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A Remark on the Stability of the Determinant in Bidimensional Homogenization

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Abstract. – *For conductivity problems in dimension $N = 2$, we prove a variant of a classical result: if a sequence A^ε of matrices H-converges to A^0 (or in other terms if A^ε converges to A^0 in the sense of homogenization) and if* $\det A^\varepsilon$ *tends to c^0 a.e., then one has* $\det A^0 = c^0$.

1. – Main result and comments.

A classical result in bidimensional homogenization of conductivity problems is the following:

THEOREM 1.1. – *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let A^ε be a sequence of matrices of $\mathcal{M}(\alpha, \beta, \Omega)$ which H-converges to a matrix A^0. Assume that*

$$\det A^\varepsilon = 1.$$

Then

$$\det A^0 = 1.$$

This result is a direct consequence of Theorem 1.3 below.

The aim of this note is to prove the following slight variant of Theorem 1.1:

THEOREM 1.2. – *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let A^ε be a sequence of matrices of $\mathcal{M}(\alpha, \beta, \Omega)$ which H-converges to a matrix A^0. Assume that*

$$\det A^\varepsilon \to c^0 \quad a.e. \ in \ \Omega \ , \tag{1.1}$$

where c^0 is a function in $L^\infty(\Omega)$. Then $\det A^0 = c^0$.

The proofs of Theorems 1.1 and 1.2 are strongly related to the following result which traces back to Dykhne [3] and Keller [5], and whose proof can be found e.g. in [4], [7], [8] or [12].

THEOREM 1.3. – *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let A^ε be a sequence of matrices of $\mathcal{M}(a, \beta, \Omega)$ which H-converges to a matrix A^0. Then*

$$\frac{A^\varepsilon}{\det A^\varepsilon} \xrightarrow{H} \frac{A^0}{\det A^0}. \tag{1.2}$$

More in general, if $a, b, c \in \mathbb{R}$ are such that $bc > a^2$ and if R is the matrix associated to the rotation of angle $\pi/2$ in the plane then

$$(aA^\varepsilon + bR)(-aI + cRA^\varepsilon)^{-1} \xrightarrow{H} (aA^0 + bR)(-aI + cRA^0)^{-1}.$$

This result is proved in [2], [8] and [12] and generalizes Theorem 1.3, which corresponds to the case where $a = 0$ and $b = c = 1$, since

$$RA^{-1}\,{}^tR = \frac{{}^tA}{\det A} \qquad \forall A \in \mathbb{R}^{2\times 2}.$$

One of the key ingredients of the proof of Theorem 1.2 is following result.

THEOREM 1.4. – *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and let A^ε be a sequence of matrices of $\mathcal{M}(a, \beta, \Omega)$ which H-converges to a matrix A^0. Assume that b^ε is a sequence of measurable functions such that*

$$m \leq b^\varepsilon(x) \leq M \qquad a.e.\ x \in \Omega, \tag{1.3}$$

where $0 < m \leq M < +\infty$ and

$$b^\varepsilon \to b^0 \quad a.e.\ in\ \Omega. \tag{1.4}$$

Then

$$b^\varepsilon A^\varepsilon \xrightarrow{H} b^0 A^0.$$

2. – H-convergence.

This section is concerned with the definition of the H-convergence. In the context of symmetric matrices the notion of H-convergence coincides with the notion of G-convergence defined in [11] (see also [10]).

DEFINITION 2.1. – *Let a and β be real numbers such that $0 < a \leq \beta < +\infty$ and let Ω be a bounded open subset of $\mathbb{R}^N$, with $N \geq 1$. We say that a $N \times N$ matrix A belongs to $\mathcal{M}(a, \beta, \Omega)$ if $A \in (L^\infty(\Omega))^{N\times N}$ and satisfies*

$$A(x)\xi\xi \geq a|\xi|^2 \qquad a.e.\ x \in \Omega \qquad \forall \xi \in \mathbb{R}^N, \tag{2.1}$$

$$A^{-1}(x)\zeta\zeta \geq \beta^{-1}|\zeta|^2 \qquad a.e.\ x \in \Omega \qquad \forall \zeta \in \mathbb{R}^N. \tag{2.2}$$

A sequence of matrices A^ε of $\mathcal{M}(a, \beta, \Omega)$ is said to H-converge to a matrix A^0 of

$\mathcal{M}(a, \beta, \Omega)$ if, for every $f \in H^{-1}(\Omega)$, the solution u^ε of the problem

$$\begin{cases} -\text{div}\, A^\varepsilon D u^\varepsilon = f & \textit{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in H_0^1(\Omega), \end{cases}$$

satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \textit{in } H_0^1(\Omega) \textit{ weakly}, \\ A^\varepsilon D u^\varepsilon \rightharpoonup A^0 D u^0 & \textit{in } (L^2(\Omega))^N \textit{ weakly}, \end{cases}$$

where u^0 is the solution of the problem

$$\begin{cases} -\text{div}\, A^0 D u^0 = f & \textit{in } \mathcal{D}'(\Omega), \\ u^0 \in H_0^1(\Omega). \end{cases}$$

In this case one writes

$$A^\varepsilon \xrightarrow{H} A^0.$$

Observe that, in view of (2.1), the matrix $A(x)$ is invertible a.e. so that $A^{-1}(x)$ exists and is measurable. Observe also that taking $\zeta = A(x)\xi$ in (2.2) one has

$$|A(x)\xi| \leq \beta|\xi| \qquad \text{a.e. } x \in \Omega \qquad \forall\, \xi \in \mathbb{R}^N. \tag{2.3}$$

The following fundamental compactness result, due to Murat and Tartar [9] and to Spagnolo [11] in the context of G-convergence, explains the interest of Definition 2.1.

THEOREM 2.1. – *Let a and β be real numbers such that $0 < a \leq \beta < +\infty$ and let Ω be a bounded open subset of $\mathbb{R}^N$, with $N \geq 1$. Any sequence of matrices A^ε of $\mathcal{M}(a, \beta, \Omega)$ admits a subsequence which H-converges to a matrix A^0 of $\mathcal{M}(a, \beta, \Omega)$.*

3. – Proofs.

PROOF OF THEOREM 1.4. – We divide the proof in two steps.

STEP 1. – Assume first that, further to (1.3) and (1.4), one has

$$b^\varepsilon \in C^1(\overline{\Omega}),\ b^0 \in C^1(\overline{\Omega}), \quad b^\varepsilon \to b^0 \quad \text{in } C^1(\overline{\Omega}) \text{ strongly}. \tag{3.1}$$

We claim that in this case the sequence $b^\varepsilon A^\varepsilon$ H-converges to $b^0 A^0$, i.e. that for every $f \in H^{-1}(\Omega)$, the solution u^ε of the problem

$$\begin{cases} -\,\text{div}(b^\varepsilon A^\varepsilon D u^\varepsilon) = f & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in H_0^1(\Omega), \end{cases} \tag{3.2}$$

satisfies

$$(3.3)\qquad \begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{in } \mathrm{H}_0^1(\Omega)\,, \\ b^\varepsilon A^\varepsilon Du^\varepsilon \rightharpoonup b^0 A^0 Du^0 & \text{in } (L^2(\Omega\,))^N, \end{cases}$$

where u^0 is the solution of the problem

$$(3.4)\qquad \begin{cases} -\mathrm{div}(b^0 A^0 Du^0) = f & \text{in } \mathcal{D}'(\Omega), \\ u^0 \in H_0^1(\Omega). \end{cases}$$

Actually it is sufficient to prove this result for $f \in L^2(\Omega)$. To this end, we observe that

$$(3.5)\qquad -\mathrm{div}(b^\varepsilon A^\varepsilon Du^\varepsilon) = -b^\varepsilon \mathrm{div}(A^\varepsilon Du^\varepsilon) - A^\varepsilon Du^\varepsilon Db^\varepsilon,$$

where $b^\varepsilon \mathrm{div}(A^\varepsilon Du^\varepsilon) \in H^{-1}(\Omega)$ is defined by

$$\langle b^\varepsilon \mathrm{div}(A^\varepsilon Du^\varepsilon), v\rangle = \langle \mathrm{div}(A^\varepsilon Du^\varepsilon), b^\varepsilon v\rangle \qquad \forall v \in H_0^1(\Omega).$$

(Note that $b^\varepsilon v \in H_0^1(\Omega)$ for every $v \in H_0^1(\Omega)$ when $b^\varepsilon \in C^1(\overline{\Omega})$; this proves that the distribution $b^\varepsilon \mathrm{div}(A^\varepsilon Du^\varepsilon)$ is well-defined as an element of $H^{-1}(\Omega)$.)

Set

$$g^\varepsilon = \frac{f + A^\varepsilon Du^\varepsilon Db^\varepsilon}{b^\varepsilon}.$$

Since u^ε is the solution of the problem (3.2), the sequence u^ε is bounded in $H_0^1(\Omega)$. We can assume that (up to a subsequence) u^ε converges to u in $H_0^1(\Omega)$ weakly for some $u \in H_0^1(\Omega)$.

Since $A^\varepsilon \in \mathcal{M}(\alpha, \beta, \Omega)$, from (2.3) it follows that $A^\varepsilon Du^\varepsilon$ is bounded in $L^2(\Omega)$. This proves that g^ε is bounded in $L^2(\Omega)$ and that (up to a subsequence) g^ε converges to g in $L^2(\Omega)$ weakly for some $g \in L^2(\Omega)$.

We now observe that u^ε is the solution of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon Du^\varepsilon) = g^\varepsilon & \text{in } \mathcal{D}'(\Omega), \\ u^\varepsilon \in H_0^1(\Omega). \end{cases}$$

Since A^ε is assumed to H-converges to A^0 and since g^ε converges to g in $L^2(\Omega)$ weakly (and therefore in $H^{-1}(\Omega)$ strongly), we deduce that (up to a subsequence)

$$(3.6)\qquad A^\varepsilon Du^\varepsilon \rightharpoonup A^0 Du \;\text{ in } (\mathrm{L}^2(\Omega\,))^{\mathrm{N}} \text{ weakly},$$

where u is the solution of the problem

$$(3.7)\qquad \begin{cases} -\mathrm{div}(A^0 Du) = g & \text{in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

In view of (3.6) and of the strong convergence (3.1), we have

$$g = \frac{f + A^0 Du Db^0}{b^0}. \tag{3.8}$$

Similarly to (3.5) we have, since $b^0 \in C^1(\overline{\Omega})$,

$$-\mathrm{div}(b^0 A^0 Du) = -b^0 \mathrm{div}(A^0 Du) - A^0 Du Db^0,$$

so that (3.7) and (3.8) imply that u is the solution of the problem

$$\begin{cases} -\mathrm{div}(b^0 A^0 Du) = f \quad \text{in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

This implies that u coincides with u^0 defined by (3.4) and that the convergences (3.3) hold for the whole sequence ε; indeed, we do not have to extract any subsequence since the limits $u, A^0 Du$ and g are uniquely defined.

We have proved the result of Theorem 1.4 when hypothesis (3.1) holds true.

STEP 2. – We now prove the assertion in the general case, i.e. when only (1.3) and (1.4) hold true. In view of Theorem 2.1 we assume that (up to a subsequence) the sequence of matrices $b^\varepsilon A^\varepsilon$ of $\mathcal{M}(am, \beta M, \Omega)$ satisfies

$$b^\varepsilon A^\varepsilon \xrightarrow{H} B^0, \tag{3.9}$$

for some B^0 of $\mathcal{M}(am, \beta M, \Omega)$.

Extend b^ε and b^0 to the whole of $\mathbb{R}^N$ by

$$b^\varepsilon(x) = b^0(x) = m \qquad \forall x \in \mathbb{R}^N \setminus \Omega.$$

Let ρ_δ be a mollifier and let $b^\varepsilon * \rho_\delta$ be the convolution of b^ε and ρ_δ. Since for $\delta > 0$ fixed we have

$$b^\varepsilon * \rho_\delta \rightarrow b^0 * \rho_\delta \quad \text{in } \mathrm{C}^1(\overline{\Omega}) \text{ strongly},$$

the result of the first step proves that for every $\delta > 0$ fixed

$$(b^\varepsilon * \rho_\delta) A^\varepsilon \xrightarrow{H} (b^0 * \rho_\delta) A^0. \tag{3.10}$$

On the other hand, since the sequence A^ε is equi-bounded in $L^\infty(\Omega)$ (see (2.3)) we have

$$|b^\varepsilon A^\varepsilon - (b^\varepsilon * \rho_\delta) A^\varepsilon| \leq \gamma_\delta^\varepsilon, \tag{3.11}$$

where $\gamma_\delta^\varepsilon$ is the function defined by

$$\gamma_\delta^\varepsilon = \beta |b^\varepsilon - (b^\varepsilon * \rho_\delta)|,$$

for every $\delta > 0$ fixed. Hypothesis (1.4) implies that

$$\gamma_\delta^\varepsilon \to \gamma_\delta^0 \quad \text{a.e. in } \Omega \ , \tag{3.12}$$

where γ_δ^0 is the function defined by

$$\gamma_\delta^0 = \beta |b^0 - (b^0 * \rho_\delta)|, \tag{3.13}$$

for every $\delta > 0$ fixed. Then (3.9), (3.10), (3.11), (3.12) and Theorem 3.1 in [1] imply for every $\delta > 0$ fixed

$$|B^0 - (b^0 * \rho_\delta) A^0| \leq \gamma_\delta^0. \tag{3.14}$$

The fact that $b^0 * \rho_\delta$ tends to b^0 a.e. as δ tends to zero, (3.13) and (3.14) imply then that $B^0 = b^0 A^0$. This concludes the proof of Theorem 1.4. □

PROOF OF THEOREM 1.2. – Define $b^\varepsilon = 1/\det A^\varepsilon$. In view of hypothesis (1.1) the sequence b^ε converges to $b^0 = 1/c^0$ a.e. in Ω. Applying the result of Theorem 1.4, the sequence $b^\varepsilon A^\varepsilon$ H-converges to $b^0 A^0 = A^0/c^0$. Since here the dimension is $N = 2$, Theorem 1.3 implies that (1.2) holds. Since the H-limit is unique, it results that $A^0/c^0 = A^0/\det A^0$, and therefore $c^0 = \det A^0$. This proves Theorem 1.2. □

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