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Transversally Pseudoconvex Foliations

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Dedicated to Aldo Andreotti

Abstract. – *We consider real analytic foliations X with complex leaves of transversal dimension one and we give the notion of transversal pseudoconvexity. This amounts to require that the transverse bundle N_F to the leaves carries a metric $\{\lambda_j\}$ on the the fibres such that the tangential* $(1,1)$*-form $\Omega = \{\lambda_j \overline{\partial}\partial\lambda_j - 2\overline{\partial}\lambda_j\partial\lambda_j\}$ is positive. This condition is of a special interest if the foliation X is* 1 *complete i.e. admits a smooth exhaustion function ϕ which is strongly plusubharmonic along the leaves. In this situation we prove that there exist an open neighbourhood U of X in the complexification $\widetilde{X}$ of X and a non negative smooth function $u : U \to \mathbb{R}$ which is plurisubharmonic in U, strongly plurisubharmonic on $U \setminus X$ and such that X is the zero set of u. This result has many implications: every compact sublevel $\overline{X}_c = \{x \in X : \phi \leq c\}$ is a Stein compact and if $\mathsf{S}(\mathsf{X})$ is the algebra of smooth CR functions on X, the restriction map $\mathsf{S}(\mathsf{X}) \to \mathsf{S}(\mathsf{X_c})$ has a dense image (Theorem* 4.1*); a transversally pseudoconvex,* 1*-complete, real analytic foliation X with complex leaves of dimension n properly embeds in $\mathbb{C}^{2n+3}$ by a CR map and the sheaf $\mathsf{S} = \mathsf{S_X}$ of germs of smooth CR functions on X is cohomologically trivial.*

1. – Introduction.

In this short note we consider real analytic foliations X with complex leaves of transversal dimension one and we give the notion of *transversal pseudoconvexity*. This amounts to require that the transverse bundle N_F to the leaves carries a metric $\{\lambda_j\}$ on the the fibres such that the tangential $(1,1)$-form $\Omega = \{\lambda_j \overline{\partial}\partial\lambda_j - 2\overline{\partial}\lambda_j\partial\lambda_j\}$ is positive (cfr. Section 3). This condition is of a special interest if the foliation X is 1 *complete* i.e. admits a smooth exhaustion function ϕ which is strongly plusubharmonic along the leaves. In this situation we prove the following result (see Theorem 3.1): there exist an open neighbourhood U of X in the complexification $\widetilde{X}$ of X and a non negative smooth function $u : U \to \mathbb{R}$ whis is plurisubharmonic in U, strongly plurisubharmonic on $U \setminus X$ and such that X is the zero set of u. This result has many implications. First of all it allows us to

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prove that every compact sublevel $\overline{X}_c = \{x \in X : \phi \leq c\}$ is a Stein compact and, denoting $\mathsf{S}(\mathsf{X})$ the algebra of smooth *CR* functions on X, the restriction map $\mathsf{S}(\mathsf{X}) \to \mathsf{S}(\mathsf{X}_c)$ has a dense image (Theorem 4.1). Then, arguing as in the in the complex case, we show that a transversally pseudoconvex, 1-complete, real analytic foliation X with complex leaves of dimension n properly embeds in $\mathbb{C}^{2n+3}$ by a CR map (Theorem 4.4). Finally, under the same hypothesis for X, in Theorem 5.1 we prove that the sheaf $\mathsf{S} = \mathsf{S}_\mathsf{X}$ of germs of smooth *CR* functions on X is cohomologically trivial.

2. – Preliminaries.

2.1 – *q-pseudoconvex foliations.*

We recall that a *foliation with complex leaves of dimension n and real codimension d* is a smooth foliation X whose local models are subdomains $U_j = V_j \times B_j$ of $\mathbb{C}^n \times \mathbb{R}^d$ and coordinates transformations are of the form

$$\begin{cases} z_j = f_{jk}(z_k, t_k) \\ t_j = g_{jk}(t_k) \end{cases} \tag{1}$$

where $f_{jk} = (f^1_{jk}, \ldots, f^n_{jk})$, $g_{jk} = (g^1_{jk}, \ldots, g^d_{jk})$ are smooth maps and f_{jk} is holomorphic with respect to z_k. If we replace $\mathbb{R}^d$ by $\mathbb{C}^d$ and we suppose that f and h are holomorphic we get the notion of *holomorphic foliation of (complex) codimension k*. A domain U_j as above is called a *distinguished coordinate domain* of X and $z^1_j, \ldots, z^n_j, t^1_j, \ldots, t^d_j$ *distinguished local coordinates*.

$\mathsf{S} = \mathsf{S}_\mathsf{X}$ denotes the sheaf of germs of smooth *CR* functions (i.e. smooth functions which are holomorphic along the leaves) on X and $\mathsf{S}_0 = \mathsf{S}_{0,X} \subset \mathsf{S}$ the subsheaf of those germs which are constant on the leaves; both are Fréchet sheaves. If X is a real analytic foliation then, by definition, $\mathcal{O}' = \mathcal{O}'_X$ is the the sheaf of germs of real analytic *CR* functions on X.

Given a subset C of X we denote $\widehat{C}_{\mathsf{S}(X)}$ the envelope of C with respect to the algebra $\mathsf{S}(X)$.

A *morphism or CR map* $F : X \to X'$ of foliations with complex leaves is a smooth map preserving the leaves and such that the restrictions to leaves are holomorphic.

A smooth function $\phi : X \to \mathbb{R}$ is said to be *strongly q-pseudoconvex (along the leaves)* if the restriction of ϕ to any leaf is q-pseudoconvex i.e. its Levi form has at least $n - q + 1$ positive eigenvalues.

A foliation X with complex leaves is said to be *strongly q-pseudoconvex* if X carries a smooth exhaustion $\phi : X \to \mathbb{R}^+$ which is strongly q-pseudoconvex outside a compact subset $K \subset X$, $1 \leq q \leq n+1$. In particular, X is said *q-complete* if $K = \varnothing$. We denote by $\overline{X}_c$ the sublevel $\{x \in X : \phi \leq c\}$.

2.2 – *Complexification.*

A real analytic foliation with complex leaves can be complexified. There exists a holomorphic foliation $\widetilde{X}$ of codimension d with a closed real analytic CR-embedding $X \hookrightarrow \widetilde{X}$. In order to construct $\widetilde{X}$ we consider a covering by distinguished domains $\{U_j = V_j \times B_j\}$ and we complexify each B_j in such a way to obtain domains $\widetilde{U}_j$ in $\mathbb{C}^n \times \mathbb{C}^d$. The domains $\widetilde{U}_j$ are patched together by the local holomorphic transformations

$$\begin{cases} z_j = \widetilde{f}_{jk}(z_k, \tau_k) \\ \tau_j = \widetilde{g}_{jk}(\tau_k) \end{cases} \tag{2}$$

obtained complexifying the variable $t_k = (t_k^1, \ldots, t_k^d)$ by $\tau_k = t_k + i\theta_k$, $\theta_k = (\theta_k^1, \ldots, \theta_k^d)$ in f_{jk} and g_{jk} (cfr. 1).

REMARK 2.1. – If X is real analytic Levi flat hypersurface in a complex manifold Z then Z and $\widetilde{X}$ are biholomorphic along X.

In [3, Theorem 2] is proved that if X is q-complete with a smooth exhaustion function ϕ then every sublevel $\overline{X}_c = \{x \in X : \phi \leq c\}$ has a fundamental system of neighbourhoods in $\widetilde{X}$ which are $(q+1)$-complete complex manifolds.

Let $\widetilde{X}$ the complexification of X. Then the structure cocycle of the (holomorphic) transverse bundle $\widetilde{N}_F$ (to the leaves of $\widetilde{X}$) is

$$\frac{\partial \widetilde{g}_{jk}(\tau_k)}{\partial \tau_k} = \frac{\partial \tau_j}{\partial \tau_k} = \left(\frac{\partial \tau_j^a}{\partial \tau_k^\beta} \right). \tag{3}$$

Let z_j, τ_j holomorphic coordinates on $\widetilde{U}_j$ and let $\theta_j = \operatorname{Im} \tau_j$. The we have $\theta_j = \operatorname{Im} \widetilde{g}_{jk}(\tau_k)$ on $\widetilde{U}_j \cap \widetilde{U}_k$ and consequently, since $\operatorname{Im} \widetilde{g}_{jk} = 0$ on X,

$$\theta^a = \sum_{\beta=1}^{d} \psi_{jk}^{a\beta} \theta_k^\beta \tag{4}$$

where $\psi_{jk} = \left(\psi_{jk}^{rs}\right)$ is an invertible $d \times d$ matrix whose entries are real analytic functions on $\widetilde{U}_j \cap \widetilde{U}_k$. Moreover, since $\widetilde{g}_{jk}$ is holomorphic and $\widetilde{g}_{jk|X} = g_{jk}$ is real, we also have $\psi_{jk|X} = \partial g_{jk} / \partial t_k$.

3. – Tranversally pseudoconvex foliations.

Let X be a foliation with complex leaves of dimension n and real codimension $d = 1$, N_F the transverse bundle to the leaves of X. A metric on the fibres of N_F is an assignement of a distinguished covering $\{U_j\}$ of X and for every j a smooth map $\lambda_j^0 : U_j \to (0, +\infty)$ such that

$$\lambda_k^0 = \left(\frac{dg_{jk}}{dt_k}\right)^2 \lambda_j^0.$$

Denoting ∂ and $\overline{\partial}$ the complex differentiation along the leaves of X the local tangential form

$$\Omega = 2\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{\lambda_j^0 \overline{\partial}\partial\lambda_j^0 - 2\overline{\partial}\lambda_j^0 \wedge \partial\lambda_j^0}{{\lambda_j^0}^3} \tag{5}$$

actually is global. The foliation X is said to be *tranversally pseudoconvex* if $\Omega > 0$.

More generally, the foliation X is said to be *tranversally q-pseudoconvex* if a metric on the fibres can be chosen in such a way that the hermitian form associated to Ω has at least $n - q + 1$ positive eigenvalues.

REMARK 3.1. – Transverse q-pseudoconvexity is a strong condition. For instance assume $q = 1$ and that X is transversally pseudoconvex. Then, due to the fact that the functions g_{jk} do not depend on z, $\omega_0 = \{\partial\overline{\partial} \log \lambda_j^0\}$ and $\eta = \{\partial \log \lambda_j^0\}$ are global tangential forms on X; moreover ω_0 is positive and exact, $\omega_0 = d\eta$, so on each leaf ω_0 gives a Kähler metric whose Kähler form is exact. In particular no positive compact complex subspace is present in X.

REMARK 3.2. – A real hyperplane X in $\mathbb{C}^n$, $n \geq 2$, is not transversally pseudoconvex. Indeed, assume $n = 2$ and let $X \subset \mathbb{C}^2$ be defined by $v = 0$, where $z = x + iy, w = u + iv$ are holomorphic coordinates. Transverse pseudoconvexity of X amounts to the existence of a positive smooth function $\lambda = \lambda(z, u)$, $(z, u) \in \mathbb{C}^2$, such that

$$\lambda\lambda_{z\bar{z}} - 2|\lambda_z|^2 > 0.$$

Consider the function λ^{-1}. Then

$$\lambda^{-1}_{z\bar{z}} = \frac{2|\lambda_z|^2 - \lambda\lambda_{z\bar{z}}}{\lambda^3} < 0$$

so, for every fixed u, the function λ^{-1} is positive and superharmonic on $\mathbb{C}_z$, hence it is constant with respect to z: contradiction.

Every domain $D \subset X = \mathbb{C}^z \times \mathbb{R}_u$ which projects over a bounded domain $D_0 \subset \mathbb{C}^z$ is strongly transversally pseudoconvex. Indeed, it sufficient to take for λ a function $\mu^{-1} \circ \pi$ where μ is a positive superharmonic on D_0 and π is the natural projection $\mathbb{C}^z \times \mathbb{R}_u \to \mathbb{C}_z$.

EXAMPLE 3.1. – Let $X \subset \mathbb{C}^n_z \times \mathbb{R}_u$ be the smooth family of n-balls of $\mathbb{C}^n_z$, $z = (z_1, \ldots, z_n)$, $w = u + iv$, defined by

$$\begin{cases} v = 0, \\ |z - a(u)|^2 < b(u)^2 \end{cases}$$

where $a = a(u)$ is a smooth map $\mathbb{R} \to \mathbb{C}^n$, $b = b(u)$ is smooth from $\mathbb{R}$ to $\mathbb{R}$ and $|a(u)|, |b(u)| \to +\infty$ as $|u| \to +\infty$. X is strongly transversally pseudoconvex with function

$$\lambda(z, u) = \frac{1}{b(u)^2 - |z - a(u)|^2}$$

and 1-complete with exhaustion function $\lambda = \phi$.

We want to prove the following

THEOREM 3.1. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension $d = 1$, $\widetilde{X}$ the complexification of X. Assume that X is transversally pseudoconvex. Then there exist an open neighbourhood U of X in $\widetilde{X}$ and a non negative smooth function $u : U \to \mathbb{R}$ with the following properties*

i) $X = \{x \in U : u(x) = 0\}$

ii) *u is plurisubharmonic in U and strongly plurisubharmonic on $U \setminus X$.*

PROOF. – Let us assume first $n = 1$. Keeping the notations of subsection 2.2 let ψ_{ij} be the cocycle defined by (4) and E the line bundle associated to ψ_{ij}; E which extends N_F on a neighbourhood of X. Let $\{\lambda^0_j\}$ be a metric on the fibres of N_F; $\{\lambda^0_j\}$ is a smooth section of the line bundle N_F^{-2} so it extends by a smooth section of E_F^{-2} giving a metric $\{\lambda_j\}$ on the fibres of E.

Now consider on $\widetilde{X}$ the smooth function u locally defined by $\lambda_j \theta_j^2$ (where $\tau_j = t_j + i\theta_j$); u is non negative and positive outside of X. Drop the subscript and compute the Levi form $L(u)$ of u. We have

$$\begin{aligned} L(u)(\xi, \eta) &= A\xi\overline{\xi} + 2\,\mathrm{Re}(B\xi\overline{\eta}) + C\eta\overline{\eta} \\ &= \lambda_{z,\overline{z}}\theta^2\xi\overline{\xi} + 2\,\mathrm{Re}\{(\lambda_{z\overline{\tau}}\theta^2 + i\lambda_z\theta)\xi\overline{\eta}\} \\ &\quad + (\lambda_{\tau\overline{\tau}}\theta^2 + i\lambda_\tau\theta - i\lambda_{\overline{\tau}}\theta + \lambda/2)\eta\overline{\eta}. \end{aligned} \tag{6}$$

Then

$$C = \lambda/2 + \theta\psi \tag{7}$$

and

$$AC - |B|^2 = \theta^2\big(\lambda_{z\bar z} - 2|\lambda_z|^2\big) + \theta^3\phi \tag{8}$$

where ψ and ϕ are smooth function. Observe that $C > 0$ when θ is small enough. The coefficient of θ^2 in (8) is nothing but that of the form $\lambda^2\Omega$ so, being X strongly transversally pseudoconvex, $L(u)$ is positive definite near each point of X and strictly positive away from X. It follows that there exists a neighbourhood U of X such that u is plurisubharmonic on U.

Assume now that n is arbitrary. Then we need to prove that given $\xi = (\xi_1, \dots, \xi_n) \neq 0$ if $(\xi_1, \dots, \xi_n, \eta) \neq 0$ and $\theta \neq 0$ then

$$\begin{aligned} L(u)(\xi,\eta) = \sum_{i,j=1}^{n} \lambda_{z_i\bar z_j}\theta^2\xi_i\bar\xi_j + 2\sum_{i=1}^{n} \mathsf{Re}\big\{\big(\lambda_{z_i\bar\tau}\theta^2 + i\lambda_{z_i}\theta\big)\xi_i\bar\eta\big\} \\ + (\lambda_{\tau\bar\tau}\theta^2 + i\lambda_\tau\theta - i\lambda_{\bar\tau}\theta + \lambda/2)\eta\bar\eta > 0 \end{aligned}$$

if θ is small enough.

If $\xi = 0$ then $\eta \neq 0$ and

$$L(u)(0,\eta) = (\lambda_{\tau\bar\tau}\theta^2 i\lambda_\tau\theta - i\lambda_{\bar\tau}\theta + \lambda/2)\eta\bar\eta > 0.$$

If $\xi \neq 0$ then $\xi_j \neq 0$ for some j. Performing the change of variable given by

$$\begin{aligned} z_i &= w_i + \frac{\xi_i}{\xi_j} w_j \quad i \neq j, \\ z_j &= w_j, \end{aligned}$$

we have the equality

$$\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial z_i} = \xi_j \frac{\partial}{\partial w_j}$$

and hence

$$\begin{aligned} L(u)(\xi,\eta) = \lambda_{w_j\bar w_j}\theta^2\xi_j\bar\xi_j + 2\,\mathsf{Re}\,\{(\lambda_{w_j\bar\tau}\theta^2 + i\lambda_{w_j}\theta)\xi_i\bar\eta\} \\ + (\lambda_{\tau\bar\tau}\theta^2 + i\lambda_\tau\theta - i\lambda_{\bar\tau}\theta + \lambda/2)\eta\bar\eta. \end{aligned}$$

The same argument given in the proof of the case $n = 1$ yields the desired result. □

REMARK 3.3. – In view of (6), the Levi form $L(u)$ at a point of X is always positive in the transversal direction η.

THEOREM 3.2. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension $d = 1$, $\widetilde X$ the complexification of X. Assume* *that X is transversally pseudoconvex and* 1-*complete. Then for every compact subset $K \subset X$ there exist an open neighbourhood V of K in $\widetilde{X}$, a smooth strongly plurisubharmonic function $v : V \to \mathbb{R}^+$ and a constant $\bar{c}$ such that*

$$K \Subset \{v < \bar{c}\} \cap X \Subset V \cap X.$$

PROOF. – Let $\phi : X \to \mathbb{R}^+$ be an exhaustion function, strongly plurisubharmonic along the leaves and a sublevel X_c of ϕ such that $K \subset X_c$. Consider $U \subset \widetilde{X}$, $u : U \to \mathbb{R}^+$ as in Theorem 3.1 and the function $v = au + \widetilde{\phi}$ where $\widetilde{\phi} : U \to \mathbb{R}^+$ is a smooth extension of ϕ to U and a a positive constant. Then, in view of Remark 3.3, it is possible choose a in such a way $L(v)(x) > 0$ for every x in a neighbourhood V of $\overline{X}_{c'}$, $c < c'$. Thus, in order to end the proof, it is sufficient to take $\bar{c} = c'$. □

REMARK 3.4. – The stament of Theorem 3.2 holds if X is a transversally pseudoconvex 1-complete real analytic Levi flat hypersurface of a complex manifold Z (cfr. 2.1).

4. – Applications.

4.1 – *Stein bases and a density theorem.*

Let X be a smooth foliation with complex leaves of dimension n and real codimension d. Let $\mathsf{S}(X)$ be the algebra of the *CR* functions in X. The $\mathsf{S}(X)$ – *envelope* of a subset C of X is the subset

$$\widehat{C}_{\mathsf{S}(X)} = \{x \in X : |f(x)| \le \|f\|_C, \forall f \in \mathsf{S}(X)\}.$$

C is said to be $\mathsf{S}(X)$ – *convex* if $C = \widehat{C}_{\mathsf{S}(X)}$.

The foliation X is said to be $\mathsf{S}(X)$ – *convex* if the $\mathsf{S}(X)$-envelope $\widehat{K}_{\mathsf{S}(X)}$ of a compact subset $K \subset X$ is also compact.

THEOREM 4.1. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension $d = 1$, $\widetilde{X}$ the complexification of X. Assume that X is transversally pseudoconvex and* 1-*complete and let $\phi : X \to \mathbb{R}^+$ be a smooth function displaying the* 1-*completeness of X. Then*

i) *$\overline{X}_c$ has in $\widetilde{X}$ a Stein basis of neighbourhoods;*
ii) *for every $c \in \mathbb{R}$ the restriction map*

$$\mathsf{S}(X) \to \mathsf{S}(X_c)$$

has a dense image.

PROOF. – In view of Theorem 3.1 we may suppose that $X = \{u = 0\}$ where $u : \widetilde{X} \to [0, +\infty)$ is plurisubharmonic and strongly plurisubharmonic on $\widetilde{X} \setminus X$.

Let U be an open neighbourhood of $\overline{X}_c$ in $\widetilde{X}$. We apply Theorem 3.2 with $K = \overline{X}_c$: there exists an open neighbourhood $V \Subset U$ of $\overline{X}_c$ in $\widetilde{X}$, a smooth strongly plurisubharmonic function $v : V \to \mathbb{R}^+$ and a constant $\bar{c}$ such that

$$\overline{X}_c \Subset \{v < \bar{c}\} \cap X \Subset V \cap X.$$

It follows that for $\varepsilon > 0$ sufficiently small $W = \{v < \bar{c}\} \cap \{u < \varepsilon\} \Subset V \Subset U$ is a Stein neighbourhood of $\overline{X}_c$.

In order to prove ii) it is sufficient to show that for every $c \in \mathbb{R}$ a CR function on a neighbourhood of $\overline{X}_c$ can be approximated in the C^∞ topology by smooth *CR* functions on X.

Let f be a smooth *CR* function on a neighbourhood V of $X_{c'}$ in X, $c < c'$, such that $\overline{X}_{c'} \subset V$. For every $j \in \mathbb{N}$ define $\overline{B}_j = \overline{X}_{c'+j}$ and choose a Stein neighbourhod U_j of $\overline{B}_j$ such that $\overline{B}_j$ has U_{j+1} a fundamental system of open neighbourhoods $W_j \Subset U_{j+1} \cap U_j$ which are Runge domains in U_{j+1}. Since $\overline{B}_0$ is the zero set of u the $\mathcal{O}(U_0)$-envelope of $\overline{B}_0$ is compact and contained in $X \cap U_0$ (cfr. [4, Theorem 4.3.4]), so we may assume that $\overline{B}_0$ is $\mathcal{O}(U_0)$-convex. Let $\|\cdot\|^{(k)}_{\overline{B}_0}$ be a C^k-norm on $\overline{B}_0$. Then, in view of the approximation theorem of Freeman (cfr. [2, Theorem 1.3]), given $\varepsilon > 0$ there exists $\widetilde{f} \in \mathcal{O}(U_0)$ such that $\|\widetilde{f} - f\|^{(k)}_{\overline{B}_0} < \varepsilon$. Now, for every $j \geq 1$ take W_j such that $\overline{W}_j$ is a Runge domain in U_{j+1} and a holomorphic function $F \in \mathcal{O}(U_j)$ satisfying

$$\|F_1 - \widetilde{f}\|^{(k)}_{\overline{W}_0} < \varepsilon/2, \quad \|F_{j+1} - F_j\|^{(k)}_{\overline{W}_j} < \varepsilon/2^{j+1};$$

the C^k function

$$g = F_1 + \sum_{j=1}^{+\infty} (F_{j+1} - F_j)$$

is *CR* on X and $\|g - f\|^{(k)}_{\overline{B}_0} \leq 2\varepsilon$. □

The construction performed in the proof of Theorem 4.1 gives, in particular, the following approximation theorem that will be used later.

THEOREM 4.2. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension $d = 1$, transversally pseudoconvex and 1-complete, $\widetilde{X}$ the complexification of X. Let K an* $\mathsf{S}(X)$*-convex compact subset of X. Then every function $f \in \mathcal{O}(K)$ can be approximated uniformly on K by functions in* $\mathsf{S}(X)$.

PROOF. – In view of Theorem 3.1 we may suppose that $X = \{u = 0\}$ where $u : \widetilde{X} \to [0, +\infty)$ is plurisubharmonic and strongly plurisubharmonic on $\widetilde{X} \setminus X$.

Let $\phi : X \to \mathbb{R}^+$ be a smooth function displaying the 1-completeness of X and $c \in \mathbb{R}$ such that $K \subset X_c$. In view of Theorem 4.1 we may assume that f is holomorphic on a Stein neighbourhood $U_c \subset U$ of K such that $\overline{X}_c \subset U_c$ and $U_c \cap X$ is holomorphically convex with $\widehat{K}_{U_c} \Subset U_c \cap X$. Moreover, since the image of the restriction map

$$\mathcal{O}(U_c) \to \mathsf{S}(U_c \cap X)$$

is everywhere dense and K is $\mathsf{S}(X)$-convex, we have

$$K \equiv \widehat{K}_{\mathsf{S}(X)} \equiv \widehat{K}_{U_c}$$

so K is $\mathcal{O}(U_c)$-convex. The classical approximation theorem for holomorphic functions (cf. [4, Corollary 5.2.9]) now implies that $f_{|K}$ is approximated by functions in $\mathsf{S}(U_c \cap X)$ whence by functions in $\mathsf{S}(X)$. □

4.2 – *An embedding theorem.*

Let X be a foliation with complex leaves of dimension n and real codimension d. Denote $\mathsf{S}(X)$ the algebra of the *CR* functions in X. We say that X is a *Stein foliation* if

i) $\mathsf{S}(X)$ separates points of X;
ii) for every $x_0 \in X$ there exist a *CR* map $f : X \to \mathbb{C}^{n+d}$ which is regular at x_0;
iii) X is $\mathsf{S}(X)$-convex.

We also denote by

$$\mathcal{CR}(X; \mathbb{C}^N) = \mathsf{S}(X)^{\oplus N} \subset \mathrm{C}^\infty(X; \mathbb{C}^N)$$

the set of all smooth CR maps $X \to \mathbb{C}^N$. Endowed with the induced topology $\mathcal{CR}(X; \mathbb{C}^N)$ is a Fréchet space.

THEOREM 4.3. – *A transversally pseudoconvex* 1*-complete real analytic foliation* X *of real codimension* $d = 1$ *is Stein.*

PROOF. – It is a consequence of Theorem 4.1. Indeed, let $\phi : \widetilde{X} \to \mathbb{R}^+$ be a function displaying the completeness of the complexification of X. Given $x, y \in X$ consider a sublevel X_c of ϕ containing x, y and a Stein neighbourhood $U \subset \widetilde{X}$ containing $\overline{X}_c$ (cfr. Theorem 4.1, i)). Then there exists a function $f \in \mathcal{O}(U)$ such that $f(x) \neq f(y)$. In order to conclude the proof of i) it is sufficient to approximate $f_{|\overline{X}_c}$ (cfr. Theorem 4.1, ii)). The proof of ii) is analogous: given $x \in X_c$ we take holomorphic functions $f_1, \ldots, f_{n+1}$ in U giving a local biholomorphism at x_0 and we approximate $f_{1|\overline{X}_c}, \ldots, f_{n+1|\overline{X}_c}$ by global *CR* functions. Finally, consider a compact subset K of X and let $K \subset X_c$. Consider $U \subset \widetilde{X}$ and $u : U \to [0, +\infty)$ plurisubharmonic such that $X = \{u = 0\}$ (cfr. Theorem 3.1). Arguing as in Theorem 3.2 we can construct a sequence $\Omega_0, \Omega_1, \ldots$ of Stein open subsets of $\widetilde{X}$ with the following properties:

1) $\Omega_\nu \supset \overline{X}_{c+\nu}, \quad \nu = 0, 1, \ldots$
2) $\Omega_n \cap \Omega_{\nu-1}$ is Runge in $\Omega_{\nu-1}$ and Ω_ν, $\nu = 1, 2, \ldots$.

Denoting $\widehat{K}_{\Omega_\nu}$ the envelope of K with respect to the algebra $\mathcal{O}(\Omega_\nu)$ we have

$$\widehat{K}_{\Omega_\nu} \supset \widehat{K}_{\mathsf{S}(X \cap \Omega_\nu)} \tag{9}$$

(cfr. [4, Theorem 4.3.4]) and

$$\widehat{K}_{\Omega_\nu} = \widehat{K}_{\Omega_\nu \cap \Omega_0}, \quad \nu = 0, 1 \ldots. \tag{10}$$

(cfr. [4, Theorem 4.3.3]). On the other hand, in view of the approximaxion theorem (cfr. Theorem 4.1) we have

$$\widehat{K}_{\mathsf{S}(X)} \cap X_{c+\nu} = \widehat{K}_{\mathsf{S}(X \cap \Omega_\nu)}. \tag{11}$$

Now (9), (10), (4.3) imply

$$\widehat{K}_{\mathsf{S}(X)} \cap X_{c+\nu} = \widehat{K}_{\mathsf{S}(X \cap \Omega_\nu)} \subset \widehat{K}_{\Omega_\nu} = \widehat{K}_{\Omega_\nu \cap \Omega_0}$$

which is a compact subset of $X \cap \Omega_0$. This prove that X is (X)-convex. □

We want to prove the following

THEOREM 4.4. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension $d = 1$. Assume that X is transversally pseudoconvex and* 1*-complete. Then X embeds in $\mathbb{C}^{2n+3}$ as a closed submanifold by a CR map.*

Let $\widetilde{X}$ be a complexification of X. We may assume that X is the zero set of nonnegative pluriharmonic function $u : \widetilde{X} \to [0, +\infty)$. Every compact subset K of X has a Stein neighbourhood V in $\widetilde{X}$ so, using again the approximation theorem (cfr. Theorem 4.1) we can extend to X the classical preparatory lemmas for the embedding of Stein manifolds (cfr. [4]):

a) for N large there is $F \in \mathcal{CR}(X; \mathbb{C}^N)$ which is regular and one-to-one on K;
b) if $F \in \mathcal{CR}(X; \mathbb{C}^N)$ and $N > n + 1$ then $F(K)$ has (Lebesgue) measure 0;
c) if $F = (F_1, \ldots, F_{N+1}) \in \mathcal{CR}(X; \mathbb{C}^{N+1})$, $N \geq 2n + 2$, is a regular map on K then, for $a = (a_1, \ldots, a_N) \in \mathbb{C}^N$ outside a set of measure 0, $F = (F_1 - a_1 F_{N+1}, \ldots, F_N - a_N F_{N+1})$ is a regular one-to-one map on K;
d) the set of all $F \in \mathcal{CR}(X; \mathbb{C}^N)$ which do not give a regular map of $X \to \mathbb{C}^N$ is of the first category if $N \geq 2n + 1$);
e) the set of all $F \in \mathcal{CR}(X; \mathbb{C}^N)$ which do not give a regular one-to-one map of $X \to \mathbb{C}^N$ is of the first category if $N \geq 2n + 3$.

Let $f_1, \ldots, f_N \in \mathsf{S}(X)$ such that

$$P = \{x \in X : |f_j| < 1, j = 1, \ldots, N\} \Subset X.$$

P is said to be a CR-*polyedron of order* N.

f) $\widehat{K}_{\mathsf{S}(X)} = K$ and W is a neighbourhood of K in X, then there exists a CR-polyedron P of order $2(n+1)$ such that $K \subset P \Subset W$.

Proof of Theorem 4.4. – The proof runs as in [4, Theorem 5.3.9] thanks to Theorem 4.2. In view of Theorem 4.3, X is $\mathsf{S}(X)$-convex. For every $F \in \mathcal{CR}(X; \mathbb{C}^{2n+3})$ we set $|F(x)| = \max_j |F_j(x)|$. According to e) above there exists a one-to-one regular *CR* map $G : X \to \mathbb{C}^{2n+3}$. In order to construct a *CR* embedding as in Theorem 4.4 it is sufficient to find $F \in \mathcal{CR}(X; \mathbb{C}^{2n+3})$ such that

$$Q = \{x \in X : |F(x)| \leq m + |G(x)|\} \Subset X \tag{12}$$

for every $m \in \mathbb{N}$.

In order to construct such an F we fix an exhaustion sequence $\{K_j\}_{j\in\mathbb{N}}$ of X by $\mathsf{S}(X)$-convex compact subsets and a sequence $\{P_j\}_{j\in\mathbb{N}}$ of polyedra satisfying $K_j \subset P_j \Subset Kj + 1$ for every $j \in \mathbb{N}$. Let

$$M_j = \sup_{P_j} |G|.$$

It is then sufficient to construct $F \in \mathcal{CR}(X; \mathbb{C}^{2n+3})$ satisfying

$$|F| \geq k \text{ in } P_{k+1} \backslash P_k \tag{13}$$

for all $k \in \mathbb{N}$. Let P_k be defined by $f_1^{(k)}, \ldots, f_{2(n+1)}^{(k)}$. Then

$$\max_{P_{k-1}} |f_j^{(k)}| < 1, \ \max_{\mathrm{b}P_k} |f_j^{(k)}| < 1$$

For every fixed $a \in \mathbb{N}$ let D^a denote any derivative of order $\leq a$ on X and set $F_j^{(k)} = (a_k f_j^{(k)})^{m_k}$ where $1 < a_k$. and $m_k \in \mathbb{N}$. We can choose a_k and m_k in such a way to have for $1 \leq j \leq 2(n+1)$

$$\max_{P_{k-1}} |D^k F_j^{(k)}| < 2^{-k},$$

$$\max_{\mathrm{b}P_k} |F_j^{(k)}| > M_{k+1} + k + 1 + \max_{\mathrm{b}P_k} \left| \sum_{s=1}^{k-1} F_j^{(s)} \right|.$$

It follows that the functions $F_1, \ldots, F_{2(n+1)}$

$$F_j = \sum_{k=1}^{+\infty} F_j^{(k)}$$

are CR and satisfy

$$\max_{bP_k} |F_j| > M_{k+1} + k. \tag{14}$$

In order to prove that (14) holds on $P_{k+1} \backslash P_k$ i.e. (12) we apply Theorem 4.2. Indeed, let

$$A_k = \Big\{ z \in P_{k+1} \backslash P_k : \max_{1 \le j \le 2(n+1)} |F_j(z)| < M_{k+1} + k \Big\}$$
$$B_k = \Big\{ z \in P_k : \max_{1 \le j \le 2(n+1)} |F_j(z)| < M_{k+1} + k \Big\}.$$

By (14) A_k, B_k are compact disjoint sets and the S-envelope of $\widehat{C}_k$ of $C_k = A_k \cup B_k$ is contained in K_{k+2}, so $\widehat{C}_k = A_k \cup B_k \cup B'_k$ where $B'_k \subset X \backslash P_{k+1}$. By virtue of Theorem 4.2 the function which is 0 on $B_k \cup B'_k$ and a large constant on A_k can be approximated by functions in $\mathsf{S}(X)$, so arguing as before we construct a sequence of functions

5. – Cohomology.

In this section we will prove the following

THEOREM 5.1. – *Let X be a real analytic foliation with complex leaves of dimension n and real codimension 1. Assume that X is 1-complete and transversally pseudoconvex. Then*

$$H^r(X, \mathsf{S}) = 0$$

for every $r \geq 1$.

PROOF. – Let us point out the main steps of the proof.

We denote $\phi : Z \to \mathbb{R}^+$ the function displaying the 1-completeness of X and set $X_c = \{\phi < c\} \cap X$, $c \in \mathbb{R}$. Then we have the following:

a) $H^r(\overline{X}_c, \mathsf{S}) = 0$ for every $r \geq 1$, $c \in \mathbb{R}^+$.

Indeed, in view of Theorem 4.1, $\overline{X}_c$ has in $\widetilde{X}$ a Stein basis of neighbourhoods. Let U such a neighbourhood and we may assume that $\overline{X}_c$ is connected and $U \backslash \overline{X}_c$ has two connected components U^+, U^-. Let $\mathcal{O}^{\pm}$ denote the sheaf of germs of holomorphic functions on $U^{\pm}$ smooth up to $U \cap X$ extended by 0 on whole U. Extending S by 0 we get an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^+ \oplus \mathcal{O}^- \xrightarrow{\mathsf{re}} \mathsf{S} \longrightarrow 0.$$

Since U is Stein we obtain the isomorphism

$$H^r(U^+, \mathcal{O}) \oplus H^r(U^-, \mathcal{O}) \cong H^r(U \cap X, \mathsf{S})$$

for $r \geq 1$.

Then, an r-cocycle ξ with values in S on a neighbourhood of $\overline{X}_c$ is represented as a difference $\omega^+ - \omega^-$ where $\omega^\pm$ is a $\overline{\partial}$-closed form on $U^\pm$ smooth up to X. In view of Kohn's theorem (cfr. [5]), $\omega^\pm = \overline{\partial}\eta^\pm$ with $\eta^\pm$ smooth on $V^\pm$ up to X ($\overline{X}_c \subset V \Subset U$, V open), so ξ is an r-coboundary.

β) For every $c \in \mathbb{R}^+$ there exists $\varepsilon > 0$ such that the natural homomorphism

$$H^r(\overline{X}_{c+\varepsilon}, \mathsf{S}) \to H^r(X_c, \mathsf{S})$$

is onto for $r \geq 1$.

Since ϕ is strongly plurisubharmonic along the leaves of X the "bump lemma" of Andreotti-Grauert applies [3, Lemma 1].

γ) a) and β) together give

$$H^r(X_c, \mathsf{S}) = 0$$

for every $r \geq 1$, $c \in \mathbb{R}^+$, and then, by a classical argument we get

$$H^r(X, \mathsf{S}) = 0$$

for every $r \geq 2$.

δ) Finally, the vanishing of the first group $H(X, \mathsf{S})$ is proved, again by a classical argument, taking into account the density theorem (cf. Theorem 4.1). □

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