad y(0)=x, \tag{2.7}$$

for all Borel measurable controls $a:[0,+\infty)\to A$. In the next statement we will also suppose that the Hamiltonian H be coercive in the following sense: we can find $\delta>0$ such that

$$f(x,A)\supset B(0,\delta),\quad \text{for all } x\in\Omega. \tag{2.8}$$

REMARK 2.1. – We can relax assumption (2.8) to

$$\overline{co}f(x,A)\supset B(0,\delta),\quad \text{for all } x\in\Omega$$

by simply using in the following relaxed controls, for instance $a(\,\cdot\,)\in L^\infty(0,+\infty;P(A))$, instead of ordinary controls, where $P(A)$ indicates the set of probability measures on the set A. We will avoid doing that for the sake of simplicity. □

REMARK 2.2. – Notice that given a trajectory $y(\,\cdot\,)$ of (2.7) we can define of by change of variables $\tilde y(t) = y\Big(\dfrac{t}{1+\|f\|_\infty}\Big)$ and obtain that

$$\dot{\tilde y}(t) = \sigma(\tilde y(t))\tilde a(t),\quad \tilde y(0)=x,$$

where $\tilde a(t) = \dfrac{f\big(\tilde y(t), a\big(\frac{t}{1+\|f\|_\infty}\big)\big)}{1+\|f\|_\infty}\in B_1(0)$ is a solution of (2.1). In particular $d(\tilde y(t),x)\le t$ and $d(y(t),x)\le t(1+\|f\|_\infty)$.

Notice moreover that if coercivity (2.8) holds, then any trajectory of (2.1) can be reproduced by a trajectory of (2.7). Indeed for any given $\tilde{a}(\cdot) \in L^\infty(0,+\infty;B_1(0))$ we can select $a(\cdot) \in L^\infty(0,+\infty;A)$ such that $f\left(\tilde{y}(t), a\left(\frac{t}{\delta}\right)\right) = \delta\tilde{a}(t)$, where $\tilde{y}$ is the solution of (2.1) corresponding to $\tilde{a}$. Therefore if we define $y(t) = \tilde{y}(\delta t)$, we get that y is the solution of (2.7) corresponding to $a(\cdot)$. In particular for any given $x, y \in \Omega$, we can find $\hat{a} \in L^\infty(0,+\infty;A)$ such that the corresponding trajectory of (2.7) satisfies $\hat{y}(0) = x, \hat{y}(t) = y$ and

$$d(x,y) = \delta t.$$

□

We prove the following result. For the definition and basic theory of viscosity solutions, the reader can consult the classical paper Crandall-Ishii-Lions [5] or [1].

PROPOSITION 2.1. – *Assume* (2.5) (2.6) *and* (2.8). *Any bounded (upper semicontinuous) viscosity subsolution of*

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) = \lambda, \quad \text{in } \Omega$$

is d-Lipschitz continuous.

PROOF. – The simple proof is based on the optimality principle as in [18] despite the fact that the HJB equation may not satisfy a comparison principle. Since u is a viscosity subsolution, for all $a(\cdot) \in L^\infty(0,+\infty;A)$ we have that for $y(s) \in \Omega$ for all $s \in (0,t)$ then

$$u(x) \leq \int_0^t e^{-\tau s} l(y(s), a(s))ds + e^{-\tau t}u(y(t)).$$

For any given $x, y \in \Omega$, as in Remark 2.2 we now choose $\hat{a}(\cdot)$ and the corresponding trajectory $\hat{y}(\cdot)$ such that $\hat{y}(0) = x, \hat{y}(t) = y, d(x,y) = \delta t$. Then we obtain

$$u(x) - u(y) \leq \|l\|_\infty t - (1 - e^{-\tau t})u(y) \leq \delta^{-1}(\|l\|_\infty + \tau\|u\|_\infty)d(x,y).$$

Reversing the roles of x, y we conclude. □

REMARK 2.3. – Proposition 2.1 still holds even if the control set A in (2.5) is unbounded, provided the functions f, l satisfy suitable coercivity conditions. To this end we can use the optimality principle as in Garavello-Soravia [12]. In particular this applies to a generic convex Hamiltonian H satisfying (2.4) and

$$H(x,p) \geq C_1|p|^r - C_2,$$

for all x, p and some $r > 1, C_1 > 0$, when we represent it in the form (2.5) by convex duality. □

The following is an easy and useful consequence of Proposition 2.1.

COROLLARY 2.2. – *If u is a viscosity subsolution of*

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) = \lambda, \quad \textit{in } \Omega$$

and $y(\cdot)$ is a trajectory of (2.7) *then the function $u(y(\cdot))$ is locally Lipschitz continuous.*

PROOF. – The following proof actually holds for any d-Lipschitz function. We use Remark 2.2 and take any interval such that $y(t) \in \Omega$ for all $t \in [r, s]$, where $y(\cdot)$ is a trajectory of (2.7). Then we get

$$|u(y(r)) - u(y(s))| \leq L_u d(y(r), y(s)) \leq L_u(1 + \|f\|_\infty)|s - r|.$$

□

3. – Viscosity and a.e. subsolutions.

This section deals with the case of subsolutions and we study the two differential inequalities

$$\tau u(x) + H(x, Xu(x)) \leq \lambda, \quad \text{a.e. } x \in \Omega,$$
$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \leq \lambda, \quad \text{in } \Omega \text{ in the viscosity sense.}$$

In our approach a crucial ingredient is the convexity of the Hamiltonian. We prove the following statement.

THEOREM 3.1. – *Assume* (2.4) *and that $\sigma_j \in W^{2,\infty}(\Omega; \mathbb{R}^n)$ for $j = 1, \ldots, m$. Let $u : \Omega \to \mathbb{R}$ be a bounded d-Lipschitz continuous function, then*

$$\tau u(x) + H(x, Xu(x)) \leq \lambda, \quad \text{a.e. } x \in \Omega$$

if and only if

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \leq \lambda, \quad \textit{in } \Omega \textit{ in the viscosity sense.}$$

We prove the two parts of the statement separately.

PROPOSITION 3.2. – *In the assumptions of Theorem* 3.1 *if*

$$\tau u(x) + H(x, Xu(x)) \leq \lambda, \quad \text{a.e. } x \in \Omega,$$

then we have

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \leq \lambda, \quad x \in \Omega,$$

in the viscosity sense.

PROOF. – We fix $x_o \in \Omega$ and prove that u is a viscosity solution of

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \le \lambda, \quad x \in B(x_o, r)$$

for $r > 0$ sufficiently small so that $B(x_o, r) \subset x_o + [-r, r]^n \subset \Omega$. Fix $\varepsilon_o > 0$ so that

$$x_o + [-r, r]^n \subset \Omega_{\varepsilon_o} = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2\varepsilon_o\}.$$

We may also suppose that $r \le \varepsilon_o/2$.

We use the standard mollification where $\rho : \mathbb{R}^n \to [0, +\infty)$ is C^∞, $\int \rho(x)dx = 1$, supp $\rho \subset B(0,1)$, $\varepsilon \in (0, \varepsilon_o)$ and $\rho_\varepsilon(x) = \dfrac{1}{\varepsilon^n}\rho\Big(\dfrac{x}{\varepsilon}\Big)$. If u is a d-Lipschitz continuous function, we define $u_\varepsilon = u * \rho_\varepsilon$ and compute, for $x \in B(x_o, r)$, $\varepsilon < \varepsilon_o$,

$$\begin{aligned}
X_j u_\varepsilon(x) &= \sum_{i=1}^n \sigma_{ij}(x) \frac{\partial}{\partial x_i} \int_\Omega u(y)\rho_\varepsilon(x-y)dy \\
&= \sum_{i=1}^n \sigma_{ij}(x) \int_\Omega u(y) \frac{\partial}{\partial x_i}\rho_\varepsilon(x-y)dy = \sum_{i=1}^n \sigma_{ij}(x) \int_\Omega \frac{u(y)}{\varepsilon^{n+1}}(\partial_i\rho)\Big(\frac{x-y}{\varepsilon}\Big)dy \\
&= \sum_{i=1}^n \int_{|z|<1} \sigma_{ij}(x) \frac{u(x-\varepsilon z)}{\varepsilon}(\partial_i\rho)(z)dz.
\end{aligned}$$

On the other hand, by the Lipschitz property of u we also compute

$$\begin{aligned}
X_j u * \rho_\varepsilon(x) &= \int_\Omega X_j u(y)\rho_\varepsilon(x-y)dy \\
&= -\int_\Omega u(y)\Big[\mathrm{div}\ \sigma_j(y)\rho_\varepsilon(x-y) + \frac{1}{\varepsilon^n}\sum_{i=1}^n \sigma_{ij}(y)\frac{\partial}{\partial y_i}\rho\Big(\frac{x-y}{\varepsilon}\Big)\Big]dy \\
&= -(\mathrm{div}\ \sigma_j)u * \rho_\varepsilon(x) + \sum_{i=1}^n \int_\Omega \frac{1}{\varepsilon^{n+1}} u(y)\sigma_{ij}(y)(\partial_i\rho)\Big(\frac{x-y}{\varepsilon}\Big)dy \\
&= -(\mathrm{div}\ \sigma_j)u * \rho_\varepsilon(x) + \sum_{i=1}^n \int_{|z|<1} \sigma_{ij}(x-\varepsilon z)\frac{u(x-\varepsilon z)}{\varepsilon}(\partial_i\rho)(z)dz.
\end{aligned}$$

We now study the convergence in $B(x_o, r/2)$ of

$$\begin{aligned}
(3.1)\quad & X_j u * \rho_\varepsilon(x) - X_j u_\varepsilon(x) \\
&= \sum_{i=1}^n \int_{|z|<1} \frac{1}{\varepsilon}(\sigma_{ij}(x-\varepsilon z) - \sigma_{ij}(x))\, u(x-\varepsilon z)(\partial_i\rho)(z)dz - (\mathrm{div}\ \sigma_j)u * \rho_\varepsilon(x).
\end{aligned}$$

The idea is to observe that, by integrating by parts in dz_i we have

$$\sum_{i=1}^n \int D\sigma_{ij}(x)\cdot z\ (\partial_i\rho(z))dz = -\sum_{i=1}^n \int \frac{\partial}{\partial x_i}\sigma_{ij}(x)\rho(z)dz = -\mathrm{div}\ \sigma_j(x)$$

and then add and subtract the term (div σ_j) u in the right hand side of (3.1).

As for the last term, we easily estimate in $B(x_o, r/2)$

$$\|(\text{div } \sigma_j)u * \rho_\varepsilon(x) - (\text{div } \sigma_j)u\|_\infty \leq \|\sigma\|_{2,\infty}(\omega(\varepsilon) + \|u\|_\infty \varepsilon),$$

where $d(x, y) \leq \omega(|x - y|)$ for all $x, y \in B(x_o, r/2)$.

For each term in the sum (3.1), we estimate for $|z| < 1$

$$\begin{aligned} &\left|\frac{1}{\varepsilon}(\sigma_{ij}(x - \varepsilon z) - \sigma_{ij}(x))\, u(x - \varepsilon z) + D\sigma_{ij}(x) \cdot z\, u(x)\right| \\ &= | - D\sigma_{ij}(x - \varepsilon_z z) \cdot z\, u(x - \varepsilon z) + D\sigma_{ij}(x) \cdot z\, u(x)| \\ &\leq \|D\sigma\|_\infty |u(x) - u(x - \varepsilon z)||z| + \|D^2\sigma\|_\infty |z|^2 \|u\|_\infty \varepsilon \\ &\leq \|\sigma\|_{2,\infty}(\omega(\varepsilon) + \|u\|_\infty \varepsilon), \end{aligned}$$

where $\varepsilon_z \in (0, \varepsilon)$ is suitably chosen.

We can then estimate in $B(x_o, r/2)$

$$\|X_j u * \rho_\varepsilon - X_j u_\varepsilon\|_\infty \leq (n\omega_n \|\rho\|_{1,\infty} + 1)\|\sigma\|_{2,\infty}(\omega(\varepsilon) + \|u\|_\infty \varepsilon), \tag{3.2}$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. In particular the family $\{X_j u_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(B(x_o, r/2))$.

We now go to the HJ equation and compute by the above, the convexity of the Hamiltonian and Jensen's Lemma

$$\begin{aligned} \tau u_\varepsilon(x) \quad &+ H(x, \sigma^t(x) D u_\varepsilon(x)) = \tau u_\varepsilon(x) + H(x, X u_\varepsilon(x)) \leq \tau u_\varepsilon(x) \\ &+ H(x, Xu * \rho_\varepsilon(x)) + o(1) \\ &\leq \tau u_\varepsilon(x) + \int \rho_\varepsilon(x - y) H(x, Xu(y)) dy + o(1) \\ &\leq \int \rho_\varepsilon(x - y)[\tau u(y) + H(y, Xu(y)) + L(1 + \|Xu\|_\infty)|x - y|] dy + o(1) \\ &\leq \lambda + o(1). \end{aligned}$$

The previous inequality holds in $B(x_o, r/2)$ pointwise and thus in the viscosity sense because u_ε is smooth. By the stability of viscosity solutions, see [5] since $u_\varepsilon \to u$ uniformly in $B(x_o, r)$, we can then conclude

$$\tau u(x) + H(x, \sigma^t(x) D u(x)) \leq \lambda, \quad B(x_o, r),$$

in the viscosity sense. □

REMARK 3.1. – We notice that in the previous proof it is shown that with the standard convolution and mollifiers we have that for a d-Lipschitz continuous function

$$X u_\varepsilon - (Xu) * \rho_\varepsilon \to 0, \quad \text{in } L^\infty_{loc}(\Omega).$$

In particular $X u_\varepsilon(x) \to Xu(x)$ in $L^1_{loc}(\Omega)$.

We also observe that the corresponding statement of Proposition 3.2 for supersolutions is false even in the Euclidean setting. For instance $u(x) = |x|$ satisfies $u'(x) = 1$ a.e. in $(-1, 1)$ nevertheless it is not a viscosity supersolution of such equation. □

We now proceed with the other part of Theorem 3.1. Here we use the nonlinear convolution, classically defined for any bounded and upper semicontinuous function $u : \Omega \to \mathbb{R}$ by setting

$$u^\varepsilon(x) = \sup_{y\in\Omega} \left\{ u(y) - \frac{1}{2\varepsilon^2}|x-y|^2 \right\}, \quad x \in \mathbb{R}^n.$$

PROPOSITION 3.3. – *In the assumptions of Theorem* 3.1 *if*

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \le \lambda, \quad x \in \Omega, \tag{3.3}$$

in the viscosity sense, then

$$\tau u(x) + H(x, Xu(x)) \le \lambda, \quad \text{a.e. } x \in \Omega.$$

PROOF. – We start with a general claim. Let $\{u^\varepsilon\}_\varepsilon \subset C(\mathbb{R}^n)$ be a family of locally Lipschitz continuous functions in the Euclidean sense such that $u_\varepsilon \to u$ locally uniformly in Ω and $\{Xu^\varepsilon\}_\varepsilon$ is locally bounded in $L^\infty(\Omega; \mathbb{R}^m)$. Then for any open $V \subset\subset \Omega$ we have that $Xu^\varepsilon \rightharpoonup Xu$ in $L^2(V)$. Indeed for all $\varphi \in C_c^\infty(V)$ we compute

$$\int_V (Xu^\varepsilon(x) - Xu(x))\varphi(x)dx = -\int_V (u^\varepsilon(x) - u(x))X^*\varphi(x)dx \to 0,$$

and then by density $\int (Xu^\varepsilon(x) - Xu(x))f(x)dx \to 0$ for all $f \in L^2(\Omega)$.

If $u : \Omega \to \mathbb{R}$ is a d-Lipschitz continuous function satisfying (3.3), we build an appropriate family of approximations by nonlinear sup-convolution. We use some well-known properties of the functions u^ε, see e.g. [1]. It is known that $u^\varepsilon \ge u$ in Ω, u^ε are semiconvex, $u^\varepsilon \to u$ locally uniformly in Ω and, for any given open set $V \subset\subset \Omega$, at each point of differentiability $x \in V$ and ε sufficiently small we have $Du^\varepsilon(x) = \dfrac{T^\varepsilon x - x}{\varepsilon^2} \in D^+u(T^\varepsilon x)$, where $T^\varepsilon x$ is such that $u^\varepsilon(x) = u(T^\varepsilon x) - \dfrac{1}{2\varepsilon^2}|x - T^\varepsilon x|^2$ and we have denoted the superdifferential

$$D^+u(x_o) = \{p \in \mathbb{R}^n : \text{there is } \varphi \in C^\infty(\Omega),\ p = D\varphi(x_o),\ x_o \in ArgMax(u-\varphi)\}.$$

One can also easily show that $\dfrac{1}{2\varepsilon^2}|x - T^\varepsilon x|^2 \le \omega^V\left(2\sqrt{\|u\|_\infty \varepsilon}\right) = o(1)$ by d-Lipschitz continuity of u.

From the assumptions we then get

$$\tau u(T^\varepsilon x) + H(T^\varepsilon x, \sigma^t(T^\varepsilon x)Du^\varepsilon(x)) \leq \lambda, \quad \text{a.e. } x \in V. \tag{3.4}$$

We pause for a second and repeat the same argument choosing as Hamiltonian $|p|$ and $\lambda = ess - \sup_{x \in V} |Xu(x)| < +\infty$ by the d-Lipschitz property of u. Then $|Xu(x)| \leq \lambda$ for a.e. $x \in V$ and therefore $|\sigma^t(x)Du(x)| \leq \lambda$ in the viscosity sense by Proposition 3.2. As above we obtain

$$|\sigma^t(T^\varepsilon x)Du^\varepsilon(x)| \leq \lambda, \quad \text{a.e. } x \in V$$

and

$$\begin{aligned} |Xu^\varepsilon(x)| &= |\sigma^t(x)Du^\varepsilon(x)| \leq |\sigma^t(T^\varepsilon x)Du^\varepsilon(x)| + |(\sigma^t(x) - \sigma^t(T^\varepsilon x))Du^\varepsilon(x)| \\ &\leq \lambda + \|D\sigma\|_\infty |x - T^\varepsilon x| |Du^\varepsilon(x)| = \lambda + o(1), \quad \text{a.e. } x \in V. \end{aligned}$$

In particular in V

$$\limsup_{\varepsilon \to 0} \|Xu^\varepsilon\|_\infty \leq \|Xu\|_\infty$$

and the family $\{Xu^\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty(V)$.

Now we go back to (3.4). From what we just observed and (2.4) we also get

$$\tau u(x) + H(x, Xu^\varepsilon(x)) \leq \lambda + o(1), \quad \text{a.e. } x \in V.$$

Now fix $\delta > 0$ and suppose ε sufficiently small so that $o(1) \leq \delta$. Then consider

$$\mathcal{A}_\lambda = \{f \in L^2(V; \mathbb{R}^m) : H(x, f(x)) \leq \lambda + \delta - \tau u(x) \text{ a.e.}\}.$$

We have that $\mathcal{A}_\lambda$ is convex and closed in $L^2(V; \mathbb{R}^m)$, hence closed in the weak topology. Moreover we showed that $\{Xu^\varepsilon\}_\varepsilon \subset \mathcal{A}_\lambda$. Thus $Xu \in \mathcal{A}_\lambda$ since $Xu^\varepsilon \rightharpoonup Xu$ in $L^2(V; \mathbb{R}^m)$ as we saw in the beginning. Threrefore

$$\tau u(x) + H(x, Xu(x)) \leq \lambda + \delta, \quad \text{a.e. } x \in V.$$

Being δ and $V \subset \Omega$ arbitrary, we can conclude. □

REMARK 3.2. – This remark may look a bit technical but it is sometimes useful, see e.g. [19], and is due to the particular nature of viscosity solutions. It is a consequence of Theorem 3.1. Notice that if

$$H(x, \sigma^t(x)Du(x)) \leq \lambda, \quad x \in \Omega, \tag{3.5}$$

in the viscosity sense, then the following inequalities are both satisfied

$$H(x, Xu(x)) \leq \lambda, \quad H(x, -X(-u(x)) \leq \lambda, \quad \text{a.e. } x \in \Omega.$$

Using again Theorem 3.1 from the latter we also obtain that

$$H(x, -\sigma^t(x)D(-u)(x)) \leq \lambda, \quad x \in \Omega, \tag{3.6}$$

in the viscosity sense. This is not obvious because while (3.5) is a condition on the viscosity superdifferential of u, (3.6) is instead a condition on the subdifferential of u since $D^+(-u) = -D^-u$, see [5, 1] for details. □

4. – Viscosity and a.e. supersolutions.

In this section we consider a Bellman Hamiltonian written in the form (2.5). We consider a locally d-Lipschitz continuous function $u \in C(\Omega)$ and suppose that it is pointwise almost everywhere differentiable with respect to the family of vector fields. Namely we assume that it satisfies the following first order Taylor expansion with respect to the horizontal gradient defined in the sense of distributions

$$u(y) = u(x) + Xu(x) \cdot (\overline{y} - \overline{x}) + o(d(y,x)), \quad \text{as } y \to x, \text{ for a.e. } x \in \Omega. \tag{4.1}$$

In the above, as a general notation we set $\overline{x} = (x_1, \ldots, x_m)$ if $x = (x_1, \ldots, x_n)$. The first general Rademacher-type theorem in Carnot groups is due to Pansu [17]. The work by Monti [15] in the case of real-valued functions, relates Pansu differential with the distributional derivatives of u and shows (4.1) in more general Carnot-Carathèodory spaces than Carnot groups. A crucial assumption that identifies the setting used in [15] (among other facts, see [15] for details) and explains why we use the difference $(\overline{y} - \overline{x})$ in the first order term of (4.1) is the following special structure of the vector fields, namely that

$$\sigma_{ij}(x) = \delta_{ij}, \quad i,j \in \{1, \ldots, m\}, \tag{4.2}$$

where the notation δ_{ij} indicates the Kronecker delta.

We prove the following.

THEOREM 4.1. – *Assume* (2.6) *and suppose that* $u \in C(\Omega)$ *is locally d-Lipschitz continuous and satisfies* (4.1) *and* (4.2). *If* u *satisfies*

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) \geq \lambda, \quad x \in \Omega, \tag{4.3}$$

in the viscosity sense, then

$$\tau u(x) + H(x, Xu(x)) \geq \lambda, \quad \text{a.e. } x \in \Omega.$$

PROOF. – Let $u \in C(\Omega)$ be a viscosity supersolution of (4.3), we work in any open $D \subset\subset \Omega$. By the optimality principle, see e.g. Soravia [18]

$$u(x) = \inf_{a(\cdot) \in L^\infty(0,+\infty;A)} \sup_{t \in [0,\tau_x)} \int_0^t e^{-\tau s}[\lambda + l(y(s), a(s))]ds + e^{-\tau t}u(y(t)),$$

where τ_x is the first exit time of the trajectory of (2.7) from D. Therefore if we choose an optimal relaxed control $\hat{a} \in L^\infty(0,+\infty;P(A))$ and the corresponding optimal trajectory $\hat{y}$ with $\hat{y}(0) = x \in D$ we have

$$
-\frac{e^{-\tau t}u(\hat{y}(t)) - u(x)}{t} - \frac{1}{t}\int_0^t e^{-\tau s} l(\hat{y}(s),\hat{a}(s))ds \geq \lambda \,\frac{(1-e^{-\tau t})}{\tau t}, \tag{4.4}
$$

for all sufficiently small $t > 0$.

On the other hand, by (4.1) we can write

$$
\frac{u(\hat{y}(t)) - u(x)}{t} = Xu(x)\cdot\frac{(\overline{\hat{y}}(t) - \overline{x})}{t} + o(1), \quad \text{as } t \to 0, \text{ for a.e. } x \in D.
$$

Therefore from (4.2) and (4.4) we deduce (notice that $\overline{\sigma f} = f$)

$$
\begin{aligned}
&\frac{(1-e^{-\tau t})}{t}\, u(\hat{y}(t)) - Xu(x)\;\cdot\frac{1}{t}\int_0^t f(\hat{y}(s),\hat{a}(s))ds - \frac{1}{t}\int_0^t l(\hat{y}(s),\hat{a}(s))ds \\
&= \frac{(1-e^{-\tau t})}{t}\, u(\hat{y}(t)) - Xu(x)\;\cdot\frac{(\overline{\hat{y}}(t)-\overline{x})}{t} - \frac{1}{t}\int_0^t e^{-\tau s} l(\hat{y}(s),\hat{a}(s))ds + o(1) \\
&\geq \lambda + o(1), \quad \text{as } t \to 0, \text{ for a.e. } x \in \Omega.
\end{aligned} \tag{4.5}
$$

We now observe the following bounds

$$
\left|\frac{1}{t}\int_0^t f(\hat{y}(s),\hat{a}(s))ds\right| \leq \|f\|_\infty, \qquad \left|\frac{1}{t}\int_0^t l(\hat{y}(s),\hat{a}(s))ds\right| \leq \|l\|_\infty.
$$

Therefore for a.e. given $x \in D$ we can find a sequence $t_n \to 0$ such that

$$
\left(\frac{1}{t_n}\int_0^{t_n} f(\hat{y}(s),\hat{a}(s))ds, \frac{1}{t_n}\int_0^{t_n} l(\hat{y}(s),\hat{a}(s))ds\right) \to (\hat{f},\hat{l}) \in co\{(f(x,a),l(x,a)) : a \in A\}.
$$

Hence by (4.5) and taking the limit $n \to +\infty$ we obtain

$$
\begin{aligned}
\tau u(x) + H(x,Xu(x)) &= \tau u(x) + \max_{a\in A}\{-f(x,a)\cdot Xu(x) - l(x,a)\} \\
&= \tau u(x) + \max_{(f,l)\in co(f(x,A),l(x,A))}\{-Xu(x)\cdot f - l\} \\
&\geq \tau u(x) - Xu(x)\cdot\hat{f} - \hat{l} \geq \lambda,
\end{aligned}
$$

for a.e. $x \in \Omega$. □

We conclude the paper with a direct consequence of the results we proved so far, an existence result of almost everywhere solutions of Hamilton-Jacobi equations in a C-C setting. This result extends those obtained by Monti [15] and Monti-Serra Cassano [16] for the eikonal equation. We add that results and methods to prove existence of continuous viscosity solutions of the most common boundary value problems for first order HJ equations are now well known, see e.g. [1].

COROLLARY 4.2. – *Assume that the vector fields* $\sigma_j \in W^{2,\infty}(\Omega;\mathbb{R}^n)$, $j = 1, \dots, m$, *are generators of a Carnot-Carathèodory, uniformly continuous distance d. Assume moreover that* (2.6), (2.5), (2.8) *and* (4.2) *are satisfied. Let* $u \in C(\Omega)$ *satisfying* (4.1), *be a viscosity solution of*

$$\tau u(x) + H(x, \sigma^t(x)Du(x)) = \lambda, \quad \textit{in } \Omega.$$

Then u satisfies

$$\tau u(x) + H(x, Xu(x)) = \lambda, \quad \textit{for } \text{a.e. } x \in \Omega.$$

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