Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **3** (2010), n.1, p. 93–109.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2010_9_3_1_93_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2010.

Bollettino U. M. I.
(9) **III** (2010), 93-109

Projective Geometry Related to the Singularities of Theta Divisors of Jacobians

C. CILIBERTO - E. SERNESI

Dedicated to the memory of Aldo Andreotti

Abstract. – *By studying the higher order focal properties of a family of linear spaces naturally associated to the singular locus of the theta divisor of a non-hyperelliptic jacobian of genus $g \geq 5$, we give a new proof of the classical theorem of Torelli. Related questions and open problems are also discussed.*

Introduction.

Let C be a non-hyperelliptic curve of genus $g \geq 5$ over $\mathbb{C}$, and let $(J(C), \Theta)$ be its principally polarized jacobian. The Riemann theta divisor Θ has a singular locus of pure dimension $g-4$, which is irreducible unless C is either trigonal or bielliptic, in which cases it consists of two irreducible components. In any event, each component of $\mathrm{Sing}(\Theta)$ is reduced, and the projectivized tangent cone to Θ at a general point of such a component is a rank four quadric containing the canonical curve $\kappa(C) \subset \mathbb{P}^{g-1}$. The beautiful geometry of this family of rank four quadrics has been exploited in the epochal paper [1], in which A. Andreotti and A. Mayer proposed an extremely original attack to Schottky's problem. Indeed they proved that the jacobian locus $\mathcal{J}_g$ in $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g, is an irreducible component of the so-called *Andreotti-Mayer locus* $\mathcal{N}_{g-4}$, i.e. the locus of points in $\mathcal{A}_g$ corresponding to all pairs (A, Θ) such that $\dim(\mathrm{Sing}(\Theta)) \geq g-4$. It is still an open problem however to characterize $\mathcal{J}_g$ among the various components of $\mathcal{N}_{g-4}$. For informations, conjectures and recent results on this subject see [5].

In [6, 7], rather than the aforementioned family of rank four quadrics through the canonical curve, we studied the family of $\mathbb{P}^{g-3}$'s in $\mathbb{P}^{g-1}$, contained in such quadrics, which are $(g-1)$–secant to $\kappa(C)$ (see also [8]). Our main tool was the so–called *theory of foci*, a classical subject in projective differential geometry, which goes back to the second half of XIX century. Indeed, one of our sources of inspiration was the paper [14] by C. Segre. We prove in fact that the focal locus in the general space Λ of the above family of $\mathbb{P}^{g-3}$'s is a non–degenerate curve F of degree $g-3$. This is a rational normal curve, unless C is *exceptional*, i.e. trigonal, bielliptic or a smooth plane quintic, in which case F is reducible. With a rather subtle analysis we were able to reconstruct the canonical curve from the family of focal curves, thus giving a new proof of Torelli's theorem, which in our view was conceptually different form the existing ones.

In the present paper we go back to this approach. Actually, we go deeper into the study of focal properties, by exploiting *higher order foci*, i.e. the foci of the family of first order rational normal focal curves, in case C is not exceptional. It turns out that the canonical curve can be reconstructed from higher order foci, thus providing a new and more transparent proof of Torelli's theorem in this way.

However, were only for another proof of Torelli's theorem, we would have probably not invested time and energy in writing the present note. The fact is that we believe that the interest of the present higher order focal approach goes well beyond the scope of this paper. Indeed, what our construction suggests is that a central object in the geometry of a canonical curve is the *focal locus* Φ described as the closure of the union of all focal curves $F \subset \Lambda$, when Λ varies among all $\mathbb{P}^{g-3}$'s which are $(g-1)$–secant to $\kappa(C)$. We conjecture that, if C has general moduli, Φ is a hypersurface and $\kappa(C)$ is an irreducible component of its singular locus. This conjecture (which holds for $g=5,6$, see [4]) suggests a new attack to the Schottky's problem in the spirit of Andreotti–Mayer. Indeed, the same focal construction can be made any time we have a principally polarized abelian variety (A,Θ), with $\mathrm{Sing}(\Theta)$ reduced of dimension $g-4$, i.e. for the general point of any *well behaved* component of $\mathcal{N}_{g-4}$. It is then reasonable to conjecture also that $\mathcal{J}_g$ is the only component of $\mathcal{N}_{g-4}$ such that, for its general point (A,Θ) the corresponding focal locus Φ is a hypersurface whose singular locus has a (genus g, canonical) curve as an irreducible component. An affirmative answer to this conjecture would be, in our view, a fulfilment of Andreotti–Mayer's original project. It should finally be the case that the Brill–Noether theory of C is reflected in geometric properties of the focal variety. In conclusion, the present approach opens up a whole uncharted territory, which we plan to explore in the next future. The present paper should be seen as a warming up for this.

This paper is organized as follows. In § 1 we recall the general theory of foci. In § 2 we concentrate on focal properties of $(r-2)$–dimensional families of $\mathbb{P}^{r-2}$'s in $\mathbb{P}^r$ for $r \geq 4$ The short § 3 is devoted to the particular example of families of $\mathbb{P}^{r-2}$'s contained in the rank four quadrics of a family of dimension $r-3$ in $\mathbb{P}^r$. In § 4 we consider the main case of interest for us, i.e. the family of $\mathbb{P}^{g-3}$'s which are $(g-1)$–secant to a canonical curve of genus g. Finally § 5 is devoted to the proof of Torelli's theorem.

1. – The general theory of foci.

In this paper we work over the complex field. Consider a flat family of closed subschemes of a projective scheme Y:

$$\begin{tikzcd} \Xi \arrow[r, hook] \arrow[d, "f"'] & B \times Y \\ B & \end{tikzcd} \tag{1}$$

parametrized by a scheme B, where f is induced by the projection

$$q_2 : B \times Y \longrightarrow Y.$$

We will denote by $\Xi(b)$ the fibre of $f : \Xi \to B$ over $b \in B$.

Denote by

$$\mathcal{N} := N_{\Xi/B\times Y}$$

the normal sheaf of Ξ in $B \times Y$, and let

$$T(q_2)_{|\Xi}$$

be the restriction to Ξ of the tangent sheaf along the fibres of q_2. The *global characteristic map* of the family (1) is the homomorphism:

$$\chi : T(q_2)_{|\Xi} \longrightarrow \mathcal{N}$$

defined by the following exact and commutative diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & \\ & & T(q_2)_{|\Xi} \arrow[r, "\chi"] \arrow[d] & \mathcal{N} \arrow[d, equal] \\ 0 \arrow[r] & T_\Xi \arrow[r] \arrow[d, "df"'] & T_{B\times Y|\Xi} \arrow[r] \arrow[d] & \mathcal{N} \\ & f^*T_Y \arrow[r, equal] & f^*T_Y & \end{tikzcd} \tag{2}$$

For each $b \in B$ the homomorphism χ induces a homomorphism

$$\chi_b : T_{B,b} \otimes \mathcal{O}_{\Xi(b)} \longrightarrow N_{\Xi(b)/Y}$$

called the *characteristic map* of the family (1) at the point b. Note that the flat family (1) induces a functorial morphism

$$\varphi : B \longrightarrow Hilb_Y$$

where $Hilb_Y$ is the Hilbert scheme parametrizing closed subschemes of Y. Then the linear map:

$$H^0(\chi_b) : T_{B,b} \longrightarrow H^0(N_{\Xi(b)/Y})$$

is $d\varphi_b$, the differential of φ at the point b.

In the following we will assume that both Y and the family (1) are smooth. In this case all the sheaves in (2) are locally free. From a diagram-chasing it follows that

$$\ker(\chi) = \ker(df)$$

and therefore

PROPOSITION 1.

$$\dim[\, f(\Xi)] = \dim(\Xi) - \mathrm{rk}\,[\ker(\chi)]$$

Let's denote by $V(\chi)$ the closed subscheme of Ξ defined by the condition:

$$\mathrm{rk}(\chi) < \min\{\mathrm{rk}\,[T(q_2)_{|\Xi}],\ \mathrm{rk}(\mathcal{N})\} = \min\{\dim(B), \mathrm{codim}_{B\times Y}(\Xi)\}$$

We will call the points of $V(\chi)$ *first order foci* of the family (1). $V(\chi)$ is *the scheme of first order foci*, and the fiber of $V(\chi)$ over a point $b \in B$:

$$V(\chi)_b = V(\chi_b) \subseteq \Xi(b)$$

is *the scheme of first order foci at* b.

If χ has maximal rank, i.e. if χ is either injective or has torsion cokernel, then $V(\chi)$ is a proper closed subscheme of Ξ. If χ does not have maximal rank then $V(\chi) = \Xi$.

2. – Families of $\mathbb{P}^{r-2}$'s in $\mathbb{P}^r$.

We will now consider a special case of the situation considered above, namely $(r-2)$-dimensional families of $\mathbb{P}^{r-2}$'s in an r-dimensional projective space, with $r \geq 4$. Our motivation is a specific example of such a family which arises in connection with the geometry of canonical curves. Before considering such example in detail in § 4 below, we want to make some general remarks which will be helpful later on.

Let V be an $(r+1)$-dimensional vector space, $r \geq 4$, $\mathbb{P} = \mathbb{P}(V)$, and let

$$\Lambda \subset S \times \mathbb{P} \tag{3}$$

be a family of $(r-2)$-dimensional linear subspaces of $\mathbb{P}$, parametrized by a quasi-projective irreducible and nonsingular scheme S of dimension $\dim(S) = r-2$. Therefore

$$\dim(\Lambda) = 2r - 4$$

Let

$$\begin{tikzcd} S \times \mathbb{P} \arrow[r, "q_2"] \arrow[d, "q_1"'] & \mathbb{P} \\ S & \end{tikzcd}$$

be the projections, and

$$\begin{tikzcd} \mathbf{\Lambda} \arrow[r, "f"] \arrow[d, "q_1"'] & \mathbb{P} \\ S & \end{tikzcd}$$

the induced maps. We will assume that the functorial map

$$\varphi : S \longrightarrow G_{r-2}(\mathbb{P})$$

to the grassmannian of $(r-2)$–dimensional linear subspaces of $\mathbb{P}$ is generically finite to its image. The family (3) is called *non–degenerate* if f is dominant. Note that

$$\dim(f(\Lambda)) \geq r-1$$

since otherwise the family (3) would be constant.

LEMMA 1. – *The family* (3) *is non–degenerate if and only if* $f(\Lambda)$ *is not contained in a hyperplane.*

PROOF. – Assume that the family is degenerate. Then $\overline{f(\Lambda)}$ is a hypersurface of $\mathbb{P}$ containing an $(r-2)$-dimensional family of $\mathbb{P}^{r-2}$'s. By a theorem of B. Segre ([13], § 8), $\overline{f(\Lambda)}$ is a hyperplane, a contradiction. □

For the rest of this section we will assume that (3) is non degenerate. For each $x \in f(\Lambda)$ we have:

$$r-2 \geq \dim[f^{-1}(x)] \geq 2r-4-r = r-4$$

The first equality holds if and only if every fibre Λ_s of the family (3) contains x. Then (3) is obtained from a $(r-2)$–dimensional family Λ' of linear spaces of dimension $r-3$ in a hyperplane of $\mathbb{P}$ not containing x, by joining the linear spaces in Λ' with x. If this happens, we will say that Λ presents the *cone case*.

The second equality holds for a general choice of x. A *fundamental point* of the family (3) is a point $x \in f(\Lambda)$ such that

$$r-2 \geq \dim[f^{-1}(x)] \geq r-3$$

From the hypothesis that (3) is non degenerate and from Lemma 1 it follows that the global characteristic map χ of our family has torsion cokernel.

Let $s \in S$ and Λ_s be the corresponding element of the family (3). Then $\Lambda_s = \boldsymbol{P}(U)$, where $U \subset V$ is a subspace of codimension 2. Then:

$$N_{\Lambda_s} = \frac{V}{U} \otimes \mathcal{O}_{\Lambda_s}(1)$$

and the characteristic map

$$\chi_s : T_sS \otimes \mathcal{O}_{\Lambda_s} \to \frac{V}{U} \otimes \mathcal{O}_{\Lambda_s}(1)$$

induces the map:

$$H^0(\chi_s) : T_sS \to \mathrm{Hom}\left(U, \frac{V}{U}\right)$$

which coincides with $d\varphi_s$, the differential of φ at the point s. The assumption that φ is generically finite to its image implies that $H^0(\chi_s)$ is injective if $s \in S$ is a general point.

Since χ_s is a map of locally free sheaves of ranks $r-2$ and 2 respectively on $\Lambda_s \cong \mathbb{P}^{r-2}$, the focal scheme $V(\chi_s)$ has codimension at most $r-3$, and one may expect that if χ_s is *sufficiently general* then $V(\chi_s)$ is a rational normal curve.

The map χ_s is said to be *1-generic* if every non-zero element of T_sS is mapped via $H^0(\chi_s)$ to a surjective homomorphism $U \longrightarrow V/U$. This is a *generality assumption* on $H^0(\chi_s)$. Indeed, assuming that $H^0(\chi_s)$ is injective, which is the case if $s \in S$ is general, and projectivizing this map, we have an embedding of $\mathbb{P}(T_sS) \simeq \mathbb{P}^{r-3}$ into $\mathbb{P}\left(U^* \otimes \frac{V}{U}\right) \simeq \mathbb{P}^{2r-3}$ as a linear space Π, and the requirement is that Π should not intersect the Segre variety $\mathbb{P}(U^*) \times \mathbb{P}\left(\frac{V}{U}\right)$, which has dimension $r-1$.

The following result will be used:

PROPOSITION 2. – *$V(\chi_s)$ is a rational normal curve if and only if χ_s is 1-generic.*

For the proof we refer to [10].

We will assume from now on in this section that for a general $s \in S$ the focal scheme $V(\chi_s) \subset \Lambda_s$ is a rational normal curve F_s.

Then $V(\chi)$ has a unique irreducible component F dominating S having dimension $\dim(F) = r-1$. We set $\Phi = \overline{f(F)}$ and call it the *focal variety* of the family (3). One has:

(4) $$2 \leq \dim[\Phi] \leq r-1$$

PROPOSITION 3. – *Assume that* $\dim[\Phi] = 2$. *Then* Φ *is a rational normal scroll if* $r \geq 5$. *If* $r = 4$ *then* Φ *is either a rational normal scroll or a projection of the Veronese surface.*

PROOF. – If $r = 4$ then $\Phi \subset \mathbb{P} = \mathbb{P}^4$ is a surface containing a 2-dimensional family of conics. Therefore, by a classical theorem of Darboux [9] (for a recent reference, see for instance [12], Theorem 32) it must be a rational normal scroll or the projection of the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from an outer point.

If $r \geq 5$ then we project Φ from $r - 4$ general points. We get a surface $\Phi' \subset \mathbb{P}^4$ containing a 2-dimensional family of conics. Since Φ' contains lines, i.e. the images of the centers of projection, it must be a scroll and therefore S is a rational normal scroll. □

We can now apply the machinery of § 1 to the *focal family*:

$$F \subset S \times \mathbb{P} \tag{5}$$

for which we have a global characteristic map:

$$\xi : T(q_2)_{|F} \longrightarrow N_{F/S\times\mathbb{P}}$$

The first order foci of (5), i.e. the points of $V(\xi)$, are called *second order foci* of the family (3). We obtain a scheme of second order foci $F^{(2)} := V(\xi) \subset F$. By Proposition (1) we have

$$\dim[f(F)] = r - 1 - \operatorname{rk}[\ker(\xi)]$$

and inequalities (4) correspond to the inequalities:

$$r - 3 \geq \operatorname{rk}[\ker(\xi)] \geq 0$$

If $\dim[f(F)] < r - 1$ then $F^{(2)} = F$. If $\dim[f(F)] = r - 1$ then $F^{(2)}$ is a proper closed subscheme of F. In any case

$$\dim[f(F^{(2)})] \leq r - 2$$

Let's analize the second order foci more closely.

Let

$$F_s \subset \Lambda_s =: \Lambda = \mathbb{P}(U) \subset \mathbb{P}$$

with $s \in S$, be a general element of the family (5). Then:

$$\xi_s = \xi \otimes \mathcal{O}_{F_s} : T_sS \otimes \mathcal{O}_{F_s} \longrightarrow N_{F_s}$$

where $N_{F_s} = N_{F_s/\mathbb{P}}$. Moreover

$$\ker(\xi_s) = \ker(\xi) \otimes \mathcal{O}_{F_s}$$

because $\mathrm{Im}(\xi)$ is torsion free and therefore $\mathrm{Tor}_1(\mathrm{Im}\,(\xi), \mathcal{O}_{F_s}) = 0$. Therefore, if we want to know $\dim\,[f(F)]$ it will suffice to know $\mathrm{rk}\,[\ker\,(\xi_s)]$ for a general $s \in S$. Let's denote by L the invertible sheaf of degree 1 on F_s and consider the following exact and commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker(\rho) \arrow[r] \arrow[d] & T_sS \otimes \mathcal{O}_{F_s} \arrow[r, "\rho"] \arrow[d, "\xi_s"'] & \frac{V}{U} \otimes \mathcal{O}_{F_s} \arrow[d, equal] & \\ 0 \arrow[r] & N_{F_s/\Lambda} \arrow[r] & N_{F_s} \arrow[r, "g"'] & \frac{V}{U} \otimes \mathcal{O}_{F_s} \arrow[r] & 0 \end{tikzcd} \tag{6}$$

where the homomorphism ρ is defined by the diagram. The second row is the exact sequence of normal bundles of F_s: it splits because

$$\mathrm{Ext}^1(\mathcal{O}_{F_s}(1), N_{F_s/\Lambda}) \cong H^1(F_s, N_{F_s/\Lambda}(-1)) = 0$$

since $N_{F_s/\Lambda} \cong (L^r)^{\oplus(r-3)}$ and therefore $N_{F_s/\Lambda}(-1) \cong (L^2)^{\oplus(r-3)}$.

LEMMA 2. – $\mathrm{Im}\,(\rho)$ *is an invertible sheaf of degree* $r-3$.

PROOF. – Since F_s is a focal curve, one has $\mathrm{rk}[\rho] \leq 1$ everywhere on F_s. On the other hand

$$H^0(\rho) : T_sS \longrightarrow \mathrm{Hom}\left(U, \frac{V}{U}\right)$$

is injective, hence $\rho \neq 0$. Therefore $\mathrm{Im}\,(\rho)$ is an invertible sheaf. The injectivity of $H^0(\rho)$ implies that $h^0(\mathrm{Im}\,(\rho)) \geq r-2$, and therefore $\deg[\mathrm{Im}\,(\rho)] \geq r-3$. Moreover, since:

$$\frac{V}{U} \otimes \mathcal{O}_{F_s}(1) \cong \mathcal{O}_{F_s}(1) \oplus \mathcal{O}_{F_s}(1)$$

composing with the two projections we get at least one non-trivial homomorphism

$$\mathrm{Im}\,(\rho) \longrightarrow \mathcal{O}_{F_s}(1)$$

and therefore:

$$\deg\,[\mathrm{Im}\,(\rho)] \leq \deg\,[\mathcal{O}_{F_s}(1)] = r-2$$

Let's assume that $\deg\,[\mathrm{Im}\,(\rho)] = r-2$. Then

$$\mathrm{Im}\,(\rho) = \langle v \rangle \otimes \mathcal{O}_{F_s}(1)$$

for some $v \in \dfrac{V}{U}$. But then $H^0(\rho)(T_sS) \subset \mathrm{Hom}(U, \langle v \rangle)$ does not consist of surjective homomorphisms, contradicting 1-genericity. Therefore $\deg\,[\mathrm{Im}\,(\rho)] = r-3$. $\square$

From Lemma 2 it follows that

$$\mathrm{Im}\,(\rho) \cong L^{r-3};$$

moreover, since $H^0(\ker(\rho)) = 0$, $\deg[\ker(\rho)] = -(r-3)$ and $\ker(\rho)$ is locally free of rank $r-3$, it follows that:

$$\ker(\rho) \cong \bigoplus^{r-3} L^{-1}$$

Denote by $E = g^{-1}(\mathrm{Im}\,(\rho))$, which is a locally free subsheaf of rank $r-2$ of N_{F_s}. Recalling that

$$N_{F_s/\Lambda} \cong \bigoplus^{r-3} L^{r}$$

we deduce from (6) the following commutative and exact diagram:

$$\text{(7)} \qquad \begin{tikzcd} 0 \arrow[r] & \bigoplus^{r-3} L^{-1} \arrow[r] \arrow[d, "\zeta_s"'] & T_sS \otimes \mathcal{O}_{F_s} \arrow[r, "\rho"] \arrow[d, "\xi_s"'] & L^{r-3} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bigoplus^{r-3} L^{r} \arrow[r] & E \arrow[r] & L^{r-3} \arrow[r] & 0 \end{tikzcd}$$

The second row splits and

$$E \cong \bigoplus^{r-3} L^{r} \oplus L^{r-3}$$

The second order foci in Λ are defined by $\det(\xi_s)$, which, if not identically zero (i.e. if ξ_s is injective), has degree

$$\deg(\det(E)) = (r-3)(r+1)$$

Note that the degrees of the entries of the *focal matrix* defining ξ_s are as follows:

$$\text{(8)} \qquad \begin{pmatrix} r & r & \cdots & r \\ \cdots & \cdots & \cdots & \cdots \\ r & r & \cdots & r \\ r-3 & r-3 & \cdots & r-3 \end{pmatrix}$$

The injectivity of ξ_s is equivalent to that of ζ_s: if they hold then $\det(\zeta_s) = \det(\xi_s)$. In particular from the above discussion we obtain:

PROPOSITION 4. – *If the focal variety Φ is a hypersurface, then, for $s \in S$ general, the scheme of second order foci in Λ_s is a subscheme $F_s^{(2)} \subset F_s$ of finite lenght equal to $(r-3)(r+1)$.*

The next proposition explains the role of fundamental points.

PROPOSITION 5. – *Assume that Φ is a hypersurface. Let $x \in f(\Lambda)$ be a fundamental point of the family* (3). *Then x is a second order focus which counts with multiplicity at least $r-3$ in each $F_s^{(2)}$ to which it belongs.*

PROOF. – Let $z \in f^{-1}(x)$ and $s = q_1(z)$. The hypothesis implies that $\dim(f^{-1}(x)) = r-4+h$ with $1 \leq h \leq 2$. Then

$$\dim[\ker(\chi_s)] = \dim[\ker(df_z)] \geq r-4+h$$

and therefore $z \in V(\chi) = F$.

It follows that $f^{-1}(x) \subset F$, and that

$$\dim[\ker(\xi_s)] = \dim[\ker(df_z)] \geq r-4+h$$

Therefore we see that

$$\mathrm{rk}_z[\mathrm{Im}(\xi_s)] \leq r-2-(r-4+h) = 2-h$$

This means that a lower bound for the multiplicity of x in $F_s^{(2)}$ is

$$\mathrm{cork}_z(\xi_s) \geq r-4+h \geq r-3$$

□

If Φ is not a hypersurface then $F_s^{(2)} = F_s$ for general $s \in S$. This means that $\mathrm{rk}(\xi_s) \leq r-3$ identically on F_s. In this case we define *the scheme of 3-rd order foci $F_s^{(3)}$* $\subset F_s$ as the subscheme defined by the condition

$$\mathrm{rk}(\xi_s) \leq r-4$$

i.e. by the submaximal minors of ξ_s. It is still possible that this condition is identically satisfied on F_s. Then we proceed to define higher order foci. Precisely, let $r-k = \dim[\Phi]$, $k \geq 2$, be the rank of ξ_s at the general point of F_s. Then *the scheme of foci of order k* is the closed subscheme $F_s^{(k)}$ of F_s defined by the condition

$$\mathrm{rk}(\xi_s) \leq r-k-1$$

PROPOSITION 6. – (i) $\deg(F_s^{(k)}) \leq (r-k-1)r+r-3$ *for each $s \in S$ such that $F_s^{(k)}$ is defined.*

(ii) *If $x \in f(\Lambda)$ is a fundamental point of the family* (3) *then x is a focus of order k which counts with multiplicity at least $r-k-1$ in each $F_s^{(k)}$ to which it belongs.*

PROOF. – The degree of $F_s^{(k)}$ is bounded by the minimum degree of a non-identically zero minor of order $r-k$ of the matrix (8) defining ξ_s. We can always arrange that such a minor includes non–zero entries of the last row of (8), because this row is not identically zero on F_s (see the proof of Lemma 2). Therefore, recalling (8), we deduce (i). The proof of part (ii) is the same as in Proposition 5. □

3. – An example: families of quadrics of rank 4.

Important special cases of the families of $\mathbb{P}^{r-2}$'s in $\mathbb{P}^r$ considered in § 2 arise from families of quadrics. Consider a family of quadrics in $\mathbb{P}^r$:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, hook] \arrow[d, "q"] & \Sigma \times \mathbb{P}^r \\ \Sigma & \end{tikzcd} \tag{9}$$

parametrized by an irreducible and nonsingular quasi-projective scheme Σ of dimension $r-3$. Assume that all fibres of q are quadrics of rank 4. Each of them contains two 1-dimensional rulings of $\mathbb{P}^{(r-2)}$'s. When $s \in \Sigma$ varies we obtain an $(r-2)$-dimensional family of $\mathbb{P}^{(r-2)}$'s: it is the q-relative Hilbert scheme of linear spaces of dimension $r-2$:

$$\begin{tikzcd} & \mathbf{\Lambda} \arrow[r, hook] \arrow[dl, "f"'] \arrow[d, "q_1"] & \mathbb{Q} \arrow[r, hook] \arrow[d, "q"] & \Sigma \times \mathbb{P}^r \\ \mathbb{P}^r & S \arrow[r, "\alpha"'] & \Sigma & \end{tikzcd} \tag{10}$$

S is quasi-projective and nonsingular of dimension $r-2$, and f denotes the projection. The morphism a has degree 2. We have the following useful fact:

PROPOSITION 7. – *Assume that the family q_1 is non-degenerate and that:*

– *for a general $s \in S$, the focal scheme $F_s \subset \Lambda_s$ is a rational normal curve.*
– *the focal variety Φ is a hypersurface.*

Let v_s be the vertex of the quadric $Q_{a(s)}$. Then $v_s \cap F_s$ contains at least $r-3$ points for a general $s \in S$.

PROOF. – See [7], p. 889. □

4. – Secant spaces to canonical curves.

In this section we will consider a special remarkable example of a family (9). We start by introducing the basic terminology and notation. For what not expressely proved here we refer ther reader to § 1 of [7].

Let C be a projective irreducible nonsingular nonhyperelliptic curve of genus $g \geq 5$ and let $\mathbb{P} = \mathbb{P}(V)$, where $V = H^1(C, \mathcal{O}_C)$. Denote by

$$\kappa : C \longrightarrow \mathbb{P}$$

the canonical embedding of C, defined by the complete canonical linear series $|K|$.

As customary, we will use the symbol g^r_n to mean "a linear series of dimension r and degree n". We denote by C_n the n-th symmetric product of C and by $W_n(C)$ the image of the Abel-Jacobi map

$$a : C_n \longrightarrow \mathrm{Pic}^n(C)$$

In particular we denote as usual by

$$\Theta = W_{g-1}(C) \subset \mathrm{Pic}^{g-1}(C)$$

the theta divisor. Moreover we let

$$C^r_n = \{D \in C_n : h^0(D) \geq r+1\}$$

and

$$W^r_n = W^r_n(C) = a_n(C^r_n) = \{L \in \mathrm{Pic}^n(C) : h^0(L) \geq r+1\}$$

We will consider W^r_n with its natural scheme structure. By the Riemann singularity theorem we have

$$\mathrm{Sing}(\Theta) = W^1_{g-1}$$

and the double point locus of Θ is

$$\mathrm{Sing}_2(\Theta) = W^1_{g-1} \backslash W^2_{g-1}$$

$\mathrm{Sing}_2(\Theta)$ has a scheme structure and its smooth locus is a dense open subset Σ of pure dimension $g-4$. We let

$$S = a^{-1}_{g-1}(\Sigma)$$

which is an open and dense subset of $C^1_{g-1} \backslash C^2_{g-1}$. The restriction of a_{g-1} to S is a $\mathbb{P}^1$-bundle:

$$a : S \longrightarrow \Sigma$$

In particular S is smooth of pure dimension $g-3$.

We have a family of linear spaces of codimension 2 in $\mathbb{P}$ parametrized by S which is naturally defined as follows.

Consider the universal divisor of degree $g-1$

$$\mathbb{D}_{g-1} \subset C_{g-1} \times C$$

and let $\mathbb{D}_S := \mathbb{D}_{g-1} \cap (S \times C)$. Denoting by $p : S \times C \longrightarrow S$ the projection, we have a homomorphism of locally free sheaves:

$$R^1 p_* \mathcal{O}_{S\times C} \longrightarrow R^1 p_* \mathcal{O}_{S\times C}(\mathbb{D}_S) \longrightarrow 0$$

whose kernel is a locally free subsheaf $\mathcal{F}$ of

$$R^1 p_* \mathcal{O}_{S\times C} \cong H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_S$$

of rank $g-2$. Taking associated projectivized bundles we obtain

$$\Lambda := \mathbb{P}(\mathcal{F}) \subset \mathbb{P}(R^1 p_* \mathcal{O}_{S\times C}) = S \times \mathbb{P} \tag{11}$$

The fibre Λ_s over a point $s \in S$ is the linear span $\langle D_s \rangle$ of the divisor D_s of degree $g-1$ parametrized by s. Therefore (11) is a non–degenerate family of linear spaces of codimension 2 in $\mathbb{P}$ which are $(g-1)$-secant to the canonical curve $\kappa(C)$.

Note that the family Λ arises from a family of rank 4 quadrics, as described in § 3. Indeed, given a linear series ξ corresponding to a point of Σ with residual ξ' with respect to $|K|$, then

$$Q_\xi = \bigcup_{D\in\xi} \langle D \rangle$$

is a quadric of rank 4, unless $\xi = \xi'$, in which case it has rank 3. If Q_ξ has rank 4, the other ruling of Q_ξ is given by all spaces $\langle D' \rangle$, with $D \in \xi'$.

Keeping the notation of § 2 we denote by $V(\chi) \subset \Lambda$ the focal scheme of the family (11).

THEOREM 1. – *If $s \in S$ is a general point of an irreducible component, and D_s is the divisor of degree $g-1$ cut on C by Λ_s, then $F_s = V(\chi_s) \subset \Lambda_s$ is a rational normal curve if and only if the pencil $|D_s|$ is base point free.*

PROOF. – See Corollary 2.6 of [7]. □

For simplicity we will call C *exceptional* if it is either trigonal, or bielliptic, or a nonsingular plane quintic. These are precisely the cases in which $W^1_{g-1} = \mathrm{Sing}(\Theta)$ has an irreducible component consisting of g^1_{g-1}'s having base points (see [7], Theorem 1.4). Moreover, if C is not exceptional, the ideal of $\kappa(C)$ is generated by quadrics (see [2]). Hence the family (11) does not present the cone case, because, by a theorem of M. Green [11], the rank four quadrics containing $\kappa(C)$ generate the quadratic part of the ideal of $\kappa(C)$ if C is non–hyperelliptic.

If C is not exceptional then S is irreducible and nonsingular of dimension $g-3$ and for a general $s \in S$ the pencil $|D_s|$ is a base point free ${}^1_{g-1}$. Therefore we see that the cases in which the conclusion of Theorem 1 does not hold are precisely when C is exceptional. Let's briefly recall what happens in the exceptional cases, which we have studied in [7].

(a) If C is trigonal then $W^1_{g-1}(C) = W \cup W'$ has two irreducible components, which are interchanged by residuation with respect to ω_C. One of them, say W, consists generically of linear series of the form $L = h + P_1 + \cdots P_{g-4}$, where h is the g^1_3 and $P_1, \ldots, P_{g-4} \in C$ are general points. In particular all such L's have $g-4$ fixed points. If $s \in a^{-1}(L) \subset S$ then the focal scheme is reducible into $g-3$

distinct lines:

$$F_s = r_1 \cup \cdots \cup r_{g-3}$$

where r_1 is a trisecant of $\kappa(C)$, and $r_2, \ldots, r_{g-3}$ meet r_1 in $g-4$ distinct points. The curve F_s has only nodes as singularities and arithmetic genus zero.

(b) If C is bielliptic then $W^1_{g-1}(C) = W \cup W'$ has two irreducible components, which are mapped into themselves by residuation with respect to ω_C. One of them, say W, consists generically of linear series having $g-5$ base points, the movable part being the pull back on C of a g^1_2 on the elliptic curve 2-to-1 covered by C. For a general $s \in S$ such that $a(s) \in W$ the focal curve F_s consists of $g-3$ generating lines of the elliptic cone containing $\kappa(C)$.

(c) If C is a nonsingular plane quintic, one has $g = 6$ and the general g^1_5 on C has a base point Q, the movable part being a g^1_4 cut out on C by a pencil of lines through a point $P \in C$ with $P \neq Q$. If $D = P_1 + \ldots + P_4 + Q$ is a divisor of such a series, with $P_1, \ldots, P_4$ on a line ℓ, then D corresponds to a point s of the unique irreducible component of S and

$$F_s = \Gamma \cup r$$

where r is a line, Γ is the conic image of ℓ on the Veronese surface in $\mathbb{P}^5$ on which $\kappa(C)$ sits, and r is a chord of the Veronese surface, meeting in one point each Γ and $\kappa(C)$.

From Theorem 1 it follows that if C is not exceptional then $V(\chi)$ has a unique irreducible component F dominating S; F has dimension $g-2$, and its fibre F_s over a general point $s \in S$ is the rational normal curve $F_s = V(\chi)_s$. If s corresponds to the reduced divisor $D = P_1 + \ldots + P_{g-1}$, F_s can be described as the unique rational normal curve in $\langle D \rangle$, containing $P_1, \ldots, P_{g-1}$ which is in addition $(g-3)$–secant the vertex of the quadric Q_ξ, with $\xi = |D|$.

5. – A proof of Torelli's Theorem.

We keep the same notation as in the previous section. In particular we let C be a projective irreducible nonsingular nonhyperelliptic curve of genus $g \geq 5$. The open subset $\Sigma \subset \mathrm{Sing}_2(\Theta)$ parametrizes the family of projectivized tangent cones of Θ along Σ:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, hook] \arrow[d, "q"] & \Sigma \times \mathbb{P} \\ \Sigma & \end{tikzcd} \tag{12}$$

and this is a family of quadrics of rank four in $\mathbb{P} \cong \mathbb{P}^{g-1}$. Precisely, this is the family of rank four quadrics containing the canonical curve $\kappa(C)$ (see [3]).

We have the following:

THEOREM 2. – *The family* (12), *and a fortiori the pair* $(\mathrm{Pic}^{g-1}(C), \Theta)$, *determines the family* (11) *and the morphism* $a : S \longrightarrow \Sigma$ *up to composition with the involution* $j : \Sigma \longrightarrow \Sigma$ *induced by residuation with respect to* ω_C.

PROOF. – As we did in § 3 we can consider the relative Hilbert scheme of linear spaces of dimension $g-3$ contained in the fibres of q. It is not difficult to show that the universal family (10) is the family (11). For details see [7], Theorem 1.2. □

Using Theorem 2 and studying the geometry of first order foci of the family (12), in [7] we gave a proof of the classical Torelli's theorem. We will now give another, simpler and more transparent, proof which uses higher order foci.

THEOREM 3 (Torelli). – *Let* C *be a projective irreducible nonsingular non-hyperelliptic and non-exceptional curve of genus* $g \geq 5$. *Then* C *can be reconstructed from the pair* $(\mathrm{Pic}^{g-1}(C), \Theta)$.

PROOF. – Thanks to Theorem 2, it suffices to prove that C can be reconstructed from the family (11). Let $g-1-k = \dim[\Phi]$ be the rank of ξ_s at a general point of F_s and let, as usual, $F_s^{(k)}$ be the scheme of order k foci.

Note that $\dim[\Phi] > 2$. Otherwise, by Proposition 3, Φ would either be a rational normal scroll or a projection of the Veronese surface in $\mathbb{P}^4$. Then the parameter space S of the family (11) would be rational, contrary to the fact that S dominates $\mathrm{Sing}(\Theta)$ which sits in the jacobian of C.

By Proposition 6, we have

$$\deg(F_s^{(k)}) \leq (g-k-2)(g-1) + g - 4$$

Moreover the $g-1$ points of $D_s = \kappa(C) \cap \Lambda_s$ are fundamental points: therefore, again by the same proposition, they are foci of order k, and each of them counts with multiplicity at least $g-k-2 \geq 2$ in $F_s^{(k)}$. Therefore there are at most $g-4$ foci of order k different from those in D_s. The algebraic set

$$Z := \overline{\bigcup_{s \in S^\circ} F_s^{(k)}}$$

(where $S^\circ \subset S$ denotes the open set where foci of order k are defined) contains the curve $\kappa(C)$ as a component, and it cannot contain any other canonical curve as a component. In fact if $Y \subset Z$ is another canonical curve then the family (11) is also the family of $\mathbb{P}^{g-3}$'s which are $(g-1)$-secant to Y and therefore it is also obtained from the pair $(\mathrm{Pic}^{g-1}(Y), \Theta_Y)$. This implies that, for $s \in S$ general, $Y \cap \Lambda_s$ contains $g-1$ points of multiplicity at least $g-4$ for $F_s^{(k)}$. This implies that $Y \cap \kappa(C) \cap \Lambda_s \neq \emptyset$ for $s \in S^\circ$ general. Therefore $Z = \kappa(C)$. □

REMARK 1. – The proof of Theorem 3 does not apply to exceptional curves. However, even in these cases, the family (11) can be used to reconstruct the curve.

If C is either trigonal or bielliptic, the parameter space S has two components. For one of the resulting families, the general focal curve is reducible. For the other it is irreducible and the proof of Theorem 3 applies.

If C is a smooth plane quintic, the general focal curve is of the form $\Gamma \cup r$, where Γ describes the Veronese surface V on which $\kappa(C)$ sits and r cuts V in two points, one on Γ, the other on $\kappa(C)$. When r varies, the latter point describes $\kappa(C)$, which can be thus reconstructed from the family (11).

Even in the hyperelliptic case the focal theory can be applied to prove Torelli's theorem. Let us briefly explain how. In this case $\mathrm{Sing}(\Theta)$ is irreducible of dimension $g-3$, its general point corresponding to a linear series of the form $P_1 + \ldots + P_{g-3} + \xi$, where $P_1, \ldots, P_{g-3}$ are general point on the curve C and ξ is the hyperelliptic involution. The corresponding tangent cones to Θ give rise to a family $\mathbb{Q} \to \Sigma$ of rank four quadrics parametrized by the smooth locus of $\mathrm{Sing}(\Theta)$. As in § 3, this defines a family $\Lambda \to S$, of linear subspaces of dimension $g-3$ in $\mathbb{P}$. Contrary to the non–hyperelliptic case, the parameter space now has dimension $g-2$. Applying the general theory of foci recalled in § 1, one sees that the first order focal scheme $F_s \subset \Lambda_s$ for $s \in S$ general, consists of $g-1$ points. These are precisely the intersections of Λ_s with the rational normal curve $\kappa(C)$, which can be thus reconstructed. Finally one has to locate the branch points of ξ on $\kappa(C)$. This can be done by looking at the ramification of the natural map of S to the family of $(g-1)$–secant $\mathbb{P}^{g-3}$ to $\kappa(C)$.

Note that, in any event, the cases C hyperelliptic or a smooth plane quintic are easy (see [6], Corollary (3.5)).

REFERENCES

[1] A. ANDREOTTI - A. MAYER, *On period relations for abelian integrals on algebraic curves*, Ann. Scuola Norm. Sup. Pisa, **21** (3) (1967), 189-238.

[2] E. ARBARELLO - M. CORNALBA - PH. GRIFFITHS - J. HARRIS, *Geometry of algebraic curves*, Volume I, Grundlehren der math. Wisensch, **267** (Springer-Verlag, 1984).

[3] E. ARBARELLO - J. HARRIS, *Canonical curves and quadrics of rank 4*, Compositio Math., **43** (1981), 145-179.

[4] A. BAJRAVANI, *Focal varieties of curves of genus 6 and 8*, preprint.

[5] C. CILIBERTO - G. VAN DER GEER, *Andreotti-Mayer Loci and the Schottky Problem*, Documenta Math., **13** (2008), 453-504.

[6] C. CILIBERTO - E. SERNESI, *Singularities of the theta divisor and congruences of planes*, J. Algebraic Geom., **1** (1992), 231-250.

[7] C. CILIBERTO - E. SERNESI, *Singularities of the Theta Divisor and Families of Secant Spaces to a Canonical Curve*, J. Algebra, **171** (1995), 867-893.

[8] C. CILIBERTO - E. SERNESI, *On the geometry of canonical curves of odd genus*, Comm. in Alg., **28** (12) (2000), 5993-6002.

[9] G. DARBOUX, *Sur le contact des courbes et des surfaces*, Bull. Soc. math. de France, **4** (2) (1880).

[10] D. EISENBUD, *Linear sections of determinantal varieties*, American J. of Math., **110** (1988), 541-575.

[11] M. GREEN, *Quadrics of rank four in the ideal of the canonical curve*, Invent. Math., **75** (1984), 85-104.

[12] E. ROGORA, *Varieties with many lines*, Manuscripta math., **82** (1994), 207-226.

[13] B. SEGRE, *Sulle* V_n *contenenti più di* $\infty^{n-k}S_k$, Nota II, Rend. Accad. Naz. Lincei, **5** (8) (1948), 275-280.

[14] C. SEGRE, *Sui fochi di* 2^o *ordine dei sistemi infiniti di piani, e sulle curve iperspaziali con una doppia infinità di piani plurisecanti*. Atti R. Accad. Lincei, **30** (5) (1921), 67-71.

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Received September 14, 2009 and in revised form December 7, 2009