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Robert Černý

Abstract. – *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with a Lipschitz boundary and let $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a Carathéodory function satisfying usual growth assumptions.*

Then the functional

$$\Phi(u) = \int_\Omega g\big(x, u(x)\big)\, dx$$

is lower semicontinuous with respect to the weak topology on $W^{1,p}(\Omega)$, $1 \le p \le \infty$, if and only if g is convex in the second variable for almost every $x \in \Omega$.

1. – Introduction.

Let $\Omega \subset \mathbb{R}^N$ be an open set.

In the Calculus of Variations it is a well-studied problem to find sufficient conditions for a Carathéodory integrand $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ so that the functional

$$\Phi(u) = \int_\Omega g\big(x, u(x), \nabla u(x)\big)\, dx$$

is sequentially weakly lower semicontinuous on the Sobolev space $W^{1,p}(\Omega)$, $1 \le p \le \infty$.

A classical result in the scalar case is (see [2, Section 3.3.1])

Theorem 1.1. – *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $1 \le p < \infty$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function. Let $g(x, \zeta, \cdot)$ be convex for every $x \in \Omega$ and every $\zeta \in \mathbb{R}$.*

Then the functional

$$\Phi(u) = \int_\Omega g\big(x, u(x), \nabla u(x)\big)\, dx$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$.

Another interesting result can be found in [3, Chapter I § 1].

THEOREM 1.2. – *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $f : \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function. Let us define the functional*

$$\Phi(u) = \int_\Omega f(u(x))\, dx.$$

Then Φ is sequentially lower semicontinuous with respect to the weak convergence on L^p, $1 \le p < \infty$, if and only if f is convex.

Moreover, Φ is sequentially lower semicontinuous with respect to the weak convergence on L^∞ if and only if f is convex.*

In this note we are interested in the question what happens if we replace usually considered sequentially weak convergence on $W^{1,p}(\Omega)$ by the weak convergence on $W^{1,p}(\Omega)$, i.e. by the convergence in the weak topology on $W^{1,p}(\Omega)$, for the functional of the type

$$\Phi(u) = \int_\Omega g\big(x, u(x)\big)\, dx\ .$$

Let us note that these integrands independent of the third variable are automatically convex in the third variable and thus Theorem 1.1 implies the sequential weak lower semicontinuity. Nevertheless for the convergence in the weak topology these assumptions are not enough to ensure the lower semicontinuity as can be seen from the main result of this paper (where the assertion is rather Theorem 1.2-like).

THEOREM 1.3. – *Assume $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with a Lipschitz boundary, $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function for almost every $x \in \Omega$ satisfying*

$$\begin{aligned}
&|g(x,y)| \le C_1 + C_2|y|^{\frac{Np}{N-p}} && \textit{for} \quad p \in [1,N)\ ,\\
&|g(x,y)| \le C_1 + C_2|y|^q, \quad \textit{for some } q \in [1,\infty) && \textit{for} \quad p = N\ ,\\
&g(x,y) \textit{ is bounded on bounded sets} && \textit{for} \quad p \in (N,\infty]\ .
\end{aligned}$$

Then the following statements are equivalent:

(i) *The functional $\Phi(u) = \int_\Omega g\big(x, u(x)\big)\, dx$ is lower semicontinuous with respect to the weak topology on $W^{1,p}(\Omega)$.*

(ii) *The function $g(x,\cdot)$ is convex for almost every $x \in \Omega$.*

For the definition of the lower semicontinuity with respect to the weak topology on $W^{1,p}(\Omega)$ see the Preliminaries.

Throughout the paper we write that $g(x, y)$ is bounded on bounded sets for a.e. $x \in \Omega$, if there is a set $\tilde{\Omega} \subset \Omega$ such that $\mathcal{L}^N(\Omega \setminus \tilde{\Omega}) = 0$ and for every bounded $A \subset \Omega$ and every $L > 0$ there is $M > 0$ such that $|g(x, y)| < M$ for every $|y| < L$ and every $x \in A \cap \tilde{\Omega}$.

The proof of Theorem 1.3 uses construction introduced in paper [1] (see also [4]), where the following result concerning the weak continuity on $W^{1,p}(\Omega)$ is proven.

THEOREM 1.4. – *Assume* $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}^N$ *is an open bounded set,* $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *is a Carathéodory function satisfying the growth condition*

$$\begin{aligned} &|g(x, y)| \leq C_1 + C_2 |y|^{\frac{Np}{N-p}} && \text{for} \quad p \in [1, N) \, , \\ &|g(x, y)| \leq C_1 + C_2 |y|^q, \text{ for some } q \in [1, \infty) && \text{for} \quad p = N \, , \\ &g(x, y) \text{ is bounded on bounded sets} && \text{for} \quad p \in (N, \infty] \, . \end{aligned}$$

Then the following statements are equivalent:

(i) *The functional* $\Phi(u) = \int_\Omega g(x, u(x))\, dx$ *is continuous with respect to the weak topology on* $W^{1,p}(\Omega)$.

(ii) *There are measurable bounded functions* $k_1, k_2: \Omega \mapsto \mathbb{R}$, *such that for a.e.* $x \in \Omega$

$$g(x, y) = k_1(x) + k_2(x) y$$

is satisfied for every $y \in \mathbb{R}$.

(iii) *There is* $f_0 \in W^{1,p}(\Omega)$ *such that the functional* $\Phi(u) = \int_\Omega g(x, u(x))\, dx$ *is continuous at* f_0 *with respect to the weak topology* $W^{1,p}(\Omega)$.

Note that a statement similar to (iii) from Theorem 1.4 cannot be added to our Theorem 1.3 (see Example 4.2).

2. – Preliminaries.

For $x \in \mathbb{R}^N$, we write x^i for the i-th coordinate, i.e. $x = [x^1, x^2, \ldots, x^N]$. Sometimes we represent a point in $\mathbb{R}^{N+1}$ in a form $[x, y]$ where $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$.

For $x_0 = [x_0^1, \ldots, x_0^N] \in \mathbb{R}^N$ and $\eta > 0$ we use the following notation for a cube

$$Q(x_0, \eta) = \{x \in \mathbb{R}^N : |x^i - x_0^i| \leq \eta, i = 1, \ldots, N\} \, .$$

An open ball centered at the origin with the radius $r > 0$ is denoted by $B(0, r)$.

The N-dimensional Lebesgue measure of a measurable set A is denoted by $\mathcal{L}^N(A)$.

For a finitely-additive measure μ on $\mathbb{R}^N$ and a Borel set A, $|\mu|(A)$ is the total variation of μ on A.

Let $1 \le p \le \infty$. We use the usual notation $W^{1,p}(\Omega)$ for the Sobolev space on $\Omega \subset \mathbb{R}^N$, i.e. functions in $L^p(\Omega)$ whose distributional partial derivatives are also in $L^p(\Omega)$.

The space $W^{1,p}(\Omega)$, $p \le 1 < \infty$, is equipped with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega |f(x)|^p \, dx + \sum_{m=1}^N \int_\Omega \Big| \frac{\partial f(x)}{\partial x^m} \Big|^p \, dx \Big)^{1/p}$$

and on $W^{1,\infty}(\Omega)$ we have

$$\|f\|_{W^{1,\infty}(\Omega)} = \max\Big(\|f\|_{L^\infty(\Omega)}, \Big\|\frac{\partial f}{\partial x^1}\Big\|_{L^\infty(\Omega)}, \dots, \Big\|\frac{\partial f}{\partial x^1}\Big\|_{L^\infty(\Omega)} \Big) \ .$$

For an introduction to Sobolev spaces see e.g. [5].

Recall that a functional Φ is *weakly lower semicontinuous* on $W^{1,p}(\Omega)$ if for every $\varepsilon > 0$ and $f_0 \in W^{1,p}(\Omega)$ there is a weak neighborhood U of f_0 such that

$$\Phi(f) > \Phi(f_0) - \varepsilon \qquad \text{for all} \quad f \in U \ .$$

A set $U \subset W^{1,p}(\Omega)$ is a *weak neighborhood of* f_0 if we can find $k \in \mathbb{N}$ and continuous linear functionals $\Lambda_1, \dots, \Lambda_k \in (W^{1,p}(\Omega))^*$ such that

$$\big\{ f \in W^{1,p}(\Omega)\colon \ |\Lambda_i(f - f_0)| < 1 \text{ for every } i \in \{1, \dots, k\} \big\} \subset U \ .$$

Above definition of the weak lower semicontinuity is equivalent to:

$$F_\lambda = \{ f \in W^{1,p}(\Omega) : \Phi(f) \le \lambda \} \quad \text{is a weakly closed set for every } \lambda \in \mathbb{R} \ .$$

For every continuous linear functional Λ on $W^{1,\infty}(\Omega)$ there are finitely additive measures μ^m, $m = 0, \dots, N$, such that

$$\Lambda(f) = \int_\Omega f(x) d\mu^0(x) + \sum_{m=1}^N \int_\Omega \frac{\partial f(x)}{\partial x^m} d\mu^m(x) \ .$$

Note that $|\mu^m|$ restricted to a bounded measurable set $A \subset \Omega$ is bounded and for every continuous function $h : A \mapsto \mathbb{R}$ we have

$$\Big| \int_A g(x) d\mu^m(x) \Big| \le |\mu^m|(A) \sup_{x \in A} |g(x)| \ .$$

For every continuous linear functional Λ on $W^{1,p}(\Omega)$, $1 \le p < \infty$, there are functions $g^m \in L^{p'}(\Omega)$, $m = 0, \dots, N$ such that

$$\Lambda(f) = \int_\Omega f(x) g^0(x) dx + \sum_{m=1}^N \int_\Omega \frac{\partial f(x)}{\partial x^m} g^m(x) \, dx \ .$$

In this case it is convenient for us to represent considered linear functional by measures $\mu^m = g^m \mathcal{L}^N$, $m = 0, \dots, N$.

We say that $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a *Carathéodory function* if $g(\cdot, y)$ is measurable for every $y \in \mathbb{R}$ and $g(x, \cdot)$ is continuous for almost every $x \in \Omega$.

3. – Weak convergence implies convexity.

The proof of the implication (i) $\Rightarrow$ (ii) from Theorem 1.3 follows from the following proposition. We do not need to suppose that Ω is bounded in this proposition.

PROPOSITION 3.1. – *Assume* $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}^N$ *is an open set and* $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *is a Carathéodory integrand, bounded on bounded sets a.e. in* Ω. *If the functional* $\Phi(u) = \int_\Omega g(x, u(x))\, dx$ *is lower semicontinuous with respect to the weak topology on* $W^{1,p}(\Omega)$ *then* $g(x, \cdot)$ *is convex for almost every* $x \in \Omega$.

We devote the rest of this section to the proof of Proposition 3.1. This proof is just a simpler version of the proof of [1, Theorem 5.1] (the technique when proving convexity by contradiction is basically the same as when proving linearity, moreover we have more restrictive assumptions on the function g and hence we were able to simplify the arguments from the previous paper).

We need the following technical lemma showing that we can find a suitable perturbation function $\psi : Q(x_0, \eta) \mapsto \mathbb{R}$ which lies in the given weak neighborhood of the zero function (see (1) and (2) below), attains value r pretty often (see (3)) and does not attain other values than $\{-r, 0, r\}$ often (see (4)).

LEMMA 3.2. – *Let* $r > 0$, $\varepsilon = 2^{-\sigma}$, $2 \leq \sigma \in \mathbb{N}$ *and* $k, N \in \mathbb{N}$. *Suppose* $x_0 = [x_0^1, \dots, x_0^N] \in \mathbb{R}^N$, $\eta \in (0, 1]$ *and* μ_l^m, $l = 1, \dots, k$, $m = 0, \dots, N$, *are finitely-additive measures on* $\mathbb{R}^N$, *bounded on* $Q(x_0, \eta)$.

Then there is a smooth function $\psi: \mathbb{R}^N \mapsto [-r, r]$ *such that* $\operatorname{spt} \psi \subset (x_0, \eta)$,

$$\text{(1)} \qquad \left| \int_{\mathbb{R}^N} \frac{\partial \psi}{\partial x^m} d\mu_i^m \right| < \frac{1}{N+1} \text{ for every } i \in \{1, \dots, k\}\ , m \in \{1, \dots, N\}\ ,$$

$$\text{(2)} \qquad \left| \int_{\mathbb{R}^N} \psi d\mu_i^0 \right| < \frac{1}{N+1} \text{ for every } i \in \{1, \dots, k\}\ ,$$

$$\text{(3)} \qquad \mathcal{L}^N\left(\psi^{-1}(r)\right) \geq \frac{\mathcal{L}^N(Q(x_0, \eta))}{60 \cdot 2^N} = \frac{\eta^N}{60}$$

and

$$\mathcal{L}^N\left(\psi^{-1}(\mathbb{R}\setminus\{-r,0,r\})\right)\le 2N\varepsilon\mathcal{L}^N(Q(x_0,\eta))\ . \tag{4}$$

For the proof of Lemma 3.2 see [1, Lemma 3.1.] and [1, Remark 5.2.(i)].

PROOF OF PROPOSITION 3.1. – For contradiction, let us suppose that Φ is weakly lower semicontinuous and it is not true that g is convex in the second variable for a.e. $x\in\Omega$.

If $x\in\Omega$ and $g(x,\cdot)$ is not convex then using the continuity of $g(x,\cdot)$ it is easy to see that there are rational numbers $A>0$, $a\in\mathbb{R}$ and $r>0$ such that

$$g(x,a-r)+g(x,a+r)-2g(x,a)< -2A\ .$$

Since it is not true that $g(x,\cdot)$ is convex in the second variable for a.e. $x\in\Omega$ and as the Carathéodory functions are measurable in the first variable, we find rational numbers $A>0$, $a\in\mathbb{R}$ and $r>0$ such that the set

$$\tilde{G}=\{x\in\Omega: g(x,a-r)+g(x,a+r)-2g(x,a)< -2A\}$$

satisfies

$$\mathcal{L}^N(\tilde{G})>0\ .$$

Let us find a radius $\rho>0$ large enough so that we have

$$\mathcal{L}^N(G)>0,\qquad \text{where}\quad G=\tilde{G}\cap B(0,\rho)\ .$$

In the rest of the proof we care about $\Omega\cap B(0,\rho)$ only. Without loss of generality we can suppose $g(x,a-r)=g(x,a+r)$ on $\Omega\cap B(0,\rho)$. Otherwise we use the fact that functionals corresponding to linear Carathéodory integrands are weakly continuous on $W^{1,p}(\Omega\cap B(0,\rho))$ (see Theorem 1.4) and we replace g with

$$\tilde{g}(x,y)=g(x,y)-\frac{g(x,a_0+r_0)-g(x,a_0-r_0)}{2r_0}\cdot y$$

(note that in the sequel we consider functions f and f_0 that vanish on $\Omega\setminus B(0,\rho)$).

From $g(x,a-r)=g(x,a+r)$ on $\Omega\cap B(0,\rho)$ we obtain for every $x\in G$

$$g(x,a-r)<g(x,a)-A\quad\text{and}\quad g(x,a+r)<g(x,a)-A\ . \tag{5}$$

Now let us pick $x_0\in G$ a point of density of G and find $\eta\in(0,1]$ small enough so that $Q(x_0,\eta)\subset\Omega\cap B(0,\rho)$ and

$$\mathcal{L}^N(Q(x_0,\eta)\setminus G)\le\frac{A}{480K}\eta^N\ . \tag{6}$$

As g is bounded on bounded sets a.e. in Ω, there is $K>0$ large enough so that

$$|g(x,y)|<K\qquad\text{for almost every } x\in Q(x_0,\eta)\quad\text{and every } |y|\le|a|+|r|\ . \tag{7}$$

Let us take $f_0\in W^{1,p}(\Omega)$ such that $f_0(x)=a$ whenever $x\in Q(x_0,\eta)$ and $f_0(x)=0$ for $x\in\Omega\setminus B(0,\rho)$. As Φ is weakly lower semicontinuous at f_0, there is a weak neighborhood U of f_0 such that for $f \in U$ we have

$$\int_{\Omega} g(x,f(x))\,dx \geq \int_{\Omega} g(x,f_0(x))\,dx - \frac{A\eta^N}{150}\ . \tag{8}$$

If $1 \leq p < \infty$, from the properties of the weak topology on $W^{1,p}(\Omega)$ we can find functions $g_l^m \in L^{p'}(\Omega)$, $l = 1,\dots,k$, $m = 0,\dots,N$, such that

$$\{f \in W^{1,p}(\Omega)\colon |\Lambda_i(f - f_0)| < 1 \quad \text{for every } i = 1,\dots,k\} \subset U \tag{9}$$

where

$$\Lambda_i(f - f_0) = \int_{\Omega} (f(x) - f_0(x))g_i^0(x)\,dx + \sum_{m=1}^{N} \int_{\Omega} \frac{\partial(f(x) - f_0(x))}{\partial x^m} g_i^m(x)\,dx\ .$$

As $L^{p'}(Q(x_0,\eta)) \subset L^1(Q(x_0,\eta))$ we define $\mu_l^m = g_l^m|_{Q(x_0,\eta)}\mathcal{L}^N$, $l = 1,\dots,k$, $m = 0,\dots,N$, and we obtain finitely additive measures bounded on $Q(x_0,\eta)$.

If $p = \infty$, then we obtain the finitely additive measures immediately.

For these measures, a, r, x_0 and η chosen above and for $\varepsilon = 2^{-\sigma}$, $2 \leq \sigma \in \mathbb{N}$, satisfying additional condition

$$\varepsilon \leq \frac{A}{960KN2^N} \tag{10}$$

we apply Lemma 3.2 and we obtain a perturbation function ψ such that (1), (2), (3) and (4) are satisfied. From (1), (2) and (9) we observe that $f_0 + \psi \in U$.

Denote the set of bad points by

$$Z = \psi^{-1}(\mathbb{R} \setminus \{-r, 0, r\}) \cup (Q(x_0,\eta) \setminus G).$$

Using (4), (6) and (10) we have

$$\mathcal{L}^N(Z) \leq \frac{A}{480K}\eta^N + 2N\frac{A}{960KN2^N}2^N\eta^N = \frac{A}{240K}\eta^N\ . \tag{11}$$

Therefore from (3), (5), (7), (8) and (11) we obtain

$$\begin{aligned}
-\frac{A}{150}\eta^N &\leq \int_{\Omega} g(x,f_0(x) + \eta(x))\,dx - \int_{\Omega} g(x,f_0(x))\,dx \\
&\leq \int_{\{\eta\neq 0\}} g(x,f_0(x) + \eta(x)) - g(x,f_0(x))\,dx \\
&\leq \int_{\{\eta=r\}\cup\{\eta=-r\}\cup Z} g(x,a + \eta(x)) - g(x,a)\,dx \\
&\leq -A\frac{\eta^N}{60} + 0 + 2K\mathcal{L}^N(Z) \\
&\leq -A\frac{\eta^N}{120}\ .
\end{aligned}$$

This is the contradiction we wanted. □

4. – Proof of Theorem 1.3.

The proof of the implication (ii) $\Rightarrow$ (i) from Theorem 1.3 uses the following lemma.

LEMMA 4.1. – *Assume* $1 \le q \le \infty$, $\Omega \subset \mathbb{R}^N$ *is an open bounded set,* $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *is a Carathéodory function for almost every* $x \in \Omega$ *satisfying*

$$|g(x,y)| \le C_1 + C_2|y|^q \qquad \text{for} \quad q \in [1,\infty) ,$$
$$g(x,y) \text{ is bounded on bounded sets} \qquad \text{for} \quad q = \infty .$$

Then the functional $\Phi(u) = \int_\Omega g\big(x, u(x)\big)\,dx$ *is continuous with respect to the norm topology on* $L^q(\Omega)$.

PROOF. – Let $f_n, f \in L^q(\Omega)$ for all $n \in \mathbb{N}$ and suppose that $f_n \to f$ in the $L^q(\Omega)$-norm. We want to show that $\Phi(f_n) \to \Phi(f)$.

First, let $1 \le q < \infty$. In this case it is enough to apply Fatou's Lemma to the sequences

$$\{g(x,f_k(x)) + C_1 + C_2|f_k(x)|^q\}_{k=1}^\infty \quad \text{and} \quad \{C_1 + C_2|f_k(x)|^q - g(x,f_k(x))\}_{k=1}^\infty .$$

Indeed, by the growth assumption both sequences contain functions that are non-negative a.e. in Ω, the convergence in the $L^q(\Omega)$-norm implies $|f_n|^q \to |f|^q$ in $L^1(\Omega)$ and passing to a subsequence we can also suppose that $f_n \to f$ a.e. in Ω.

Now, let $q = \infty$. Since $f_n \to f$ in the L^∞-norm, there is a set $\tilde{B} \subset \Omega$ and $L > 0$ such that $\mathcal{L}^N(\tilde{B}) = 0$, $f_n \to f$ uniformly on $\Omega \setminus \tilde{B}$, $|f(x)| \le L$ on $\Omega \setminus \tilde{B}$ and $|f_n(x)| \le L$ on $\Omega \setminus \tilde{B}$ for all $n \in \mathbb{N}$.

As g is bounded on bounded sets a.e. in Ω there are $B \supset \tilde{B}$ and $M > 0$ such that $\mathcal{L}^N(B) = 0$ and

$$g(x,y) \le M \qquad \text{on} \quad (\Omega \setminus B) \times [-L, L] .$$

Finally, we apply Fatou's Lemma to

$$\{g(x,f_k(x)) + M\}_{k=1}^\infty \quad \text{and} \quad \{M - g(x,f_k(x))\}_{k=1}^\infty .$$

□

PROOF OF THEOREM 1.3. – Let us prove implication (ii) $\Rightarrow$ (i). Suppose that (*ii*) is satisfied. Let $p \in [1,\infty]$, $f_n, f \in W^{1,p}(\Omega)$ and suppose that $f_n \to f$ in $W^{1,p}(\Omega)$. By the Sobolev Embedding Theorem we have $f_n \to f$ in $L^q(\Omega)$ where $q = \dfrac{Np}{N-p}$ provided $p \in [1, N)$, $q \in [1,\infty)$ is arbitrary provided $p = N$ and $q = \infty$ provided $p \in (N, \infty]$. Thus Lemma 4.1 and the growth condition imply $\Phi(f_n) \to \Phi(f)$. It follows that the sets

$$F_\lambda = \{f \in W^{1,p}(\Omega): \Phi(f) \le \lambda\} , \quad \lambda \in \mathbb{R} ,$$

are closed. Since g is convex in the second variable a.e. in Ω, F_λ are convex. Finally, as any closed convex set is weakly closed, we immediately obtain (i).

The implication (i) $\Rightarrow$ (ii) follows from Proposition 3.1 and the fact that the growth condition in Theorem 1.3 implies that g is bounded on bounded sets a.e. in Ω. □

Finally we give an elementary example showing that a condition similar to (iii) from Theorem 1.4 cannot be added to Theorem 1.3.

EXAMPLE 4.2. – Let $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with a Lipschitz boundary. Then there is a Carathéodory function $g\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ such that for every $x \in \Omega$ we have $g(x,\cdot)$ is not convex and

$$\begin{aligned} &|g(x,y)| \leq C_1 + C_2|y|^{\frac{Np}{N-p}} && \text{for} \quad p \in [1,N)\ , \\ &|g(x,y)| \leq C_1 + C_2|y|^q,\ \text{for some } q \in [1,\infty) && \text{for} \quad p = N\ , \\ &g(x,y) \text{ is bounded on bounded sets} && \text{for} \quad p \in (N,\infty]\ , \end{aligned}$$

and there is $f_0 \in W^{1,p}(\Omega)$ such that the functional $\Phi(u) = \int_\Omega g\big(x,u(x)\big)\,dx$ is lower semicontinuous at f_0 with respect to the weak topology on $W^{1,p}(\Omega)$.

PROOF. – First let $p \in [1,N)$. Let us pick a continuous function $\varphi\colon \mathbb{R} \mapsto [0,\infty)$ such that

$$\varphi(y) = |y|^{\frac{Np}{N-p}} \qquad \text{for } y \in \mathbb{R} \setminus (1,3)$$

and φ is not convex. We set $g(x,y) = \varphi(y)$. Then g is a Carathéodory integrand satisfying the demanded growth condition. Finally we set $f_0 \equiv 0$, hence $\Phi(f_0) = 0$. Since $g(x,y) \geq 0$ everywhere, we have

$$\Phi(f) \geq 0 = \Phi(f_0) \qquad \text{for every } f \in W^{1,p}(\Omega)$$

(and $\Phi(f) < \infty$ by the Sobolev Embedding Theorem and Lemma 4.1) and thus for every $f \in U$ we have $\Phi(f) \geq \Phi(f_0)$ no matter how the weak neighborhood looks like. Hence we have the weak lower semicontinuity at f_0.

If $p \geq N$, then we set $\varphi(y) = y^2$ on $\mathbb{R} \setminus (1,3)$ and we continue the same way as in the previous case. □

REMARK 4.3. – (i) If the boundary of the set Ω is not nice enough to use the embedding theorem, then we replace the growth condition in Theorem 1.3 by

$$\begin{aligned} &|g(x,y)| \leq C_1 + C_2|y|^p && \text{for } 1 \leq p < \infty \\ &|g(x,y)| \text{ is bounded on bounded sets} && \text{for } p = \infty \end{aligned}$$

a.e. in Ω. In the proof of such a version of Theorem 1.3 we use Lemma 4.1 with $q = p$ now.

(ii) We can consider the space $W_0^{1,p}$, $1 \le p \le \infty$, in Theorem 1.3 without any assumption concerning the boundary of Ω. The proof of implication (ii) $\Rightarrow$ (i) does not require any changes in this case. Similarly for the proof of (i) $\Rightarrow$ (ii), because each continuous linear functional Λ on $W_0^{1,p}$, $1 \le p < \infty$ is represented by

$$\Lambda(f) = \sum_{m=1}^{N} \int_{\Omega} \frac{\partial f(x)}{\partial x^m} g^m(x)\, dx \ ,$$

where $g^m \in L^{p'}(\Omega)$, $m = 1, \dots, N$. Similarly for every linear functional Λ on $W_0^{1,\infty}(\Omega)$ there are bounded finitely-additive measures μ^m, $m = 1, \dots, N$, on the algebra of measurable subsets of Ω such that

$$\Lambda(f) = \sum_{m=1}^{N} \int_{\Omega} \frac{\partial f(x)}{\partial x^m} d\mu^m(x) \ .$$

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REFERENCES

[1] R. Černý - S. Hencl - J. Kolář, *Integral functionals that are continuous with respect to the weak topology on* $W_0^{1,p}(\Omega)$, Nonlinear Anal., **71** (2009), 2753-2763.

[2] B. Dacorogna, *Direct Methods in the Calculus of Variations* (Springer, 1989).

[3] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity of Non-Linear Functionals* (Springer, 1982).

[4] S. Hencl - J. Kolář - O. Pankrác, *Integral functionals that are continuous with respect to the weak topology on* $W_0^{1,p}(0,1)$, Nonlinear Anal., **63** (2005), 81-87.

[5] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, **120** (Springer-Verlag, New York, 1989).

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