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On Rational Elliptic Surfaces with Mordell-Weil Group of Rank Five

DAVIDE FUSI - ANDREA LUIGI TIRONI

Abstract. – *Let $E(K)$ be the Mordell-Weil group of a rational elliptic surface and let r be its rank. In this note we classify all the rational elliptic surfaces with Mordell-Weil group of rank $r = 5$ over an algebraically closed field of arbitrary characteristic and using the theory of Mordell-Weil lattices, we find systems of generators for $E(K)$ in the coordinate-free situation.*

Introduction.

In [7], Shioda gives the foundation of the theory of the Mordell-Weil lattices. The key idea is to see the Mordell-Weil group of an elliptic curve over a function field, or of an elliptic surface, as a lattice with respect to a suitable pairing.

Let S be a rational elliptic surface with Mordell-Weil group $E(K)$ of rank r. From [7], we know that $r \leq 8$ and if equality holds, then we can look at S as $\mathbb{P}^2$ blown-up at nine base points of a suitable linear pencil of cubic curves. Moreover, these base points have to be in general position, (that is, nine distinct points, no six on a conic and no three on a line) and if we take one of them as zero in $E(K)$, it turns out that the subgroup generated by the other eight is of index three in $E(K)$, as shown in [4] and [7].

This paper is a continuation of [3] in which the first named author considered the next cases $r = 7$ and $r = 6$.

So, following the ideas of Shioda [7] and using arguments and techniques inspired by [3], we push forward to the $r = 5$ case. More precisely, we classify rational elliptic surfaces with Mordell-Weil group of rank five, showing that all of them come from suitable linear pencils of plane cubic curves with nine base points not in general position (see Theorem 3.1). We would like to note that for $r \geq 5$ the Mordell-Weil group is torsion free, unlike for $r \leq 4$. Moreover, when $r = 5$, the possible configurations of the singular fibers are still handleable by technical tools and results similar to those used in [3]. These are the main motivations which led us to consider also case $r = 5$.

The paper is organized as follows. In § 1, we fix some notation and background. In § 2, confining to the rank five case, we produce several examples of linear pencils of plane cubic curves and in § 3 we show that when $r = 5$ the only possible constructions up to Cremona transformations are these examples.

Finally, in § 4 we exhibit explicit generators for the Mordell-Weil group $E(K)$ in case $r = 8$ (see Proposition 4.1). Combining this with [3] we have a complete panorama for all rational elliptic surfaces with Mordell-Weil group of rank $r \geq 5$.

1. – Notation and background material.

The ground field k is an algebraically closed field of arbitrary characteristic. By rational elliptic surface, which we will call briefly RES, we mean a smooth rational projective surface S with a relatively minimal elliptic fibration $f : S \longrightarrow \mathbb{P}^1$, that is a surjective morphism such that i) almost all fibers are elliptic curves, and ii) no fibers contain an exceptional curve of the first kind. In what follows we always assume that f has a global section O. Moreover, note that f is not smooth, that is, there is at least one singular fiber. For more details and some useful facts about RES, we refer to [7].

Let E denote the generic fiber of f. Then E is an elliptic curve defined over the function field of $\mathbb{P}^1$, $K = K(\mathbb{P}^1)$, given with a K-rational point O. Let $E(K)$ denote the group of K-rational points of E, with the origin O. The global sections of f are in a natural one-to-one correspondence with the K-rational points of E. In this setting we know that $E(K)$ is a finitely generated abelian group ([7, Theorem 1.1]). The group $E(K)$ is called the Mordell-Weil group of the elliptic curve E/K, or of the elliptic surface $f : S \longrightarrow \mathbb{P}^1$, briefly MW group. In the bijection above, for every point $\widetilde{P} \in E(K)$ we denote with P the correspondent global section.

Denote by U the subgroup of the Neron-Severi group NS(S) generated by the global section O and all irreducible components of fibers, by R the set $\{v \in \mathbb{P}^1 | F_v := f^{-1}(v) \text{ is reducible}\}$ and by F the class of the generic fiber of f in NS(S). For every fiber F_v with $v \in R$, we set $\Theta_{v,0}$ the component which meets the O section, and we set $\Theta_{v,i}$, for $i = 1, \ldots, m_v - 1$, the others, where m_v is the number of components of F_v. For $v \in R$, T_v is the sublattice of the Neron-Severi lattice generated by $\Theta_{v,i}$ for $i = 1, \ldots, m_v - 1$. We denote with A_v the Gram matrix of the intersection form restricted to T_v.

Recall that rkNS(S) $= 10$ ([7, Lemma 10.1]), and that $\mathrm{NS}(S)/U \simeq E(K)$ ([7, Theorem 1.3]). Then we have the following

$$\text{rk } E(K) = 8 - \sum_{v \in R} (m_v - 1). \tag{1}$$

DEFINITION 1.1. – *The Mordell-Weil lattice of the elliptic curve E/K, or of the elliptic surface $f : S \longrightarrow \mathbb{P}^1$, is the lattice $(E(K)/E(K)_{\mathrm{tor}}, \langle , \rangle)$.*

The form $\langle , \rangle$, called the height pairing, *is defined as follows:*

$$\langle \widetilde{P}, \widetilde{Q} \rangle = -(\phi(\widetilde{P}) \cdot \phi(\widetilde{Q})),$$

where $\phi : E(K) \to \mathrm{NS}(S) \otimes \mathbb{Q}$ *is the group homomorphism and* $\phi(\widetilde{P})$ *is given by the following expression* (*see* [3]):

$$P - O - (1 + P \cdot O)F + \sum_{v \in R} (\Theta_{v,1}, .., \Theta_{v,m_v-1})(-A_v^{-1})(P \cdot \Theta_{v,1}, .., P \cdot \Theta_{v,m_v-1})^t.$$

Let us give the structure theorem for RES with MW group of rank 5.

THEOREM 1.2 ([6], Main Theorem). – *The Mordell-Weil group* $E(K)$ *of a rational elliptic surface with Mordell-Weil rank five is torsion free and its structure is given by the following table:*

(*)

No.	T	rk T	det T	$E(K)$	$E(K)^0$
1	A_3	3	4	D_5^*	D_5
2	$A_2 \oplus A_1$	3	6	A_5^*	A_5
3	$A_1^{\oplus 3}$	3	8	$D_4^* \oplus A_1^*$	$D_4 \oplus A_1$

where $E(K)^0$ *is the subgroup of the MW group* $E(K)$ *generated by global sections which meet* $\Theta_{v,0}$ *for all* $v \in R$ *and* $T = \oplus_{v \in R} T_v$ *is the root lattice associated with the reducible fibers* $F_v = f^{-1}(v)$*, where* $f : S \to \mathbb{P}^1$ *is the relatively minimal elliptic fibration over* $\mathbb{P}^1$.

We also recall that if L' is a sublattice of finite index in a lattice L, then

$$\det L' = (\det L)[L : L']^2. \tag{2}$$

Finally, for more on lattices we refer to [1] and [2].

2. – Constructions of linear pencils of cubic curves.

Let us give here some examples of RES with MW group $E(K)$ of rank five coming from linear pencils of cubic curves with special base points.

Construction 1. Let C_1 be an irreducible cubic in $\mathbb{P}^2$ and take three lines l_i for $i = 1, 2, 3$ such that the points $l_1 \cap C_1 = \{p_0, p_1, p_2\}, l_2 \cap C_1 = \{p_0, p_3, p_4\}$ and $l_3 \cap C_1 = \{p_5, p_6, p_7\}$ are in general position. Finally, put $C_2 = l_1 \cup l_2 \cup l_3$. Under this construction, we have that the linear pencil of cubic curves passing through the eight points p_i for $i = 0, \ldots, 7$, defined as above, has the only one reducible member C_2. Moreover, note that any plane cubic $C_{\lambda\mu} := \lambda C_1 + \mu l_1 l_2 l_3$ of our pencil has the same tangent line at p_0 for $\lambda \neq 0$. Consequently, the rational elliptic surface S, obtained by blowing-up $\mathbb{P}^2$ at these eight points p_i for $i = 0, \ldots, 7$ and successively at the point p_8 which lies on $P_0 \cap \widetilde{C_1}$, has only one reducible fiber of type I_4 in Kodaira's classification, where P_i denotes the exceptional curve arising from the blowing-up at p_i and $\widetilde{C_1}$ is the strict transform of C_1 by the first blowing-up at $p_0, \ldots, p_7$.

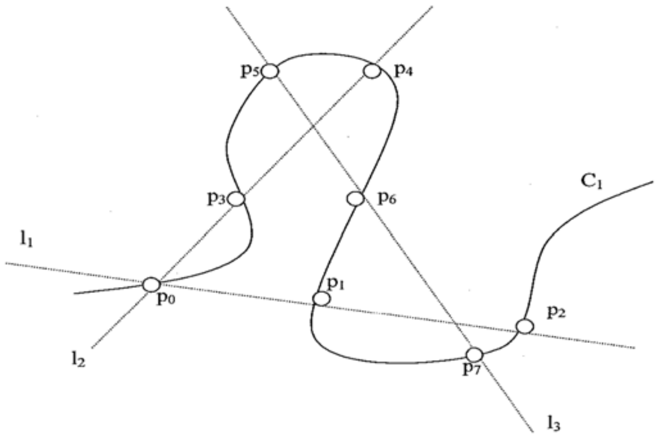

Fig. 1. – Construction 1.

Note that P_i for $i = 1, \ldots, 8$ are sections of the elliptic fibration. For simplicity, let us denote here with the same capital letters P_i the points in $E(K)$ corresponding to the exceptional curves P_i. To produce a system of generators for $E(K)$, take the set $\{P_8, P_3, P_5, P_1, P_6, P_7\}$, where P_7 is the zero in $E(K)$ and P_8 is the point in $E(K)$ corresponding to the (-1)-curve obtained by the second blowing-up at p_8. The *height pairing* of these points of $E(K)$ are computed as follows:

$$\langle P_8, P_8\rangle = 1, \langle P_8, P_3\rangle = \frac{1}{2}, \langle P_8, P_5\rangle = 1, \langle P_8, P_1\rangle = \frac{1}{2}, \langle P_8, P_6\rangle = 1,$$
$$\langle P_3, P_3\rangle = \frac{5}{4}, \langle P_3, P_5\rangle = 1, \langle P_3, P_1\rangle = \frac{3}{4}, \langle P_3, P_6\rangle = 1, \langle P_5, P_5\rangle = 2,$$
$$\langle P_5, P_1\rangle = 1, \langle P_5, P_6\rangle = 1, \langle P_1, P_1\rangle = \frac{5}{4}, \langle P_1, P_6\rangle = 1, \langle P_6, P_6\rangle = 2.$$

Consider now the sublattice H of the Mordell-Weil lattice $E(K)$ generated by the above set $\{P_8, P_3, P_5, P_1, P_6, P_7\}$. This sublattice is of finite index in $E(K)$ and it has the following Gram matrix:

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{5}{4} & 1 & \frac{3}{4} & 1 \\ 1 & 1 & 2 & 1 & 1 \\ \frac{1}{2} & \frac{3}{4} & 1 & \frac{5}{4} & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Since $\det A = \dfrac{1}{4}$, from (2) and [7, Theorem 10.4 (i)], it follows that this sublattice is actually the full Mordell-Weil group $E(K)$. □

Construction 2. Let Q be an irreducible conic in $\mathbb{P}^2$, and let l be a line. Set $C_1 := Q \cup l$. Let l_1, l_2 be two lines such that $l_1 \cup l_2$ meets C_1 in six distinct points.

Set $l_1 \cap C_1 = \{p_0, p_1, p_2\}$ and $l_2 \cap C_1 = \{p_3, p_4, p_5\}$. Choose a line l_3 such that: *i)* $l_1 \cup l_2 \cup l_3$ meets C_1 in nine distinct points and put $l_3 \cap C_1 = \{p_6, p_7, p_8\}$; *ii)* p_i, for $i = 6, 7, 8$, does not belong to any other line (conic) through two (four or five) points of the set $\{p_0, \ldots, p_5\}$. Put $C_2 = l_1 \cup l_2 \cup l_3$.

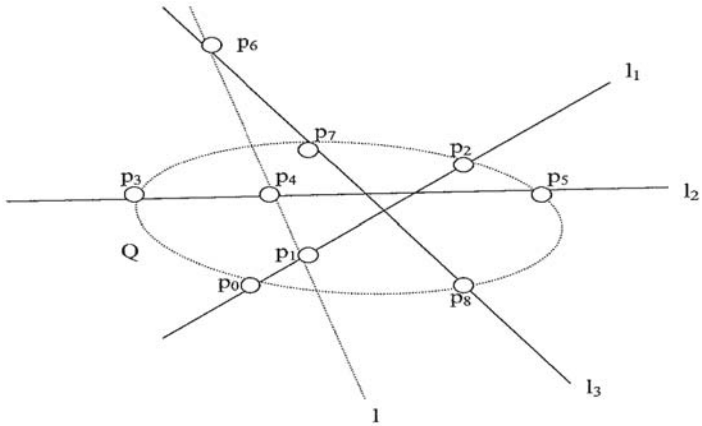

Fig. 2. – Construction 2.

It turns out that the linear pencil of cubic curves generated by C_1 and C_2 determines a RES with only two reducible fibers: the first one of type I_2, (if l meets Q in two distinct points), or of type III (if l is tangent to Q), and the second one of type I_3, (if $l_1 \cap l_2 \cap l_3 = \emptyset$), or of type IV, (if the three lines $l_i, i = 1, 2, 3$, have a common point), in Kodaira's classification. To have a system of generators, take $\{P_0, P_3, P_1, P_6, P_7, P_8\}$, where P_0 is the zero section. The *height pairing* of this set is computed as follows:

$$\langle P_3, P_3\rangle = \frac{4}{3}, \langle P_3, P_1\rangle = 1, \langle P_3, P_7\rangle = \frac{2}{3}, \langle P_3, P_8\rangle = \frac{2}{3}, \langle P_3, P_6\rangle = \frac{2}{3},$$

$$\langle P_1, P_1\rangle = \frac{3}{2}, \langle P_1, P_7\rangle = 1, \langle P_1, P_8\rangle = 1, \langle P_1, P_6\rangle = \frac{1}{2}, \langle P_7, P_7\rangle = \frac{4}{3},$$

$$\langle P_7, P_8\rangle = \frac{1}{3}, \langle P_7, P_6\rangle = \frac{1}{3}, \langle P_8, P_8\rangle = \frac{4}{3}, \langle P_8, P_6\rangle = \frac{1}{3}, \langle P_6, P_6\rangle = \frac{5}{6}.$$

The determinant of the corresponding Gram matrix is $\frac{1}{6}$ and as in Construction 1, it follows that the set $\{P_0, P_3, P_1, P_7, P_8\}$ generates all the Mordell-Weil group $E(K)$. □

Construction 3. Let Q_1, Q_2 be two conics meeting in four distinct points $\{p_0, p_1, p_2, p_3\}$. Consider now two lines l_1 and l_2 such that: (i) the intersection points $l_1 \cap Q_2 = \{p_4, p_5\}$ and $l_2 \cap Q_1 = \{p_6, p_7\}$ are in general position; (ii) the point $p_8 = l_1 \cap l_2$ is collinear to p_1 and p_2. Put $C_i = l_i \cup Q_i$ for $i = 1, 2$. Then the linear pencil of C_1 and C_2 has only three reducible cubic curves which are union of a line and a conic.

Thus the rational elliptic surface S, obtained by the blowing-up of $\mathbb{P}^2$ at these nine points p_i for $i = 0, \ldots, 8$, has exactly three reducible fibers: these are the proper transforms of the three reducible cubics; moreover, these are either of type I_2, (if the line and the conic in $\mathbb{P}^2$ meets in two points), or of type III, (if the line is tangent to the conic), in Kodaira's classification. To find a system of generators, with the same notation as before, take P_0 as zero section, and the points $\{P_8, P_6, P_4, P_7, P_1\}$. It turns out that:

$$\langle P_1, P_1\rangle = \langle P_4, P_4\rangle = \langle P_6, P_6\rangle = \langle P_7, P_7\rangle = \frac{3}{2}, \ \langle P_8, P_8\rangle = \frac{1}{2},$$

$$\langle P_1, P_8\rangle = \langle P_4, P_8\rangle = \langle P_6, P_7\rangle = \langle P_6, P_8\rangle = \langle P_7, P_8\rangle = \frac{1}{2},$$

$$\langle P_1, P_6\rangle = \langle P_1, P_4\rangle = \langle P_1, P_7\rangle = \langle P_4, P_6\rangle = \langle P_4, P_7\rangle = 1.$$

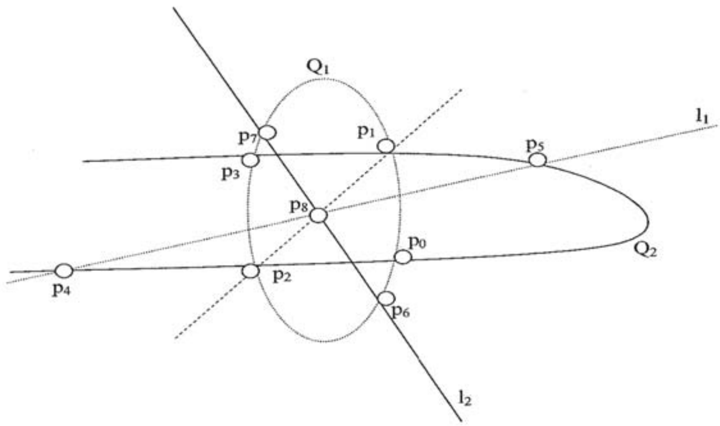

Fig. 3. – Construction 3.

Since the determinant of the corresponding Gram matrix is $\frac{1}{8}$, these points $\{P_8, P_6, P_4, P_7, P_1\}$ generate all the Mordell-Weil group $E(K)$. □

3. – RES with MW group of rank $r = 5$.

In this section, we prove that any RES with MW group of rank five can be obtained from one of the three constructions of the above section via Cremona transformations. In particular, we get the following

THEOREM 3.1. – *Let S be a rational elliptic surface of Mordell-Weil group $E(K)$ of rank 5. Then S arises from a suitable linear pencil (l.p.) Λ of cubic curves in $\mathbb{P}^2$. More precisely, up to Cremona transformations, we have the following possibilities:*

(A) if $E(K) \cong D_5^*$, then S rises from a l.p. Λ as in Construction 1;

(B) if $E(K) \cong A_5^*$, then S rises from a l.p. Λ as in Construction 2;

(C) if $E(K) \cong D_4^* \oplus A_1^*$, then S rises from a l.p. Λ as in Construction 3.

PROOF. – First of all, from [5] and [3, pp. 202-203], we know that S can be seen as $\mathbb{P}^2$ blown-up at nine points $\widehat{p}_i, i = 0, \dots, 8$. Moreover, by [3, (2.9)] we have the following two properties:

(i) a curve on S is a section if and only if it is a (-1)-curve;
(ii) there are no irreducible curves C on S with $C^2 < 0$ other than the sections ($C^2 = -1$) and the components of the reducible fibers ($C^2 = -2$).

Let $\sigma : S \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at the nine points $\widehat{p}_i, i = 0, \dots, 8$ and denote by Λ the linear pencil of cubic curves in $\mathbb{P}^2$ given by the images of the fibers F of the elliptic fibration $f : S \to \mathbb{P}^1$ by the map $\sigma : S \to \mathbb{P}^2$.

Having in mind the table of Theorem 1.1 and the above properties (i) and (ii), we proceed now with a case-by-case analysis.

NO. 1 OF THM. (1.1). – Since $E(K) \cong D_5^*$, it follows from the work of Oguiso and Shioda that S has only one reducible fiber $\widehat{F}$ of type I_4 in Kodaira's classification, that is, $\widehat{F} = L_1 \cup L_2 \cup L_3 \cup L_4$ such that $L_i^2 = -2$ for $i = 1, \dots, 4$, $L_1 \cdot L_2 = L_3 \cdot L_4 = 0$ and $L_1 \cdot L_3 = L_1 \cdot L_4 = L_2 \cdot L_3 = L_2 \cdot L_4 = 1$. Thus, after renumbering, we have the following possibilities for the nine base points $\widehat{p}_i$:

(A_1) all the $\widehat{p}_i$'s are distinct points;
(A_2) $\widehat{p}_8$ is infinitely near to $\widehat{p}_0$;
(A_3) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$ and $\widehat{p}_1$ respectively;
(A_4) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$;
(A_5) $\widehat{p}_6, \widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$.

Since at least one of the four components L_i of the unique reducible fiber $\widehat{F}$ must be contracted, Case (A_1) cannot occur.

In Case (A_2), Λ has eight base points, one of which is a double point. Moreover, Λ has only one reducible member which is the union of three lines. This proves that Λ is as in Construction 1.

In Case (A_3), Λ has seven base points, two of which are double points, say $\widehat{p}_0$ and $\widehat{p}_1$. Moreover, all the members of the pencil are smooth at its base points except for the image $\widehat{C}$ on $\mathbb{P}^2$ of the reducible fiber $\widehat{F}$ on S, which is singular at the two points $\widehat{p}_0$ and $\widehat{p}_1$.

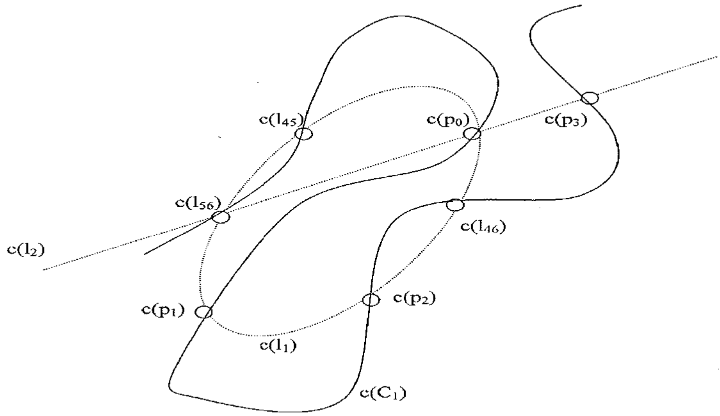

Fig. 4. – Case (A_3), where $\widehat{l} = \mathrm{c}(l_2)$, $\widehat{Q} = \mathrm{c}(l_1)$, $\widehat{p}_0 = \mathrm{c}(p_0)$ and $\widehat{p}_1 = \mathrm{c}(l_{56})$.

Note that, unless for the reducible member $\widehat{C}$ of Λ, which is the union of a line $\widehat{l}$ and a conic $\widehat{Q}$ such that $\widehat{l} \cap \widehat{Q} = \{\widehat{p}_0, \widehat{p}_1\}$, all the cubic curves have the same tangents at $\widehat{p}_0$ and $\widehat{p}_1$. From now on, let us denote by $l_{u_i u_j} := l_{ij}$ the line through the points u_i and u_j. Therefore, it is possible to obtain such a Λ (see, e.g., Figure 4) from a pencil as in Construction 1 via a Cremona transformation c: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points p_4, p_5, p_6 of Figure 1.

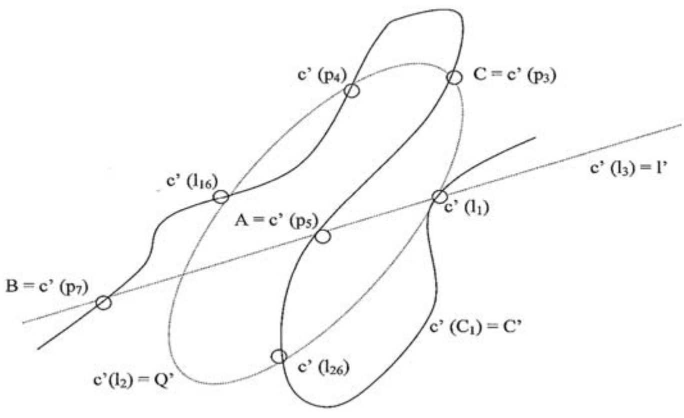

Fig. 5. – Case (A_4), where $\widehat{l} = l'$, $\widehat{Q} = Q'$ and $\widehat{p}_0 = c'(l_{4p})$.

In Case (A_4), Λ has seven base points, one of which, say $\widehat{p}_0$, is a triple point, that is, any two members in Λ have intersection multiplicity equal to three at $\widehat{p}_0$. All the members of the pencil are smooth at its base points except the image $\widehat{C}$ on $\mathbb{P}^2$ of the reducible fiber $\widehat{F}$ on S which is singular at the triple point. Note that $\widehat{C}$ is a cubic of Λ which splits into a line $\widehat{l}$ and a conic $\widehat{Q}$. Then $\widehat{p}_0 \in \widehat{l} \cap \widehat{Q}$ and since $\widehat{p}_0$ is a triple point, either $\widehat{l}$ or the tangent line t of $\widehat{Q}$ at $\widehat{p}_0$ is the common tangent to all the members of Λ.

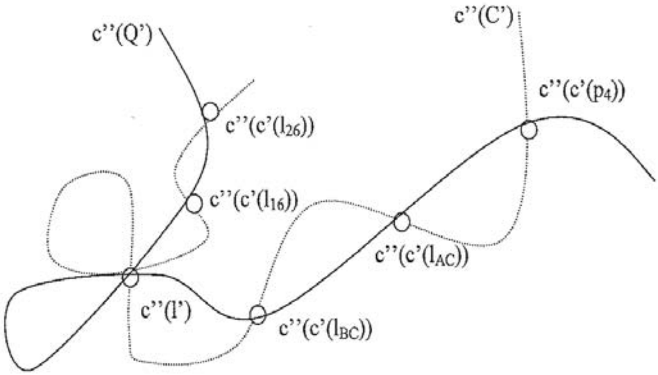

Fig. 6. – Case (A_5), where $\widehat{C}$ = c"(Q') and $\widehat{p}_0$ = c"(l').

Thus the pencil Λ (see Figure 5) can be obtained from a pencil as in Construction 1 (see Figure 1) by a Cremona transformation c': $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points p_1, p_2, p_6.

Finally, in Case (A_5), Λ has six base points, one of which, say $\widehat{p}_0$, is a point of multiplicity four, that is, any two members in Λ have intersection multiplicity equal to four at $\widehat{p}_0$. The image on $\mathbb{P}^2$ of the reducible fiber $\widehat{F}$ on S is a nodal rational curve $\widehat{C}$ with the node in $\widehat{p}_0$; moreover, except for $\widehat{C}$, all the cubic curves of Λ have either the same tangents t, t' of $\widehat{C}$ at $\widehat{p}_0$, or one of them, say t, is a common tangent with suitable multiplicity. Thus the pencil Λ (see Figure 6) comes from a linear pencil as in Case (A_4) via a Cremona transformation c": $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points c'$(p_5) = A$, c'$(p_7) = B$, c'$(p_3) = C$ of Figure 5. Summing up, we deduce that Λ can be obtained from a pencil as in Construction 1 via a composition of the two Cremona transformations c' and c".

This completes Case (A) of the statement.

No. 2 of Thm. (1.1). – Since $E(K) \cong A_5^*$, it follows from the work of Oguiso and Shioda that S has two reducible fibers: the first one comes from the union of a line and a conic, while the second one comes from the union of three lines. Thus, after renaming, we have the following possibilities for the nine base points of $\sigma : S \to \mathbb{P}^2$:

(B_1) all the $\widehat{p}_i$'s are distinct points;

(B_2) $\widehat{p}_8$ is infinitely near to $\widehat{p}_0$;

(B_3) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$;

(B_4) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$ and $\widehat{p}_1$ respectively;

(B_5) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$, and $\widehat{p}_6$ is infinitely near to $\widehat{p}_1$.

In case (B_1) the images of the fibers F of $f : S \to \mathbb{P}^1$ give rise to a pencil Λ of cubic curves with nine distinct base points. Since in this case no (-2)-curves of S are contracted by $\sigma : S \to \mathbb{P}^2$, the pencil Λ has only two reducible members, say C_2 (union of three lines) and C_1 (union of a conic Q and a line l). Note that any section of S does not meet the fibers at singularities. Therefore, all the members of Λ are smooth at the nine base points, and since C_2 have to contain the nine base points, the pencil Λ is forced to be as in Construction 2 (see Figure 2).

Case (B_2). We can have two possible configurations for the images of the fibers of the elliptic fibration: $i)$ Λ_1 is a pencil of cubic curves with a reducible member, say $\widetilde{C_1}$ (union of three lines), and an irreducible cubic, say $\widetilde{C_2}$, singular at the base point $\widehat{p}_0$ of the pencil; $ii)$ Λ_2 is a pencil with a reducible member, say D_1 (union of a conic and a line), smooth at the base points of Λ and a reducible member, say D_2 (union of a conic and a line), which is singular at the base point $\widehat{p}_0$.

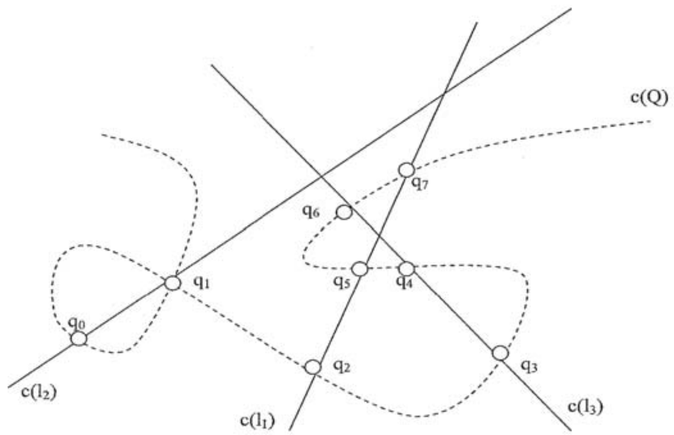

Fig. 7. – Case $(B_2)i)$, where $q_0 = \mathfrak{c}(p_3)$, $q_1 = \mathfrak{c}(p_4) = \mathfrak{c}(l)$, $q_2 = \mathfrak{c}(p_0)$, $q_3 = \mathfrak{c}(p_8)$, $q_4 = \mathfrak{c}(l_{15})$, $q_5 = \mathfrak{c}(p_2)$, $q_6 = \mathfrak{c}(p_7)$, $q_7 = \mathfrak{c}(l_{56})$.

In case $i)$, the pencil Λ_1 can be obtained from that of Construction 2 (see Figure 2) via a Cremona transformation $\mathfrak{c}$: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points p_1, p_5, p_6 (see Figure 7).

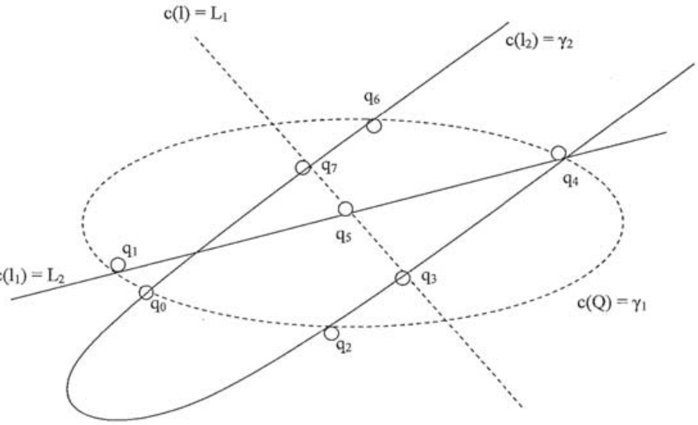

Fig. 8. – Case $(B_2)ii)$, where $q_0 = \mathrm{c}(l_{06})$, $q_1 = \mathrm{c}(p_2)$, $q_2 = \mathrm{c}(p_3)$, $q_3 = \mathrm{c}(l_{08})$, $q_4 = \mathrm{c}(p_7)$, $q_5 = \mathrm{c}(p_1)$, $q_6 = \mathrm{c}(p_5)$, $q_7 = \mathrm{c}(p_4)$.

In case $ii)$, the pencil Λ_2 can be obtained from Construction 2 (see Figure 2) by a Cremona transformation c: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points p_0, p_6, p_8 (see Figure 8).

In Case (B_3), the pencil Λ has seven base points, one of which, say $\widehat{p}_0$, is a triple point. Moreover, Λ is a pencil with a reducible member, say C_1' (union of a conic Q_1' and a line l_1'), smooth at all the base points $\widehat{p}_i$ of Λ, and an irreducible member, say C_2', singular at the base point $\widehat{p}_0$.

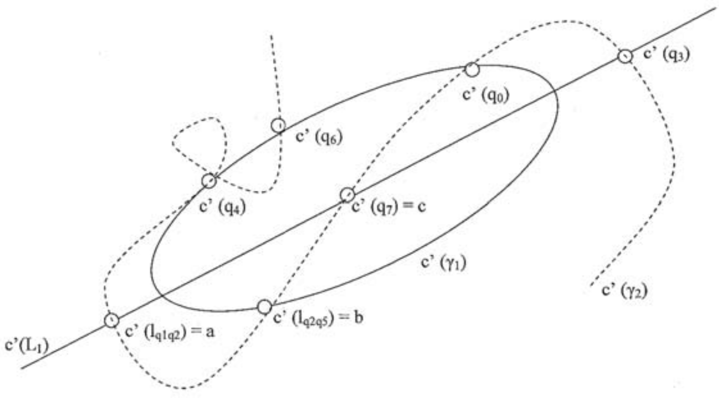

Fig. 9. – Case (B_3), where $Q_1' = \mathrm{c}'(\gamma_1)$, $l_1' = \mathrm{c}'(L_1)$, $C_2' = \mathrm{c}'(\gamma_2)$ and $\widehat{p}_0 = \mathrm{c}'(q_4)$.

Such a pencil Λ (see Figure 9) can be obtained from the pencil of Case $(B_2)ii)$ by a Cremona transformation c': $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with, e.g., the base points q_1, q_2, q_5 of Figure 8. Thus by a composition of two Cremona transformations c and c' we see that Λ can be obtained from Construction 2.

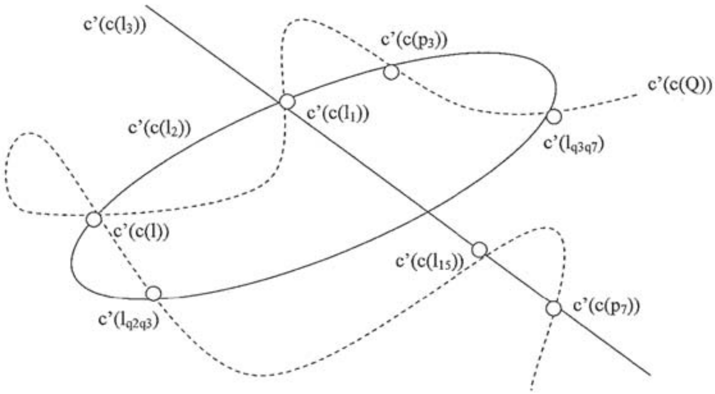

Fig. 10. – Case (B_4), where $Q_1^{'}=$ c'$(c(l_2))$, $l_1^{'}=$ c'$(c(l_3))$, $C_2^{'}=$ c'$(c(Q))$, $\widehat{p}_0=$ c'$(c(l_1))$, $\widehat{p}_1=$ c'$(c(l))$.

Case (B_4). In this case the images of the fibers F of the elliptic fibration $f: S \to \mathbb{P}^1$ give rise to a pencil Λ of cubic curves, with only one reducible member, say $C_1^{'}$ (union of a conic $Q_1^{'}$ and a line $l_1^{'}$), singular at one, say $\widehat{p}_0$, of the base points of Λ and an irreducible member, say $C_2^{'}$, which is singular at $\widehat{p}_1$. In this case the pencil has seven base points, counted without multiplicity. Then such a pencil Λ can be obtained from the pencil of Case $(B_2)(i)$ (see Figure 7) by the Cremona transformation c': $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with, e.g., base points q_2, q_3, q_7. Thus the pencil Λ can be obtained from the pencil of Construction 2 by the composition of the two Cremona transformations c and c' (see Figure 10).

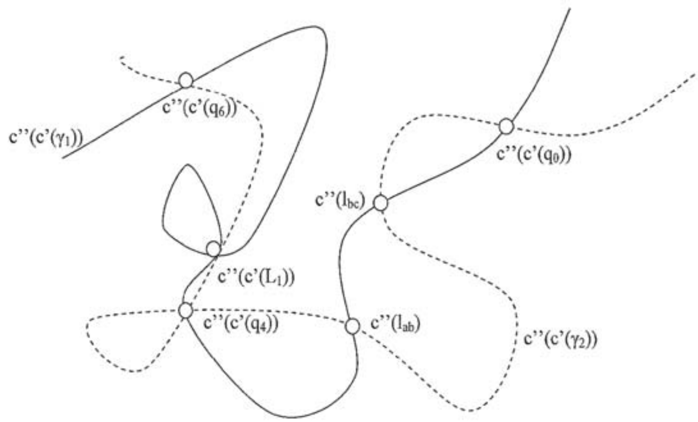

Fig. 11. – Case (B_5), where $\widehat{C_1}=$ c"(c'(γ_2)), $\widehat{C_2}=$ c"(c'(γ_1)), $\widehat{p}_0=$ c"(c'(L_1)), $\widehat{p}_1=$ c"(c'(q_4)).

Case (B_5). In this case the images of the fibers F of the elliptic fibration $f: S \to \mathbb{P}^1$ give rise to a pencil Λ of cubic curves whose elements are all irreducible.

Such a pencil has two special members: the first, say $\widehat{C_1}$, is singular at $\widehat{p}_1$, and it meets all the other members of the pencil at $\widehat{p}_1$ with multiplicity 2; the second one, say $\widehat{C_2}$, is singular at $\widehat{p}_0$, and it meets all other members of the pencil with multiplicity 3 at $\widehat{p}_0$. In this case Λ has six base points counted without multiplicity and it can be obtained (see Figure 11) from a pencil as in Case (B_3) by a Cremona transformation c": $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with, e.g., base points c'($l_{q_1q_2}$) $= a$, c'($l_{q_2q_5}$) $= b$, c'(q_7) $= c$ (see Figure 9). Therefore, by a composition of three Cremona transformation (c, c' and c"), the pencil Λ can be obtained from a pencil as in Construction 2.

This leads to Case (B) of the statement.

NO. 3 OF THM. (1.1). – Since $E(K) \cong D_4^* \oplus A_1^*$, from the work of Oguiso and Shioda we deduce that S has three reducible fibers F_i, $i = 1, 2, 3$, which come from the union of a line and a conic. Note that in this case there are at most three (-2)-curves mutually disjoint. Moreover, the components of each reducible fiber are non-singular with intersection in two points (possibly infinitely near). Therefore, after renumbering, we get the following possibilities for the nine base points:

(C_1) all the $\widehat{p}_i$'s are distinct points;
(C_2) $\widehat{p}_8$ is infinitely near to $\widehat{p}_0$;
(C_3) $\widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0$ and $\widehat{p}_1$ respectively;
(C_4) $\widehat{p}_6, \widehat{p}_7$ and $\widehat{p}_8$ are infinitely near to $\widehat{p}_0, \widehat{p}_1$ and $\widehat{p}_2$ respectively.

In Case (C_1), we see that Λ has nine base points and it contains only three reducible members $C_i = l_i \cup Q_i$, where l_i is a line and Q_i is an irreducible conic for $i = 1, 2, 3$. The points $\widehat{p}_i$ of the blowing-up are the base points of the pencil Λ. Moreover, the points where l_i meets Q_i are disjoint from all the $\widehat{p}_j$'s, since the sections do not intersect the fibers at singularities. Finally, note that we can have only four cases for the reducible fibers F_i, $i = 1, 2, 3$, on S and the corresponding members $l_i \cup Q_i$ of Λ: (a) F_i are all of type I_2 and l_i meets Q_i in two distinct points for $i = 1, 2, 3$; (b) after renaming, F_1 is of type I_2 with l_1 meeting Q_1 in two distinct points and the other fibers F_i are of type III with l_i tangent to Q_i for $i = 2, 3$; (c) after renaming, F_1 is of type III with l_1 tangent to Q_1 and the other fibers F_i are of type I_2 with l_i meeting Q_i in two distinct points for $i = 2, 3$; (d) F_i are all of type III and l_i is tangent to Q_i for $i = 1, 2, 3$. This proves that Λ is as in Construction 3.

In Case (C_2), we have that Λ has eight base points, one of which, say $\widehat{p}_0$, is a double point. In this situation, if $F_1 = \Theta_0 \cup \Theta_1$ is a reducible fiber, one component of F_1, say Θ_0, is contracted to a point giving as image either (j) a rational cubic curve with a node in $\widehat{p}_0$ if Θ_0 and Θ_1 meet transversally, or (jj) a rational curve with a cusp at $\widehat{p}_0$ if Θ_0 and Θ_1 are tangent. Note that we can obtain (j) from (a), (b) or (c) of Case (C_1) (see, e.g., Figure 12) by the Cremona transformation c: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose base points are p_1, p_3, p_8 as in Construction 3, where l_1 and Q_1

meet in two distinct points (see Figure 3). By a similar argument, it is not difficult to see that (jj) comes from (b), (c) or (d) of Case (C_1) by a Cremona transformation whose base points are p_1, p_3, p_8 as in Construction 3 (see Figure 3), assuming now that the line l_1 is tangent to the conic Q_1.

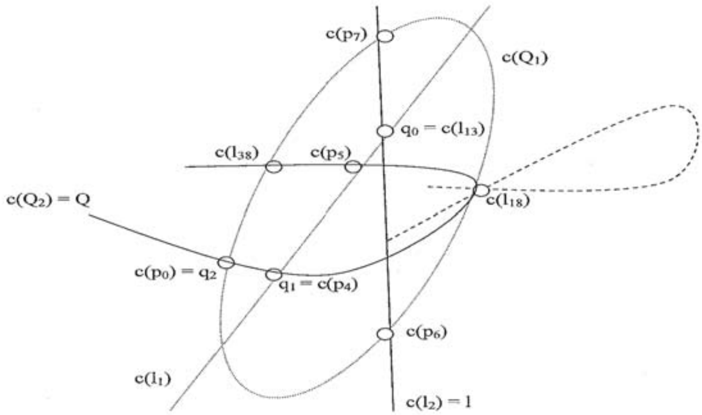

Fig. 12. – Case $(C_2)(j)$, where $\widehat{p}_0 = c(l_{18})$.

In Case (C_3), Λ has seven base points, two of which, say $\widehat{p}_0$ and $\widehat{p}_1$, are distinct double points on $\mathbb{P}^2$. In this situation, only one component of two distinct reducible fibers, say $F_1 = \Theta_0^{(1)} \cup \Theta_1^{(1)}$ and $F_2 = \Theta_0^{(2)} \cup \Theta_1^{(2)}$, is contracted to a point giving as image in $\mathbb{P}^2$ for each fiber F_i, $i = 1, 2$, either a rational cubic curve with a node at $\widehat{p}_0$ (or $\widehat{p}_1$) if $\Theta_0^{(i)}$ and $\Theta_1^{(i)}$ meet transversally, or a rational curve with a cusp at $\widehat{p}_0$ (or $\widehat{p}_1$) if $\Theta_0^{(i)}$ and $\Theta_1^{(i)}$ are tangent. Also in this case, one can easily see that all the configurations obtained come from those of Case (C_2) by a suitable Cremona transformation.

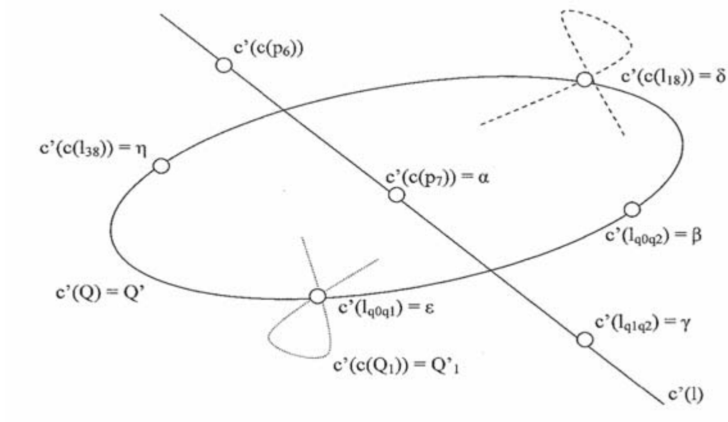

Fig. 13. – A situation of Case (C_3) in which F_1 and F_2 have as image in $\mathbb{P}^2$ two rational cubic curves with a node at $\widehat{p}_0 = \delta$ and $\widehat{p}_1 = \varepsilon$ respectively.

E.g., the situation of Case (C_3) in which F_1 and F_2 have as image in $\mathbb{P}^2$ a rational cubic curve with a node at $\widehat{p}_0$ and $\widehat{p}_1$ respectively (see Figure 13), can be obtained from the pencil of Figure 12 by a Cremona transformation c': $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points q_0, q_1, q_2. Thus, by the composition of two Cremona transformations c and c', it is possible to obtain Λ from a pencil as in Construction 3. Therefore, in a similar way, one can see that it is possible to obtain all the situations of Case (C_3) from the cases (a), (b), (c) and (d) of (C_1) by the composition of two Cremona transformations.

Finally, in Case (C_4), Λ has six base points, three of which, say $\widehat{p}_i$ for $i = 0, 1, 2$, are distinct double points on $\mathbb{P}^2$. Note that for all the three reducible fibers $F_i = \Theta_0^i \cup \Theta_1^i (i = 1, 2, 3)$ of S, only one component is contracted to a point giving as image in $\mathbb{P}^2$ for each fiber F_i a rational cubic curve with a node in $\widehat{p}_{i-1}$ if Θ_0^i and Θ_1^i meet transversally, or a rational curve with a cusp at $\widehat{p}_{i-1}$ if Θ_0^i and Θ_1^i are tangent for $i = 1, 2, 3$.

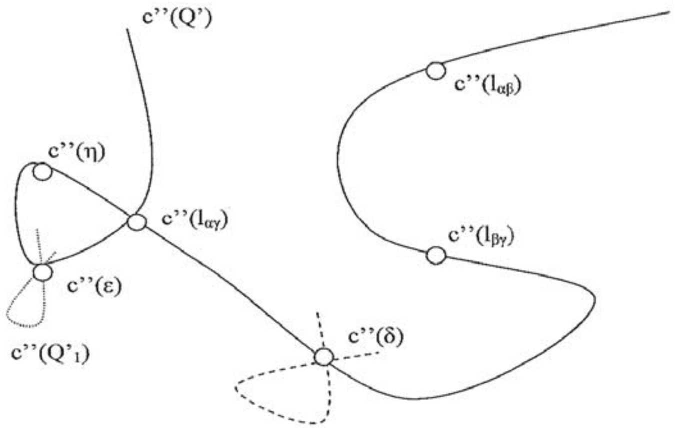

Fig. 14. – A situation of Case (C_4) in which all the F_i ($i = 1, 2, 3$) have as image in $\mathbb{P}^2$ a rational cubic curve with a node at $\widehat{p}_0 =$ c"(ε), $\widehat{p}_1 =$ c"(δ) and $\widehat{p}_2 =$ c"($l_{\alpha\gamma}$) respectively.

In this situation, one can see that all the configurations obtained come from those of Case (C_3) via a Cremona transformation. E.g., the case of (C_4) in which all the F_i ($i = 1, 2, 3$) have as image in $\mathbb{P}^2$ a rational cubic curve with a node at $\widehat{p}_{i-1}$ (see Figure 14), can be obtained from the pencil of Figure 13 by a Cremona transformation c": $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base points α, β, γ. So we can conclude that by the composition of three suitable Cremona transformations c, c' and c", it is possible to obtain all the pencils Λ of Case (C_4) from a pencil as in (a), (b), (c) and (d) of Case (C_1).

This gives Case (C) of the statement. □

4. – Appendix to the case $r = 8$.

We know that a Rational Elliptic Surface (RES) with Mordell-Weil (MW) group of rank eight arises from a linear pencil of cubic curves on $\mathbb{P}^2$ supported on nine distinct points in general position (that is, no three on a line and no six on a conic). It is well known, see [4] and [7], that if we choose one of them as zero section (in the sense of the point in $E(K)$ corresponding to the exceptional curve of the blown-up at the point on $\mathbb{P}^2$), the group generated by the other eight is of index three in $E(K)$. On the other hand, the geometry of the base points of the pencil gives us a great number of points in the Mordell-Weil group for which is really simple to compute the *height pairing*. Such points are the proper transform of the lines through two of the nine base points (these are exactly 36), and the proper transform of the conic through five of the nine base points of the pencil (these are exactly 126). It turns out that for a suitable choice of the points, it is quite easy to find a system of generators for $E(K)$. In what follows, we give a proof of this fact. With abuse of notation, we do not distinguish between exceptional curves on S and points in $E(K)$.

PROPOSITION 4.1. – *Let S be the RES arises from a linear pencil of cubic curves supported on nine distinct points in general position on $\mathbb{P}^2$, say p_i for $i = 0, .., 8$. Set P_i the exceptional curve of the blown-up at p_i. Fix P_0 as zero in $E(K)$. Let l be the proper transform of the line through p_0 and p_7. Then the set $\{P_1, \dots, P_7, l\}$ is a system of generators for the MW group $E(K)$.*

PROOF. – The *heigh pairing* of the set $\{P_1, \dots, P_7, l\}$ is computed as follows:

$$\langle P_i, P_j \rangle = \begin{cases} 2, & i = j; \\ 1, & i \neq j. \end{cases} \langle l, P_i \rangle = \begin{cases} 2, & \text{for } i = 1, .., 6; \\ 1, & i = 7. \end{cases} \langle l, l \rangle = 4.$$

The sublattice generated by $\{P_1, \dots, P_8\}$ with $l = P_8$ is of finite index in $E(K)$ and it has the Gram matrix $A = \big(\langle P_i, P_j \rangle\big)_{i,j=1,\dots,8}$. Since $\det A = 1$, from [7, Theorem 10.4 (i)], and the equation (2), it follows that this sublattice is actually the full Mordell-Weil group $E(K)$. □

REMARK 4.1. – By Theorem 3.1, Proposition 4.1 and the results of [3], we have a complete description of all RES with MW group of rank $r \geq 5$.

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