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Abstract. – *Consider a M-player game in strategic form* $G = (X_1, \ldots, X_M, g_1, \ldots, g_M)$ *where the set* X_i *is a closed interval of real numbers and the payoff function* g_i *is concave and differentiable with respect to the variable* $x_i \in X_i$, *for any* $i = 1, \ldots, M$. *The aim of this paper is to find appropriate conditions on the payoff functions under which the well-posedness with respect to the related variational inequality is equivalent to the formulation of the Tykhonov well-posedness in a game context. The idea of the proof is to appeal to a third equivalence, which is the well-posedness of an appropriate minimum problem.*

1. – Introduction.

Consider a *M-player* game $G = (X_1, X_2, \ldots, X_M, g_1, g_2, \ldots, g_M)$ where, for any fixed $i \in PL = \{1, 2, 3, \ldots, M\}$, the strategy space X_i is a nonempty closed interval of real numbers.

Let $X = \prod_{i=1}^{M} X_i$ be the set of multistrategies and $x = (x_1, x_2, \ldots, x_M) \in X$.

The behavior of the *i-th* player is described by the action on X of the function g_i which measures his payoff, associating with any multistrategy x a real number $g_i(x)$.

Denote by $-i$ the set $PL - \{i\}$. Therefore, $X = X_{-i} \times X_i$ with $X_{-i} = \prod_{j \neq i} X_j$ and for any $x \in X$ then $x = (x_{-i}, x_i)$ where $x_{-i} \in X_{-i}$ and $x_i \in X_i$.

Consider $u = (u_1, u_2, \ldots, u_M) \in \boldsymbol{R}^M$, the norm $|u| = \sum_{i=1}^{M} |u_i|$ and the usual scalar product $\langle \cdot, \cdot \rangle_M$.

In this paper, for any $i \in PL$, assume:

(1) The property of continuity on X for the function g_i;

(2) The property of concavity for the map $u \mapsto g_i(x_{-i}, u)$;

(3) The existence of the derivative $\dfrac{\partial g_i(x)}{\partial x_i}$, for any $x \in X$;

(4) The existence of a positive constant H such that $g_i(x) \leq H$, for any $x \in X$.

Let note $F = \left(\frac{\partial g_1}{\partial x_1}, \frac{\partial g_2}{\partial x_2}, \ldots, \frac{\partial g_M}{\partial x_M}\right)$ the operator acting on X such that

$$F(x) = \left(\frac{\partial g_1(x)}{\partial x_1}, \frac{\partial g_2(x)}{\partial x_2}, \ldots, \frac{\partial g_M(x)}{\partial x_M}\right) \in \boldsymbol{R}^M.$$

and write the variational inequality related to the game G as follows:

$$\text{(1.1)} \qquad \langle F(x), u - x \rangle_M = \sum_{i=1}^{M} \frac{\partial g_i(x)}{\partial x_i}(u_i - x_i) \leq 0, \text{ for any } u \in X.$$

See [9] and [2] for a complete presentation of variational inequalities and [7] for the formulation in a game context.

Remember that a multistrategy w is a Nash equilibrium for the game G when the following two conditions occur:

(1) $w \in X$;
(2) $g_i(w) \geq g_i(w_{-i}, u_i)$, for any $u_i \in X_i$ and $i \in PL$.

Note $NE(G)$ the set of the Nash equilibria for the game G.

From now on suppose $NE(G)$ nonempty and remember that because of the assumptions expressed above, $w \in NE(G)$ if and only if w is a solution of variational inequality (1.1). See [7] and [11].

For any $\varepsilon > 0$ define the set:

$$\text{(1.2)} \qquad T(\varepsilon) = \{x \in X / \langle F(x), u - x \rangle_M \leq \varepsilon |u - x|\,, \text{ for any } u \in X\}$$

and denote the diameter with $\operatorname{diam} T(\varepsilon)$.

DEFINITION 1.1. – *The well-posedness of the game G with respect to the variational inequality* (1.1) (**VI-wp**) *is defined as follows:*

(1) $T(\varepsilon) \neq \emptyset$;
(2) $\lim_{\varepsilon \to 0} \operatorname{diam} T(\varepsilon) = 0$.

See [11] for the above definition and [8] for the first formulation of well-posedness with respect to the variational inequality.

In order to extend the concept of Tykhonov well-posedness in a game context, let us recall the classical formulation.

Let Y be a set endowed with a convergence structure and $\phi: Y \to \boldsymbol{R}$ be a function. Consider the problem of minimizing $\phi(x)$ subject to $x \in Y$. Denote this problem with (Y, ϕ).

DEFINITION 1.2. – *The Tykhonov well-posedness of the minimum problem* (Y, ϕ) *is defined as follows:*

(1) *There exists a* $x^* \in Y$ *such that* $\phi(x^*) \leq \phi(x)$, *for all* $x \in Y$;
(2) *for any sequence* $x_n \in Y$ *with* $\lim_{n \to +\infty} \phi(x_n) = \phi(x^*)$, *then* $\lim_{n \to +\infty} x_n = x^*$.

See [13] and [5] for an in-depth on well-posedness.

For any $\varepsilon > 0$, define the set:

$$L_\phi(\varepsilon) = \{x \in Y / \phi(x) \leq \inf_{u \in Y} \phi(u) + \varepsilon\}$$

and note the diameter with $\operatorname{diam} L_\phi(\varepsilon)$.

According to Furi and Vignoli [6] the following statements are true:

(1) The Tykhonov well-posedness of (Y, ϕ) implies $\lim_{\varepsilon \to 0} \operatorname{diam} L_\phi(\varepsilon) = 0$.

(2) The sequentially lower semicontinuity and the lower boundedness of ϕ, the completeness of Y and the fact that $\lim_{\varepsilon \to 0} \operatorname{diam} L_\phi(\varepsilon) = 0$ imply the Tykhonov well-posedness of (Y, ϕ).

Remember that a sequence $x_n \in X$ is an asymptotically Nash equilibrium (**a-NE(G)**) if $\sup_{x_i \in X_i} \{g_i((x_{-i})_n, x_i) - g_i(x_n)\} \to 0$, for any $i \in PL$.

We are now ready to formulate the concept of Tykhonov well-posedness in a game context.

DEFINITION 1.3. – *The Tykhonov well-posedness* (**T-wp**) *of a game G is defined as follows:*

(1) $NE(G) = \{x^*\}$;
(2) *every* **a-NE(G)** *sequence converges to* x^*.

See [4] and [10].

Relate to the game G the function $\beta : X \to [0, +\infty)$ defined as follows:

$$\beta(x) = \sum_{i=1}^{M} (l_i(x) - g_i(x)), \text{ where } l_i(x) \equiv \sup_{u \in X_i} g_i(x_{-i}, u).$$

For any $\varepsilon > 0$ let $L_\beta(\varepsilon) = \{x \in X / \beta(x) \leq \varepsilon\}$.

The following proposition states a characterization of Tykhonov well-posedness in a game context.

PROPOSITION 1. – *Let* $NE(G)$ *be nonempty. The following facts are equivalent:*

(1) *the game G has the property* **T-wp**;
(2) *the problem* (X, β) *has the property of Tykhonov well-posedness;*
(3) $\lim_{\varepsilon \to 0} \operatorname{diam} L_\beta(\varepsilon) = 0$.

PROOF. – By Proposition 3.1 in [11], β is lower semicontinuous, $\inf_{x\in X} \beta(x) \geq 0$ and $\beta(w) = 0$ if and only if $w \in NE(G)$.

Therefore:

(1) $\Leftrightarrow$ (2) follows by remark 2.1 in [12];

(2) $\Leftrightarrow$ (3) follows by the Furi-Vignoli result stated above. □

See [1] and [3] to characterize the Nash equilibria of a game with a scalar function.

The following sets of conditions are determinant to establish the field of applicability of our main result. The first set identifies a class of games for which the well-posedness with respect to the variational inequality implies the Tykhonov well-posedness.

CONDITIONS 1. – (1) *There exists* $K > 0$ *such that* $\left|\frac{\partial g_i(x')}{\partial x_i} - \frac{\partial g_i(x'')}{\partial x_i}\right| \leq K|x' - x''|$ *for any* $x', x'' \in X$ *and* $i \in PL$;

(2) *Fix* $i \in PL$, *if* $X_i = (-\infty, b_i]$ *or* $X_i = [a_i, b_i]$ *or* $X_i = [a_i, +\infty)$, *then* $\frac{\partial g_i(x_{-i}, b_i)}{\partial x_i} \leq 0$ *and* $\frac{\partial g_i(x_{-i}, a_i)}{\partial x_i} \geq 0$.

Observe how condition (2) prevents the possibility for any component of the equilibrium to be known a priori.

Instead, the second set identifies a class of games for which the Tyhkonov well-posedness implies the well-posedness with respect to the variational inequality.

CONDITIONS 2. – (3) *For* $i \in PL$ *the function* $x \mapsto \frac{\partial^2 g_i(x)}{\partial (x_i)^2}$ *is well defined on* X;

(4) *There exists a positive number* $\delta > 0$ *such that* $\frac{\partial^2 g_i(x)}{\partial (x_i)^2} < -\delta$, *for any* $x \in X$.

With the aim of clarifying the consistency of these conditions, it is useful to consider the following game which satisfies both sets of conditions but does not have anyone of the well-posedness properties considered.

EXAMPLE 1. – Consider the 2-*player* game $G = (X_1, X_2, g_1, g_2)$ where $X_1 = X_2 = \boldsymbol{R}$ and $g_1(x) = g_1(x_1, x_2) = g_2(x) = g_2(x_1, x_2) = -(x_1 - x_2)^2$. Let us verify that G has both sets of conditions.

The functions g_i act continuously on X and are bounded from above by zero. They are differentiable and concave. As well as

$$\left|\frac{\partial g_1(x')}{\partial x_1} - \frac{\partial g_1(x'')}{\partial x_1}\right| \leq 2 \mid x' - x'' \mid \quad \text{and} \quad \left|\frac{\partial g_2(x')}{\partial x_2} - \frac{\partial g_2(x'')}{\partial x_2}\right| \leq 2 \mid x' - x'' \mid .$$

Since the points $(a, a) \in NE(G)$ for any $a \in \boldsymbol{R}$, then G does not have the **T-wp** property.

Fix $\varepsilon > 0$. The point $(x_1, x_2) \in T(\varepsilon)$ if and only if for any $(u, v) \in \boldsymbol{R}^2$

$$\langle (- 2(x_1 - x_2), 2(x_1 - x_2)), (u, v) - (x_1, x_2)\rangle_2 \leq \varepsilon \mid u - x_1 \mid + \varepsilon \mid v - x_2 \mid .$$

Since the points $(a, a) \in T(\varepsilon)$ for any $a \in \boldsymbol{R}$, then G does not have the **VI-wp** property. □

Now we are ready to state our main result.

THEOREM. – *Under Conditions* 1 *and* 2 *on a game G, the* **VI-wp** *is equivalent to the* **T-wp**.

Section 2 of this paper deals with the proof of: **VI-wp** implies **T-wp**, when the first set of conditions apply. Instead, section 3 deals with the proof of: **T-wp** implies **VI-wp**, when the second set of conditions apply. Section 4 presents a final remark.

2. – VI-wp implies T-wp.

First of all we need to state the Ekeland Optimality Principle, see [1].

EKELAND OPTIMALITY PRINCIPLE. – *Let* $(X, |\cdot|)$ *be a complete metric space, the function* $\vartheta : X \to [0, +\infty)$ *continuous and* λ, μ *positive constants. If for any* $x \in X$

$$\vartheta(x) \leq \inf_{u \in X} \vartheta(u) + \lambda\mu ,$$

then there exists $y \in X$ *such that:* $\vartheta(y) \leq \vartheta(x)$; $|x - y| \leq \lambda$; *and for any* $z \in X$

$$\vartheta(y) \leq \vartheta(z) + \mu|z - y|.$$

Fix x in X and define the function $X \ni v \mapsto \vartheta_x(v)$ as follows:

$$\vartheta_x(v) = - \sum_{j=1}^{M} g_j(x_{-j}, v_j) .$$

LEMMA 2.1. – *Let* x *be in* X *and fix* $\varepsilon > 0$. *The following facts are true:*

$$\beta(x) = \sup_{u \in X} \sum_{i=1}^{M} (g_i(x_{-i}, u_i) - g_i(x)); \tag{2.1}$$

the function ϑ_x *is continuous; if* $\beta(x) \leq \varepsilon$ *then*

$$\vartheta_x(x) \leq \inf_{u \in X} \vartheta_x(u) + \varepsilon. \tag{2.2}$$

PROOF. – By definition $\beta(x) \geq \sum_{i=1}^{M} (g_i(x_{-i}, u_i) - g_i(x))$, for any $u_i \in X_i$ and $i \in PL$. Therefore,

$$\beta(x) \geq \sup_{u \in X} \sum_{i=1}^{M} (g_i(x_{-i}, u_i) - g_i(x)) .$$

Vice versa, fix $\varepsilon > 0$ and for any $i \in PL$ let $v_i \in X_i$ be such that $g_i(x_{-i}, v_i) > l_i - \dfrac{\varepsilon}{M}$. Then

$$\sum_{i=1}^{M} (g_i(x_{-i}, v_i) - g_i(x)) > \sum_{i=1}^{M} (l_i(x) - g_i(x)) - \varepsilon = \beta(x) - \varepsilon .$$

This means that

$$\sup_{u \in X} \sum_{i=1}^{M} (g_i(x_{-i}, u_i) - g_i(x)) > \beta(x) - \varepsilon .$$

By the arbitrariness of ε the result (2.1) is proven.

Let $y = (y_1, y_2, \ldots, y_M)$ and $z = (z_1, z_2, \ldots, z_M)$ be two arbitrary points of X. Then

$$| \vartheta_x(y) - \vartheta_x(z) | = \left| \sum_{i=1}^{M} g_i(x_{-i}, z_i) - \sum_{i=1}^{M} g_i(x_{-i}, y_i) \right| \leq \sum_{i=1}^{M} | g_i(x_{-i}, z_i) - g_i(x_{-i}, y_i) | .$$

By the continuity of g_i, the function ϑ_x is continuous.

Fix x in X, then

$$\beta(x) = \sup_{u \in X} \sum_{i=1}^{M} (g_i(x_{-i}, u_i) - g_i(x)) = \sup_{u \in X} \sum_{i=1}^{M} g_i(x_{-i}, u_i) + \sum_{i=1}^{M} (- g_i(x_{-i}, x_i))$$
$$= \sup_{u \in X} (- \vartheta_x(u)) + \vartheta_x(x) = \vartheta_x(x) - \inf_{u \in X} \vartheta_x(u) .$$

Being $\beta(x) \leq \varepsilon$ the result (2.2) is proven. □

LEMMA 2.2. – *Fix* $\varepsilon > 0$ *and* $x \in L_\beta(\varepsilon)$. *Under Conditions* 1 *there exists* $y \in X$, *with* $| y - x | \leq \sqrt{\varepsilon}$, *such that* $y \in T((K+1)\sqrt{\varepsilon})$.

PROOF. – By equation (2.2), if $\beta(x) \leq \varepsilon$ and $\lambda = \mu = \sqrt{\varepsilon}$ then we can apply the Ekeland Optimality Principle to the function ϑ_x. Since $\vartheta_x(x) \leq \inf_{u \in X} \vartheta_x(u) + \varepsilon$, there exists $y \in X$ such that $| y - x | \leq \sqrt{\varepsilon}$, $\vartheta_x(y) \leq \vartheta_x(x)$ and

$$\vartheta_x(y) \leq \vartheta_x(z) + \sqrt{\varepsilon}\, | y - z | , \tag{2.3}$$

for any z in X.

Fix $i \in PL$ and h in $\boldsymbol{R}$. In (2.3) let z be the point $(y_{-i}, y_i + h)$. Then

$$\vartheta_x(y_{-i}, y_i) \leq \vartheta_x(y_{-i}, y_i + h) + \sqrt{\varepsilon} \mid (y_{-i}, y_i) - (y_{-i}, y_i + h) \mid,$$

$$-\vartheta_x(y_{-i}, y_i + h) + \vartheta_x(y_{-i}, y_i) \leq \sqrt{\varepsilon} \mid h \mid,$$

$$\sum_{j \neq i} g_j(x_{-j}, y_j) + g_i(x_{-i}, y_i + h) - \sum_{j \neq i} g_j(x_{-j}, y_j) - g_i(x_{-i}, y_i) \leq \sqrt{\varepsilon} \mid h \mid,$$

$$g_i(y_{-i}, y_i + h) - g_i(y_{-i}, y_i) \leq \sqrt{\varepsilon} \mid h \mid .$$

Considering $h > 0$, there are three possible different situations.

FIRST CASE. – Let $y_i + h \in X_i$ and $y_i - h \notin X_i$. Then

$$\frac{g_i(x_{-i}, y_i + h) - g_i(x_{-i}, y_i)}{h} \leq \sqrt{\varepsilon}.$$

Due to the arbitrariness of h and by condition 1.2, we have

$$0 \leq \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \leq \sqrt{\varepsilon}. \tag{2.4}$$

SECOND CASE. – Let $y_i + h \notin X_i$ and $y_i - h \in X_i$. Then, by similarity,

$$\frac{g_i(x_{-i}, y_i - h) - g_i(x_{-i}, y_i)}{-h} \geq -\sqrt{\varepsilon}.$$

Due to the arbitrariness of h and by condition 1.2, we have

$$0 \geq \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \geq -\sqrt{\varepsilon}. \tag{2.5}$$

THIRD CASE. – Let $y_i + h \in X_i$ and $y_i - h \in X_i$. Then,

$$-\sqrt{\varepsilon} \leq \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \leq \sqrt{\varepsilon}. \tag{2.6}$$

By (2.4), (2.5) and (2.6) there exists $y \in X$ with $\mid y - x \mid \leq \sqrt{\varepsilon}$ such that

$$\left| \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \right| \leq \sqrt{\varepsilon},$$

for any $i \in PL$. Therefore, by condition 1.1 it follows:

$$\frac{\partial g_i(y)}{\partial x_i}(u_i - y_i) = \left[\frac{\partial g_i(y_{-i}, y_i)}{\partial x_i} - \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \right] (u_i - y_i) + \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i}(u_i - y_i)$$

$$\leq \left| \frac{\partial g_i(y_{-i}, y_i)}{\partial x_i} - \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \right| \mid u_i - y_i \mid + \left| \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i} \right| \mid u_i - y_i \mid$$

$$\le \left|\frac{\partial g_i(y_{-i}, y_i)}{\partial x_i} - \frac{\partial g_i(x_{-i}, y_i)}{\partial x_i}\right| \mid u_i - y_i \mid + \sqrt{\varepsilon} \mid u_i - y_i \mid$$

$$\le K \mid (y_{-i}, y_i) - (x_{-i}, y_i) \mid\mid u_i - y_i \mid + \sqrt{\varepsilon} \mid u_i - y_i \mid$$

$$\le K \mid y - x \mid\mid u_i - y_i \mid + \sqrt{\varepsilon} \mid u_i - y_i \mid \le (K+1)\sqrt{\varepsilon} \mid u_i - y_i \mid,$$

for any $u_i \in X_i$ and $i \in PL$.

Adding with respect to i:

$$\langle F(y), u - y \rangle_M = \sum_{i=1}^{M} \frac{\partial g_i(y)}{\partial x_i}(u_i - y_i)$$

$$\le (K+1)\sqrt{\varepsilon} \sum_{i=1}^{M} \mid u_i - y_i \mid = (K+1)\sqrt{\varepsilon} \mid u - y \mid,$$

for any $u \in X$. That is $y \in T((K+1)\sqrt{\varepsilon})$. □

The following is the proof of how under Conditions 1, the **VI-wp** implies the **T-wp**, which is a straightforward application of the two lemmas and the Proposition 1.

Let $S \neq \emptyset$ be a subset of $\boldsymbol{R}^M$ and for any $\rho > 0$ define

$$I(\rho, S) = \{p \in \boldsymbol{R}^M / \mid u - p \mid \le \rho, \text{ for some } u \in S\}.$$

Fix $\varepsilon > 0$ and $x \in L_\beta(\varepsilon)$. By Lemma 2.2, there exists a point $y \in X$ such that $\mid x - y \mid \le \sqrt{\varepsilon}$ and $y \in T((K+1)\sqrt{\varepsilon})$. Therefore, $x \in I(\sqrt{\varepsilon}, T((K+1)\sqrt{\varepsilon}))$ and

$$L_\beta(\varepsilon) \subset I(\sqrt{\varepsilon}, T((K+1)\sqrt{\varepsilon})). \tag{2.7}$$

By definition, the property of well-posedness with respect to the variational inequality implies that $\operatorname{diam} T(\varepsilon) \to 0$ and by (2.7), $\operatorname{diam} L_\beta(\varepsilon) \to 0$. Therefore, by Proposition 1, the game G has the property of **T-wp**.

3. – T-wp implies VI-wp.

Let us fix $0 < \varepsilon \le \delta$ and prove that $T(\varepsilon) \subset L_\beta(M\varepsilon)$.

Our assumption that $NE(G)$ is nonempty, implies $T(\varepsilon)$ is nonempty as well.

Let x be an element of this set and fix $i \in PL$. If $u \in X$ is such that $u_j = x_j$ for $j \neq i$, then for any $u_i \in X_i$ by (1.2) and the concavity of g_i we have

$$g_i(x_{-i}, u_i) - g_i(x_{-i}, x_i) \le \frac{\partial g_i(x)}{\partial x_i}(u_i - x_i) \le \varepsilon \mid u_i - x_i \mid.$$

As a consequence,

$$\sup_{u_i \in X_i \text{ and } |u_i - x_i| \le 1} \{g_i(x_{-i}, u_i) - g_i(x_{-i}, x_i)\} \le \varepsilon.$$

Proceeding by contradiction, let $\sup_{u_i \in X_i} \{g_i(x_{-i}, u_i) - g_i(x_{-i}, x_i)\} > \varepsilon$. Then there exists $v_i \in X_i$ such that $|v_i - x_i| > 1$ and

$$g_i(x_{-i}, v_i) - g_i(x_{-i}, x_i) > \varepsilon \,. \tag{3.1}$$

By hypothesis, there exists c_i between x_i and v_i such that

$$g_i(x_{-i}, v_i) = g_i(x_{-i}, x_i) + \frac{\partial g_i(x_{-i}, x_i)}{\partial x_i}(v_i - x_i) + \frac{1}{2}\frac{\partial^2 g_i(x_{-i}, c_i)}{\partial (x_i)^2}(v_i - x_i)^2 \,. \tag{3.2}$$

By equations (3.1), (3.2) and condition 2.4 we have

$$\varepsilon < g_i(x_{-i}, v_i) - g_i(x_{-i}, x_i) = \frac{\partial g_i(x_{-i}, x_i)}{\partial x_i}(v_i - x_i) + \frac{1}{2}\frac{\partial^2 g_i(x_{-i}, c_i)}{\partial (x_i)^2}(v_i - x_i)^2$$

$$\leq \varepsilon \mid v_i - x_i \mid - \frac{1}{2}\delta \mid v_i - x_i \mid^2 \,.$$

Let a be the positive number such that $\mid v_i - x_i \mid = 1 + a$. Then

$$\varepsilon < \varepsilon(1+a) - \frac{\delta(1+a)^2}{2} = \varepsilon + \varepsilon a - \frac{\delta}{2} - \frac{\delta a^2}{2} - \delta a \leq \varepsilon + \varepsilon a - \frac{\varepsilon}{2} - \frac{\varepsilon a^2}{2} - \varepsilon a < \varepsilon$$

which is a contradiction. Therefore, for any $x \in T(\varepsilon)$ and $0 < \varepsilon \leq \delta$

$$l_i(x) - g_i(x_{-i}, x_i) \leq \varepsilon \,.$$

Adding with respect to i, we have $\beta(x) \leq M\varepsilon$. This means $x \in L_\beta(M\varepsilon)$ and

$$T(\varepsilon) \subset L_\beta(M\varepsilon) \,. \tag{3.3}$$

By the Tykhonov well-posedness of G, Proposition 1 and (3.3) we have the **VI-wp** of the game.

4. – Remark.

In this section we present a game which does not verify condition 2.4, with the property **T-wp** and without the property **VI-wp**.

EXAMPLE 2. – Consider the 2-*player* game $G = (X_1, X_2, g_1, g_2)$ where $X_1 = X_2 = [1, +\infty)$, $g_1(x_1, x_2) = -\dfrac{x_2 - 1}{x_1}$ and $g_2(x_1, x_2) = -\dfrac{x_1 - 1}{x_2}$. The related function β is defined as follows:

$$\beta(x_1, x_2) = \frac{x_2 - 1}{x_1} + \frac{x_1 - 1}{x_2} \,.$$

Now $\beta(x_1, x_2) = 0$ if and only if $\dfrac{x_2 - 1}{x_1} + \dfrac{x_1 - 1}{x_2} = 0$ which is $(x_1, x_2) = (1, 1)$. Therefore, the game G has only one Nash equilibrium point.

By Proposition 1, the fact that this game has the property of **T-wp**, it is equivalent to say that (X, β) has the property of Tykhonov well-posedness. This is what we are going to prove now.

Let $((x_1)_n, (x_2)_n) \in X$ be a sequence such that

$$\lim_{n\to+\infty} \beta((x_1)_n, (x_2)_n) = \lim_{n\to+\infty} \left(\frac{(x_2)_n - 1}{(x_1)_n} + \frac{(x_1)_n - 1}{(x_2)_n} \right) = 0 . \tag{4.1}$$

We must verify that $\lim\limits_{n\to+\infty} ((x_1)_n, (x_2)_n) = (1,1)$. Being the space X in question $[1, +\infty) \times [1, +\infty)$ this is equivalent to prove that

$$\max_{n\to+\infty}\lim ((x_1)_n, (x_2)_n) = (1,1) .$$

Let $((x_1)_k, (x_2)_k)$ be a subsequence of $((x_1)_n, (x_2)_n)$ with

$$\lim_{k\to+\infty} ((x_1)_k, (x_2)_k) = \max_{n\to+\infty}\lim ((x_1)_n, (x_2)_n) .$$

There are four different cases to consider.

CASE 1. – $\lim\limits_{k\to+\infty} ((x_1)_k, (x_2)_k) = (l, m)$, where $l, m \geq 1$. In this case,

$$\lim_{k\to+\infty} \beta((x_1)_k, (x_2)_k) = \lim_{k\to+\infty} \left(\frac{(x_2)_k - 1}{(x_1)_k} + \frac{(x_1)_k - 1}{(x_2)_k} \right) = \frac{m-1}{l} + \frac{l-1}{m} .$$

By (4.1) we have $\dfrac{m-1}{l} + \dfrac{l-1}{m} = 0$, that is $m = 1$ and $l = 1$.

CASE 2. – $\lim\limits_{k\to+\infty} ((x_1)_k, (x_2)_k) = (l, +\infty)$, where $l \geq 1$. In this case,

$$\lim_{k\to+\infty} \left(\frac{(x_2)_k - 1}{(x_1)_k} + \frac{(x_1)_k - 1}{(x_2)_k} \right) = +\infty$$

which is not compatible with (4.1).

CASE 3. – $\lim\limits_{k\to+\infty} ((x_1)_k, (x_2)_k) = (+\infty, m)$, where $m \geq 1$. In this case,

$$\lim_{k\to+\infty} \left(\frac{(x_2)_k - 1}{(x_1)_k} + \frac{(x_1)_k - 1}{(x_2)_k} \right) = +\infty$$

which is again not compatible with (4.1).

CASE 4. – $\lim\limits_{k\to+\infty} ((x_1)_k, (x_2)_k) = (+\infty, +\infty)$. In this case, let $((x_1)_h, (x_2)_h)$ be a subsequence of the sequence $((x_1)_k, (x_2)_k)$, with

$$\lim_{h\to+\infty} \frac{(x_2)_h - 1}{(x_1)_h} = \min_{k\to+\infty}\lim \frac{(x_2)_k - 1}{(x_1)_k} \geq 0$$

and

$$\lim_{h\to+\infty}\frac{(x_1)_h-1}{(x_2)_h}=\min_{k\to+\infty}\lim\frac{(x_1)_k-1}{(x_2)_k}\geq 0\,.$$

By (4.1), we have:

$$0=\lim_{k\to+\infty}\left(\frac{(x_2)_k-1}{(x_1)_k}+\frac{(x_1)_k-1}{(x_2)_k}\right)=\min_{k\to+\infty}\lim\left(\frac{(x_2)_k-1}{(x_1)_k}+\frac{(x_1)_k-1}{(x_2)_k}\right)$$

$$\geq\min_{k\to+\infty}\lim\frac{(x_2)_k-1}{(x_1)_k}+\min_{k\to+\infty}\lim\frac{(x_1)_k-1}{(x_2)_k}\geq 0\,.$$

Therefore,

$$\lim_{h\to+\infty}\frac{(x_2)_h-1}{(x_1)_h}=\min_{k\to+\infty}\lim\frac{(x_2)_k-1}{(x_1)_k}=0$$

and

$$\lim_{h\to+\infty}\frac{(x_1)_h-1}{(x_2)_h}=\min_{k\to+\infty}\lim\frac{(x_1)_k-1}{(x_2)_k}=0\,.$$

Finally,

$$\lim_{h\to+\infty}(x_1)_h=+\infty;\qquad\lim_{h\to+\infty}(x_2)_h=+\infty;$$

and

$$\lim_{h\to+\infty}\frac{(x_2)_h-1}{(x_1)_h}=0;\qquad\lim_{h\to+\infty}\frac{(x_1)_h-1}{(x_2)_h}=0\,.$$

These conditions are incompatible. Therefore, G has the property **T-wp**.

Let now show that the game G does not have the property **VI-wp**.

Fix $0<\varepsilon<1$ and choose $0<\lambda\leq\varepsilon$. Consider the point $(x_1,x_2)=(\lambda^{-1},\lambda^{-1})$. Therefore,

$$\left\langle\left(\frac{\partial g_1(\lambda^{-1},\lambda^{-1})}{\partial x_1},\frac{\partial g_2(\lambda^{-1},\lambda^{-1})}{\partial x_2}\right),((u,v)-(\lambda^{-1},\lambda^{-1}))\right\rangle_2$$

$$=\langle((\lambda^{-1}-1)\lambda^2,(\lambda^{-1}-1)\lambda^2),(u-\lambda^{-1},v-\lambda^{-1})\rangle_2$$

$$=\langle(\lambda-\lambda^2,\lambda-\lambda^2),(u-\lambda^{-1},v-\lambda^{-1})\rangle_2=(\lambda-\lambda^2)(u-\lambda^{-1})+(\lambda-\lambda^2)(v-\lambda^{-1})$$

$$\leq\lambda\mid u-\lambda^{-1}\mid+\lambda\mid v-\lambda^{-1}\mid\leq\varepsilon\mid u-\lambda^{-1}\mid+\varepsilon\mid v-\lambda^{-1}\mid,$$

for any $(u,v)\in X_1\times X_2$.

Hence, $(\lambda^{-1},\lambda^{-1})\in T(\varepsilon)$ and $\operatorname{diam}T(\varepsilon)=+\infty$. □

At the moment, the authors do not have available a significant case of a game which does not verify Conditions 1, with the property **VI-wp** and without the property **T-wp**.

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