Bollettino Unione Matematica Italiana

L. Di Persio

Anomalous Behaviour of the Correction to the Central Limit Theorem for a Model of Random Walk in Random Media

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **3** (2010), n.1, p. 179–206.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2010_9_3_1_179_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2010.

Bollettino U. M. I.
(9) **III** (2010), 179-206

Anomalous Behaviour of the Correction to the Central Limit Theorem for a Model of Random Walk in Random Media

L. DI PERSIO

Abstract. – *We give a result concerning the correction to the Central Limit Theorem for a Random Walk on the lattice $\mathbb{Z}^2$ which interacts with a random environment under a small randomness condition. Our main theorem close a gap which dates back to seminal works by Boldrighini, Minlos and Pellegrinotti, see* [3], [8] *and* [9]. *Asymptotic behaviour of the corrections to the average and the covariance matrix in dimension $\nu = 1, 2$ are also presented.*

1. – Introduction.

1.1 – *General overview.*

In the present paper we consider a particle moving in a ν-dimensional infinite lattice which evolves stochastically in discrete time t as stated in [3, 8, 9].

The environment is described by a random field $\xi \equiv \{\xi_t(x) : x \in \mathbb{Z}^\nu, t \in \mathbb{Z}^+\}$, i.e. ξ is the result of independent copies of the same random variable taking value in some finite set $\mathscr{S}$.

The one step transition probability from a site x at time $t \geq 0$ to a site y at time $t + 1$ for a given realization of the environment ξ is as follows

$$P(X_{t+1} = y \mid X_t = x, \xi) = P_0(y - x) + \varepsilon c(y - x; \xi) ,$$

hence it is a sum of two terms, namely a free homogeneous random walk $P_0(\cdot)$ and a random perturbation $c(\cdot;\cdot)$, while the parameter ε controls coupling intensity between them.

A finite range assumption on $P_0(\cdot)$ and $c(\cdot;\cdot)$ is made. Under further standard technical conditions results in [3, 8, 9] include the Central Limit Theorem (CLT) for the displacement $X_t - X_0$ almost everywhere in the realization of the environment and in any dimension.

Moreover in [8] is proved that the time behaviour of the corrections to the CLT for the Random Walk (RW) X_t in dimension $\nu \geq 3$, depends on the environment and the traditional expansion in inverse powers of $T^{\frac{1}{2}}$ is reduced to only a finite number of terms, more precisely it holds up to the term of order $T^{-\frac{k}{2}}$, where $k \leq \left[\frac{(\nu-1)}{2}\right]$ is the largest integer smaller than $\frac{\nu}{2}$.

Analogous conclusions are shown in [8] for the cumulants of the first and second order in dimendion $\nu \geq 3$. In [12] is proved that in dimension $\nu = 1$ the correction to the CLT is an environment-depending term of order $T^{-\frac{1}{4}}$ which, if normalized, tends to a random Gaussian variable as $T \to \infty$.

In this work we give the correct term of normalization in dimension $\nu = 2$, which is $\sqrt{\frac{T}{\ln T}}$, proving that the correction to the CLT tends to a limiting centered Gaussian variable for which we are able to write the dispersion by an explicit integral expression.

Moreover we prove that the corrections to the average and to the covariance matrix in dimension $\nu = 1, 2$ have similar anomalous behaviour.

It is important to note that in [10] the almost sure validity of the CLT for the quenched model is proved without assuming a small stochasticity condition, but only that an obvious non-degeneracy condition is fulfilled. The proof is based on the analysis of a suitable generating function, which allows to estimate L^2 norms by contour integrals. Similar arguments are used in [11] to prove the CLT when the RW starts out at a fixed point of the lattice $\mathbb{Z}^\nu$ both in the quenched case and for the annealed model. Nevertheless results about the asymptotic behaviour for the corrections seems to require the small randomness assumption.

We would also like to mention further selected recent works related to the subject of RW in random media.

In [6, 7] asymptotic decay of correlations for RW in interaction with a random environment independent in space and with a Markov evolution in time are given.

In [5] is considered the case of a particle moving accordingly to a jump Markov process and interacting with an evolving random environment represented by a stationary Glauber type dynamics in the continuum. Under some assumptions on the Glauber dynamic and on the coupling between particle and environment, the authors give the large time asymptotics for the particle position distribution.

In [4] numerical results about some models of $1-$dimensional RW in fluctuating random environment are given.

In [13, 14, 15] the author consider a RW in a stationary random medium, defined by an ergodic dynamical system, in the case when the possible jumps are $\{-L, \ldots, -1, 0, +1\}$ for some fixed integer L. A recurrence criterion expressed in terms of the sign of the maximal Lyapounov exponent is given together with the existence of the absolutely continuous invariant measure for the Markov chain and , in the transient cases, the presence of a nonzero drift. Study of the validity of the CLT in the transient cases is made using the notion of harmonic coordinates introduced by Kozlov. Previous results are considered in the context of a random medium defined by an irrational rotation on the circle and their realization in terms of regularity and Diophantine approximation are given. In the case of $1-$dimensional RW with bounded steps in a stationary and ergodic random medium the author show that the algebraic structure of the RW is given by geometrical invariants related to the description of a space of harmonic functions and prove a recurrence criterion similar to Key's Theorem. In the same context it is also shown the validity of the Law of Large Numbers. Moreover a fine analysis of the geometrical properties of the central left and right Lyapunov eigenvectors of the random matrix naturally associated with the random walk is provided.

In [17] a quenched CLT for random walks with bounded increments in a randomly evolving environment on $\mathbb{Z}^\nu$ is proved provided that the transition probabilities of the walk weakly depend on the environment. Moreover the evolution of the environment is assumed to be Markovian with strong spatial and temporal mixing properties.

In [1] author proves that, in the nearest-neighbour case, when the averaged random walk is symmetric, the almost sure CLT holds for an arbitrary level of randomness.

We would also like to cite [20] were the author gives an almost complete review of various model of RW in random media together with available results and techniques.

1.2 – *Plan of the work.*

In Section 2 we will describe the model and state the main results which will be proved in Section 3 using Cluster Expansion's techniques. Some details of the proofs will be shifted to the Appendix A.

2. – Definitions and main results.

We denote by $X_t \in \mathbb{Z}^\nu$, with $t \in \mathbb{Z}$, the position of a particle which is moving in a ν-dimensional infinite lattice. Time is discrete and the particle's probability to jump from one site to another depends on the state of environment.

More precisely we put independent copies of the same discrete random variable on each site of the grid, this variable takes values in a finite set $\mathscr{S} \equiv \{s_1, \dots, s_n\}$ with a non degenerate probability π. We will define $\hat{\Omega} \equiv \mathscr{S}^{\mathbb{Z}^{\nu+1}}$ as the set of all possible configurations of the environment equipped with the natural product measure $\Pi \equiv \pi^{\mathbb{Z}^{\nu+1}}$.

In the following $\langle \cdot, \cdot \rangle$ and $\mathbb{E}(\,\cdot\,)$ indicate expectations w.r.t. the distribution Π (or w.r.t. the measure π for a single point $(x, t) \in \mathbb{Z}^{\nu+1}$) and over the trajectories $\{X_t\}$ respectively.

Once a configuration $\xi \in \hat{\Omega}$ of the environment is fixed and for $\varepsilon \in [0, 1)$, we define one step transition probabilities as follows

$$P(X_{t+1} = y \mid X_t = x, \xi) \equiv P_0(y - x) + \varepsilon c(y - x; \xi_t(x)). \tag{2.1}$$

hence they are defined as a sum made of a homogeneous random walk $P_0(u)$ plus a random term $c(u;s)$ which, with no loss of generality, is supposed to have zero average (i.e. $\langle c(u;\cdot)\rangle = 0$) and to be such that $\sum_{u\in\mathbb{Z}^2} c(u;s) = 0$.

Further assumptions are the following

(1) $0 \le P_0(u) + \varepsilon c(u,s) < 1$

(2) $\exists D \ge 1 : P_0(u) = c(u,s) = 0 \quad \forall u \in \mathbb{Z}^2 \ : \| u \|_2 > D, \forall s \in \mathscr{S}$

(3) The characteristic function associated to P_0

$$\tilde{p}_0(\lambda) = \sum_{u\in\mathbb{Z}^\nu} P_0(u) e^{i(\lambda,u)}, \quad \lambda \equiv (\lambda_1, \dots, \lambda_\nu) \in T^\nu$$

where T^ν is the usual $\nu-$dimensional torus, satisfies

(3a) $\mid \tilde{p}_0(\lambda) \mid < 1, \quad \forall \lambda \neq 0$

As a consequence of (2) and (3a) we also have that the quadratic term which appears in the following Taylor expansion

$$\ln \tilde{p}_0(\lambda) = \imath \sum_{k=1}^{\nu} b_k \lambda_k - \frac{1}{2} \sum_{i,j=1}^{\nu} \mathfrak{c}_{ij} \lambda_i \lambda_j + \cdots ,$$

around $\lambda = 0$, is strictly positive for $\lambda \neq 0$.

We want to prove that, in dimension $\nu = 2$, an anomalous correction to the CLT for the displacement $X_t - X_0$ appears.

For a given $\nu \ge 1$, let us define with P_0^T the convolution of T copies of P_0, the quantity

$$Q_T(x \mid \xi) \equiv P(X_T = x \mid X_0 = 0; \xi) - P_0^T(x),$$

and let us also define $\boldsymbol{b} \equiv (b_1, \dots, b_\nu)$ and $C \equiv (\mathfrak{c}_{ij})_{ij=1,\dots,\nu}$, then the following theorem holds

THEOREM 2.1. – *If* $\nu = 2$ *and* $\varepsilon \in [0,1)$ *is small enough, then for every function* $f \in \mathscr{C}^{2,lim}(\mathbb{R}^2)$ *the sequence of functionals*

$$\hat{\mathscr{Q}}_T(f \mid \xi) \equiv \sqrt{\frac{T}{\ln T}} \sum_{x\in\mathbb{Z}^2} Q_T(x \mid \xi) f\left(\frac{x - \boldsymbol{b}T}{\sqrt{T}}\right), \tag{2.2}$$

tends in distribution, for $T \to \infty$ *and some constants* $\tilde{c}_0, \mathfrak{M}_{ij}(i,j = 1,2)$, *to a centered Gaussian variable with dispersion*

$$\frac{\tilde{c}_0}{2} \sum_{ij=1}^{2} \mathfrak{M}_{ij} \left(\int K_C(1,v) f_i(v) dv \right) \left(\int K_C(1,v) f_j(v) dv \right), \tag{2.3}$$

which depends only on the position reached by the particle at the final time and *where*

$$K_C(s,v) \equiv \frac{\sqrt{C}}{2\pi s} \cdot e^{-\frac{\mathcal{A}(v)}{2s}} \quad , \quad f_i \equiv \frac{\partial f}{\partial x_i} \tag{2.4}$$

with $\mathcal{A} = \{\mathfrak{a}_{ij}\} = \{\mathfrak{c}_{ij}\}^{-1}$, *so that* $\mathcal{A}$ *defines, for all* $v \in \mathbb{R}^2$, *the quadratic form* $\mathcal{A}(v) \equiv \sum_{i,j=1}^{2} \mathfrak{a}_{ij} v_i v_j$.

The same techniques used to prove previous result allow us to investigate time asymptotics for the correction to the average and to the covariance matrix in dimension $\nu = 1, 2$. Let us define for $i, j = 1, \ldots, \nu$ and $\nu = 1, 2$ the average vector components and the covariance matrix elements respectively as follows

$$\mathscr{E}_i^{(T)}(\xi) \equiv \mathbb{E}((X_t)_i \mid X_0 = 0, \xi)) - b_i T$$

$$\mathscr{C}_{ij}^{(T)}(\xi) \equiv \mathbb{E}\big((X_T - \boldsymbol{b}T)_i (X_T - \boldsymbol{b}T)_j \mid X_0 = 0, \xi\big) - \mathfrak{c}_{ij} T$$

where $\boldsymbol{b} \equiv (b_1, \ldots, b_\nu)$ represents the drif term, then the following results hold

THEOREM 2.2. – *For* $\nu = 1$, $\varepsilon \in [0,1)$ *small enough and setting* $S_T \equiv \langle (\mathscr{E}^{(T)})^2 \rangle^{\frac{1}{2}}$, *the sequence*

$$\frac{\mathscr{E}^{(T)}(\xi)}{S_T}$$

converges in distribution, for $T \to \infty$, *to a standard Gaussian variable. Moreover we have* $S_T \asymp T^{\frac{1}{4}}$.

THEOREM 2.3. – *For* $\nu = 2$, $\varepsilon \in [0,1)$ *small enough and setting* $S_T \equiv \langle (\mathscr{E}^{(T)})^2 \rangle^{\frac{1}{2}}$, *the sequence*

$$\frac{\mathscr{E}^{(T)}(\xi)}{S_T}$$

converges in distribution, for $T \to \infty$, *to a centered Gaussian variable with covariance matrix*

$$\varSigma \equiv \{\mathfrak{b}_{ij}\} = \{\langle \mathfrak{b}_i(\,\cdot\,)\mathfrak{b}_j(\,\cdot\,)\rangle\}$$

where $\mathfrak{b}_i(\,\cdot\,) \equiv \sum_{u \in \mathbb{Z}^2} u_i \mathfrak{c}(u; \cdot)$. *Moreover* $S_T \asymp (\ln T)^{\frac{1}{2}}$.

THEOREM 2.4. – *For* $\nu = 1$, $\varepsilon \in [0,1)$ *small enough and setting* $\tilde{S}_T \equiv \langle (\mathscr{C}^{(T)})^2 \rangle^{\frac{1}{2}}$, *the sequence*

$$\frac{\mathscr{C}^{(T)}(\xi)}{\tilde{S}_T}$$

converges in distribution, for $T \to \infty$, *to a standard Gaussian variable. Moreover* $\tilde{S}_T \asymp T^{\frac{3}{4}}$.

THEOREM 2.5. – *For* $\nu = 2$, $\varepsilon \in [0,1)$ *small enough and setting* $\tilde{S}^{(T)}_{ij} \equiv \langle (\mathscr{C}^{(T)}_{ij})^2 \rangle^{\frac{1}{2}}$, *the sequence*

$$\frac{\mathscr{C}^{(T)}_{ij}(\xi)}{\tilde{S}^{(T)}_{ij}}$$

converges in distribution, for $T \to \infty$, *to a standard Gaussian variable. Moreover* $\tilde{S}^{(T)}_{ij} \asymp T^{\frac{1}{2}}$.

We would like to mention that results in Theorems 2.2, 2.3, 2.4 and 2.5 are also proved in [2] with different techniques, namely using a CLT result for martingale differences contained in [16], Sec. 9.3, Th.1.

3. – Proofs.

Our model is characterized by a space-time invariance so there is no loss of generality in assuming that the random walk always starts at the origin at time $t = 0$.

We can rewrite (2) as

$$Q_T(x \mid \xi) = \sum_{0 \le t_1 \le t_2 \le T-1} \sum_{y_1, y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_*(t_2 - t_1, y_2 - y_1; \xi_{(t_1, y_1)}) \times h^{T-t_2}(x - y_2; \xi_{t_2}(y_2)) \,,$$

where

$$h^t(y; s) \equiv \sum_{u \in \mathbb{Z}^2} c(u; s) P_0^{t-1}(y - u) \,,$$

$$M_*(t, y; \xi) \equiv \sum_{B:(0,0) \to (t,y)} \varepsilon^{|B|} M^*_B(\xi), \; M^*_B(\xi) \equiv \prod_{i=0}^{n-1} h^{\tau_i}(z_i; \xi_{t_i}(y_i)) \,,$$

and $\xi_{(t,y)}$ is the shifted environment, i.e.

$$\xi_{(t,y)}(z, \tau) \equiv \xi_{\tau - t}(z - y) \,.$$

Sums in the definition of $M_*(t, y; \xi)$ run over all subsets $B = \{(t_1, y_1), \ldots, (t_n, y_n)\}$ from $(0, 0)$ to (t, y).

Quantities τ_i and positions z_i are defined as $\tau_i \equiv t_{i+1} - t_i$, $z_i \equiv y_{i+1} - y_i$ respectively, besides we assume $P_0^0(y) \equiv \delta_{y0}$ and $M_*(0, y; \xi) \equiv \varepsilon \delta_{y0}$.

Setting $\mathfrak{b}(s) \equiv \sum_{u \in \mathbb{Z}^2} u \, c(u; s)$ and indicating with $\mathscr{H}_f(x)$ the Hessian matrix of the function f evaluated at a certain point $x \in \mathbb{R}^2$, we have that for all $y \in \mathbb{R}^2$ there exists $\zeta \in \mathbb{R}^2$ with $\|\zeta\|_2 \le D$ such that

$$\sum_{u \in \mathbb{Z}^2} c(u; s) f\left(\frac{y + u}{\sqrt{T}}\right) = \frac{1}{\sqrt{T}} \left(\mathfrak{b}(s) \nabla f\left(\frac{y}{\sqrt{T}}\right)\right) + r_T(y; s),$$

where

$$r_T(y,s) \equiv \frac{1}{2T} \sum_{u \in \mathbb{Z}^2} c(u;s).\mathscr{H}_f(\zeta) \cdot (u,u).$$

In the following we will work only with function $f \in \mathcal{C}^{2,lim}(\mathbb{R}^2)$ with a norm defined by

$$\| f \| \equiv \| f \|_\infty + \| \nabla(f) \|_\infty + \| \mathscr{H}_f \|_\infty ,$$

where

$$\| \nabla(f) \|_\infty \equiv \max_{x \in \mathbb{R}^2} \left\{ \left| \frac{\partial f}{\partial x_1}(x) \right|, \left| \frac{\partial f}{\partial x_2}(x) \right| \right\} ,$$

$$\| \mathscr{H}_f \|_\infty \equiv \max_{x \in \mathbb{R}^2} \left\{ \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| : i,j = 1,2 \right\} .$$

We have

(1) $\langle r_T(y;\cdot) \rangle = 0$, $\forall y \in \mathbb{R}^2$

(2) $| r_T(y;s) | \leq \dfrac{const \| \mathscr{H}_f \|_\infty}{T} \overset{T\to\infty}{\longrightarrow} 0$, $\forall (y,s) \in \mathbb{R}^2 \times \mathscr{S}$

Let be

$$\begin{aligned} \delta_T(t,y;s) &\equiv \sum_{x \in \mathbb{Z}^2} h^t(x;s) f\left(\frac{y+x}{\sqrt{T}}\right) - \frac{\mathfrak{b}(s)}{\sqrt{T}} \cdot \sum_{z \in \mathbb{Z}^2} P_0^{t-1}(z) \nabla f\left(\frac{y+z}{\sqrt{T}}\right) \\ &= \sum_{z \in \mathbb{Z}^2} P_0^{t-1}(z) \left[\sum_{u \in \mathbb{Z}^2} c(u;s) f\left(\frac{y+u+z}{\sqrt{T}}\right) - \frac{\mathfrak{b}(s)}{\sqrt{T}} \nabla f\left(\frac{y+z}{\sqrt{T}}\right) \right] , \end{aligned} \tag{3.1}$$

then $\delta_T(t,y;s)$ has zero average and $\forall (s,y) \in \mathscr{S} \times \mathbb{R}^2$, it satisfies

$$\begin{aligned} |\delta_T(t,y;s)| &= \left| \sum_{z \in \mathbb{Z}^2} P_0^{t-1}(z) \left\{ \sum_{u \in \mathbb{Z}^2} c(u;s) \left[f\left(\frac{y+z}{\sqrt{T}}\right) \right.\right.\right. \\ &\quad \left. + \nabla f\left(\frac{y+z}{\sqrt{T}}\right) \frac{u}{\sqrt{T}} + \frac{\mathscr{H} f(\zeta) \cdot (u,u)}{T} \right] \\ &\quad \left.\left. - \frac{\mathfrak{b}(s)}{\sqrt{T}} \nabla f\left(\frac{y+z}{\sqrt{T}}\right) \right\} \right| \\ &= \left| \sum_{z \in \mathbb{Z}^2} P_0^{t-1}(z) r_T(y+z,s) \right| \leq \frac{const \| \mathscr{H}_f \|_\infty}{T} \overset{T\to\infty}{\longrightarrow} 0 . \end{aligned} \tag{3.2}$$

Proof of our main theorem, i.e. Th. (2.1), will be split into several results stated as lemmas and propositions. We will use throughout what follows the notation *const* in order to denote different constants which may or may not depend on the coupling parameter ε, besides we will use the lower case t as well as the capital T to indicate the (discrete) time variable.

PROOF (of Th. 2.1). – If we define the vector $M_\sharp(t, y \mid \xi) \equiv M_*(t, y \mid \xi)\mathfrak{b}(\xi_t(y))$, then we have

PROPOSITION 3.1. – *For $i = 1, 2$ and $\varepsilon \in [0, 1)$ small enough there exists a positive constant such that*

$$\sum_{y\in\mathbb{Z}^2}\left\langle\left(\left(M_\sharp(t,y\mid\cdot)\right)_i\right)^2\right\rangle \le \frac{const\cdot\varepsilon^2}{(t+1)^2}$$

PROOF. – By definition of $c(u; s)$ the contribution given by two subsets of points which do not coincide in space and time is equal to zero, then setting

$$b_i \equiv \max_{s\in\mathscr{S}} \mid (\mathfrak{b}(s))_i \mid \ \text{ for } i = 1, 2 \ ,$$

we have

$$\begin{aligned}\sum_{y\in\mathbb{Z}^2}\left\langle\left(\left(M_\sharp(t,y\mid\cdot)\right)_i\right)^2\right\rangle &= \sum_{y\in\mathbb{Z}^2}\left\langle\left(\sum_{B:(0,0)\to(t,y)}\varepsilon^{|B|}M_B^*(\,\cdot\,)(\mathfrak{b}(\,\cdot\,))_i\right)^2\right\rangle \\ &\le (\varepsilon b_i)^2\sum_{y\in\mathbb{Z}^2}\sum_{n=1}^{t}\varepsilon^{2n}\sum_{\substack{t_1+\cdots+t_n=t\\ \min\{t_2,\ldots,t_n\}>0}}\sum_{x_1,\ldots,x_n}\prod_{i=1}^{n}\langle(h^{t_i}(x_i;\cdot))^2\rangle\ .\end{aligned} \tag{3.3}$$

Since

$$\int_{\mathbb{R}^\nu} e^{-c\frac{(y-\boldsymbol{b}t)^2}{t}}dy = \int_{\mathbb{R}^\nu} t^{\frac{\nu}{2}}e^{-cx^2}dx \le const\cdot t^{\frac{\nu}{2}}\ , \tag{3.4}$$

by appendix A of [8], we have

$$\max_{s\in\mathscr{S}}\sum_{y\in\mathbb{Z}^2}(h^t(y;s))^2 \le \sum_{y\in\mathbb{Z}^2}A_1\frac{e^{-a\frac{(y-\boldsymbol{b}t)^2}{t}}}{(t+1)^3} \le \frac{const}{t^2}\ ,$$

then the quantity on the second line of (3.3) is bounded by

$$\begin{aligned}&(\varepsilon b_i)^2\sum_{y\in\mathbb{Z}^2}\sum_{n=1}^{t}\varepsilon^{2n}\sum_{\substack{t_1+\cdots+t_n=t\\ \min\{t_2,\ldots,t_n\}>0}}\sum_{x_1,\ldots,x_n}\prod_{i=1}^{n}A_1\frac{e^{-a\frac{(x_i-\boldsymbol{b}t_i)^2}{t_i}}}{(t_i+1)^3}\\ &\qquad\le (\varepsilon b_i)^2\sum_{n=1}^{t}\varepsilon^{2n}\sum_{\substack{t_1+\cdots+t_n=t\\ \min\{t_2,\ldots,t_n\}>0}}\frac{const}{\prod_{i=1}^{n}(t_i+1)^2}\ .\end{aligned}$$

Iterating the following inequality

$$\sum_{t_1=1}^{T-1} \frac{1}{t_1^a} \cdot \frac{1}{(T-t_1)^a} \leq \frac{K(a)}{T^a} \ ,$$

which is valid for all $a > 1$ and some constant $K(a) > 0$, by the small randomness condition, i.e. $\varepsilon < 1$, we can sum over n to obtain the result.

Let be

$$\mathscr{Q}_T^{(1)}(f \mid \xi) \equiv \frac{1}{\sqrt{T}} \sum_{\substack{t_1+t_2+t_3=T-1 \\ x, y_1, y_2 \in \mathbb{Z}^2}} P_0^{t_1}(y_1) M_\sharp(t_2, y_2 - y_1 \mid \xi_{(t_1,y_1)}) P_0^{t_3}(x - y_2) \cdot \nabla f\left(\frac{x - \boldsymbol{b}T}{\sqrt{T}}\right),$$

assuming

$$\hat{\mathscr{Q}}_T^{(1)}(f \mid \xi) \equiv \sqrt{\frac{T}{\ln T}}\, \mathscr{Q}_T^{(1)}(f \mid \xi) \ ,$$

then for the sequence of functionals $\hat{\mathscr{Q}}_T(f \mid \xi)$ defined in (2.2) we have

LEMMA 3.1. – *For* $\varepsilon \in [0,1)$ *small enough*

$$\left\langle \left(\hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) \right)^2 \right\rangle \stackrel{T \to \infty}{\longrightarrow} 0$$

PROOF. – By the definition of $\delta_T(t, y; s)$ in (3.1), we have that the difference between functionals

$$\hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) \ ,$$

can be written as

$$\left(\frac{T}{\ln T}\right)^{\frac{1}{2}} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_*(t_2, y - y_1 \mid \xi_{(t_1,y_1)}) \delta_T(T - t, y - \boldsymbol{b}T; \xi_t(y)) \ .$$

By $L^2(\Pi)$-orthogonality of the terms

$$M_*(t_2, y - y_1 \mid \xi_{(t_1,y_1)}) \delta_T(T - t, y - \boldsymbol{b}T; \xi_t(y)) \ ,$$

and using (3.2) we find

$$\begin{aligned} &\left\langle \left(\hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) \right)^2 \right\rangle \\ &\leq \frac{const \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y_2 \in \mathbb{Z}^2} (P_0^{t_1}(y_1))^2 \langle M_*^2(t_2, y_2 \mid \cdot) \rangle \ . \end{aligned} \tag{3.5}$$

Using again the inequality (3.4) and Prop. (3.1) we have that the right-hand side of

(3.5) is bounded by

$$\frac{const \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \frac{const \cdot \varepsilon^2}{(t_1+1)(t_2+1)^2} \leq \frac{const \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=1}^{T-1} \frac{1}{t} ,$$

as

$$\sum_{t_1+t_2=t} \frac{1}{t_1+1} \frac{1}{(t_2+1)^2} \leq \frac{const}{t} ,$$

and

$$\sum_{t=1}^{T} \frac{1}{t} \asymp \log T ,$$

then

$$\frac{const \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=1}^{T-1} \frac{1}{t} \leq \frac{const \parallel \mathscr{H}_f \parallel_\infty^2}{T} \stackrel{T\to\infty}{\longrightarrow} 0 .$$

In order to specify the constants

$\mathfrak{M}_{ij}$ introduced in (2.3) let us denote by $(\mathfrak{b}(\xi_t(y)))_i$ the i-th component of the vector $\mathfrak{b}$ for $i = 1, 2$, then the following hold

PROPOSITION 3.2. – *For $i, j = 1, 2$, if $\varepsilon \in [0, 1)$ is small enough, the sequence*

$$\mathfrak{C}_{ij}^{(T)}(\xi) \equiv \sum_{t=0}^{T-1} \sum_{y \in \mathbb{Z}^2} (M_*(t, y; \xi))(\mathfrak{b}(\xi_t(y)))_i (\mathfrak{b}(\xi_t(y)))_j \tag{3.6}$$

converges for $T \to \infty$ to a limiting functional $\mathfrak{C}_{ij}$ in L^2 as well as Π-a.s. in the realization of the environment.

PROOF. – By Prop. (3.1) and using the $L^2(\Pi)$-orthogonality of the terms $M_*(t, y; \xi)$, for $T' > T$ we have

$$\left\langle \left(\mathfrak{C}_{ij}^{T'}(\cdot) - \mathfrak{C}_{ij}^{(T)}(\cdot) \right)^2 \right\rangle \leq \sum_{t=T}^{T'-1} \frac{const}{(t+1)^2} \leq const \left(\frac{1}{T} - \frac{1}{T'} \right) ,$$

and we can conclude the proof using the result contained in the appendix A of [9].

We will show later that the constants $\mathfrak{M}_{ij}$, which appear in (2.3), are exactly the second moments of the limiting functionals $\mathfrak{C}_{ij}$.

Let us define the following quantities

$$T_1 \equiv [T^\beta], \beta \in (0, 1), \quad T_* \equiv [\ln_+ T], \quad \ln_+ T \equiv \max\{1, \ln T\} ,$$

and

$$H_T(t,y) = \sum_{z\in\mathbb{Z}^2} P_0^{T-t-1}(z)\nabla f\left(\frac{y+z-\boldsymbol{b}T}{\sqrt{T}}\right),$$

then the functional

$$\hat{\mathcal{Q}}_T^{(2)}(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2,y_2)$$

is obtained removing those terms that are relative to large t_1 and large differences $t_2 - t_1$ and for $\hat{\mathcal{Q}}_T^{(1)}$ the following result holds

LEMMA 3.2. – *For* $\varepsilon \in [0,1)$ *small enough we have*

$$\left\langle \left(\hat{\mathcal{Q}}_T^{(1)}(f \mid \cdot) - \hat{\mathcal{Q}}_T^{(2)}(f \mid \cdot) \right)^2 \right\rangle \stackrel{T\to\infty}{\longrightarrow} 0\,.$$

PROOF. – First, considering the large t_1 values, we define

$$\bar{\mathcal{Q}}(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t_1=T-T_1+1}^{T-1} \sum_{t_2=t_1}^{T-1} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2,y_2)\,.$$

Proceeding as in the proof of lemma (3.1) we have

$$\langle (\bar{\mathcal{Q}}(f \mid \cdot))^2 \rangle \le \frac{const}{T\ln T} \parallel \nabla f \parallel_\infty^2 \sum_{t=1}^{T} \frac{1}{t} = \frac{const}{T} \parallel \nabla f \parallel_\infty^2 \stackrel{T\to\infty}{\longrightarrow} 0\,.$$

The contribution for large differences $t_2 - t_1$ can be rewritten as

$$\widetilde{\mathcal{Q}}(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1+T_*+1}^{T-1} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2,y_2)\,,$$

so we obtain

$$\begin{aligned}\langle (\widetilde{\mathcal{Q}}(f \mid \cdot))^2 \rangle &\le \frac{const \parallel \nabla f \parallel_\infty^2}{\ln T} \sum_{t_1=0}^{T-T_*} \frac{1}{(t_1+1)} \sum_{t'=T_*}^{T-t_1} \frac{1}{(t'+1)^2} \\ &\le \frac{const \parallel \nabla f \parallel_\infty^2}{T_* \cdot \ln T} \sum_{t_1=1}^{T} \frac{1}{t_1} \le \frac{const \parallel \nabla f \parallel_\infty^2}{T_*} \stackrel{T\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

LEMMA 3.3. – *For* $\varepsilon \in [0,1)$ *small enough we have*

$$\langle (\hat{\mathcal{Q}}_T^{(2)}(f \mid \cdot))^2 \rangle \stackrel{T\to\infty}{\longrightarrow} \frac{\tilde{c}_0}{2} \sum_{ij=1}^{2} \mathfrak{M}_{ij} \left(\int K_C(1,v) f_i(v)dv \right) \left(\int K_C(1,v) f_j(v)dv \right),$$

where $\mathfrak{M}_{ij} = \langle (\mathfrak{C}_{ij}(\,\cdot\,))^2 \rangle$ *and* $\mathfrak{C}_{ij}$ *are the same limiting functionals that appear in Prop.* 3.2, *while* K_C *is the heat kernel defined in* (2.4).

PROOF. – Hypotheses on our model imply, see e.g. [18], that, around the point $\lambda = 0$ in the ν–dimensional torus, the Taylor expansion of the characteristic function of P_0 is

$$\ln \tilde{p}_0(\lambda) = \imath \sum_{k=1}^{\nu} b_k \lambda_k - \frac{1}{2} \sum_{i,j=1}^{\nu} \mathfrak{c}_{ij} \lambda_i \lambda_j + \cdots .$$

In the bidimensional case the Local Limit Theorem (LLT) implies

$$P_0^t(x) = \frac{\sqrt{C}}{(2\pi t)} e^{-\frac{1}{2}\mathcal{A}(\frac{x - \boldsymbol{b}t}{\sqrt{t}})} \cdot \left(1 + O\left(\frac{1}{\sqrt{t}}\right)\right) ,$$

then for all $f \in \mathcal{C}^{2,lim}(\mathbb{R}^2)$ we have

$$\left| \sum_z \left(P_0^t(z) - \frac{\sqrt{C} e^{-\frac{1}{2}\mathcal{A}(\frac{z - \boldsymbol{b}t}{\sqrt{t}})}}{2\pi t} \right) \cdot f\left(\frac{y + z - \boldsymbol{b}T}{\sqrt{T}} \right) \right|$$
$$= \left| \sum_z \frac{\sqrt{C} e^{-\frac{1}{2}\mathcal{A}(\frac{z - \boldsymbol{b}t}{\sqrt{t}})}}{2\pi t} \cdot O\left(\frac{1}{\sqrt{t}}\right) \cdot f\left(\frac{y + z - \boldsymbol{b}T}{\sqrt{T}} \right) \right| \le \frac{const \cdot \| f \|_\infty}{\sqrt{t}} .$$

We want to control the error that occurs replacing sums with integrals, i.e. we want to evaluate the asymptotic of the following Riemann sum

$$\frac{1}{t} \sum_{z \in \mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)} f\left(\frac{y - \boldsymbol{b}(T - t)}{\sqrt{T}} + \sqrt{\frac{t}{T}} \frac{z - \boldsymbol{b}t}{\sqrt{t}} \right) .$$

If $Q_t(z)$ is the square centered in $\dfrac{z - \boldsymbol{b}t}{\sqrt{t}}$, with sides parallel to the cartesian axes and of length $t^{-\frac{1}{2}}$, defining $R \equiv \dfrac{y - \boldsymbol{b}(T - t)}{\sqrt{T}}$ we have

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R + \sqrt{\frac{t}{T}}\, x \right) - \frac{1}{t} \sum_{z \in \mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)} f\left(R + \sqrt{\frac{t}{T}} \frac{z - \boldsymbol{b}t}{\sqrt{t}} \right) . \tag{3.7}$$

Let be

$$G(x) \equiv e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R + \sqrt{\frac{t}{T}}\, x \right) ,$$

then the integral over $Q_t(z)$ can be written as

$$\Delta(z) \equiv \int_{Q_t(z)} \left[G(x) - G\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}} \right) \right] dx . \tag{3.8}$$

Writing the second order Taylor expansion of $G(x)$ in (3.8) at $x = \frac{z - \boldsymbol{b}t}{\sqrt{t}}$, we have that the term of zero order is cancelled by $G\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)$.

For the first order term we have

$$\nabla G\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}}\right) \cdot \int_{Q_t(z)} \left(x - \frac{z - \boldsymbol{b}t}{\sqrt{t}}\right) dx = 0$$

by symmetry and the first term that survives is the second order one

$$\begin{aligned} |\,\varDelta(z)\,| &= \frac{1}{2} \left| \int_{Q_t(z)} dx \sum_{j,k=1,2} \left[\frac{\partial^2 G}{\partial x_j \partial x_k}\right]_{x=\bar{x}} \left(x - \frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)_j \left(x - \frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)_k \right| \\ &\leq \max_{x \in Q_t(z)} \max_{j,k=1,2} \left| \frac{\partial^2 G(x)}{\partial x_j \partial x_k} \right| \frac{const}{t^2} . \end{aligned}$$

Now we recognize that

$$\begin{aligned} \frac{\partial^2 G(x)}{\partial x_j \partial x_k} &= \frac{t}{T} f_{jk} \left(R + \sqrt{\frac{t}{T}}\, x\right) e^{-\frac{\mathcal{A}(x)}{2}} + \sqrt{t}\, T f_j \left(R + \sqrt{\frac{t}{T}}\, x\right) \frac{\partial e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_k} \\ &+ \sqrt{\frac{t}{T}} f_k \left(R + \sqrt{\frac{t}{T}}\, x\right) \frac{\partial e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_j} + f\left(R + \sqrt{\frac{t}{T}}\, x\right) \frac{\partial^2 e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_j \partial x_k} , \end{aligned}$$

and denoting by z_t^* the point where the function $Q_t(z)$ reaches its maximum, we have

$$\left| \frac{\partial^2 G(x)}{\partial x_j \partial x_k} \right| \leq const \,\|\, f \,\|\, \max_{x \in Q_t(z)} e^{-\frac{\mathcal{A}(x)}{4}} = const \,\|\, f \,\|\, e^{-\frac{\mathcal{A}(z_t^*)}{4}} .$$

Since the sum

$$\sum_{z \in \mathbb{Z}^2} \frac{1}{t} e^{-\frac{\mathcal{A}(z_t^*)}{4}} ,$$

is a bounded Riemann sum, hence the difference in (3.7) can be bounded as follows

$$\text{(3.9)} \quad \begin{aligned} &\left| \int \mathbb{R}^2 e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R + \sqrt{\frac{t}{T}}\, x\right) - \frac{1}{t} \sum_{z \in \mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z - \boldsymbol{b}t}{\sqrt{t}}\right)} f\left(R + \sqrt{\frac{t}{T}}\, \frac{z - \boldsymbol{b}t}{\sqrt{t}}\right) \right| \\ &\qquad \leq const \cdot \frac{\|\, f \,\|}{t} . \end{aligned}$$

Again by the LLT we have

$$H_T(t,y) = \sum_{z\in\mathbb{Z}^2} P_0^{T-t}(z)\nabla f\left(\frac{y-\boldsymbol{b}t+z-\boldsymbol{b}(T-t)}{\sqrt{T}}\right)$$
$$= \sum_{z\in\mathbb{Z}^2} \sqrt{C}\,\frac{e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\boldsymbol{b}(T-t)}{\sqrt{T-t}}\right)}}{2\pi t}\nabla f\left(\frac{y-\boldsymbol{b}t+z-\boldsymbol{b}(T-t)}{\sqrt{T}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{T-t}}\right),$$

then by (3.9) we obtain

$$\sum_{z\in\mathbb{Z}^2} \sqrt{C}\,\frac{e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\boldsymbol{b}(T-t)}{\sqrt{T-t}}\right)}}{2\pi t}\nabla f\left(\frac{y-\boldsymbol{b}t+z-\boldsymbol{b}(T-t)}{\sqrt{T}}\right)$$
$$= \int \frac{\sqrt{C}e^{-\frac{1}{2}\mathcal{A}(x)}}{2\pi}\nabla f\left(\frac{y-\boldsymbol{b}t}{\sqrt{T}} + \sqrt{1-\frac{t}{T}}x\right)dx + \mathcal{O}\left(\frac{1}{T-t}\right),$$

by the change of variable $v \equiv x\sqrt{1-\frac{t}{T}}$ and setting:

$$H_T^*(t,y,\nabla f) \equiv \int_{\mathbb{R}^2} K_C\left(1-\frac{t}{T},v\right)\nabla f\left(\frac{y-\boldsymbol{b}t}{\sqrt{T}}+v\right)dv$$

so that

$$H_T^*(t,y,\nabla f) = \left(H_T^*(t,y,\frac{\partial f}{\partial y_1}), H_T^*(t,y,\frac{\partial f}{\partial y_2})\right) = (H_T^*(t,y,f_1), H_T^*(t,y,f_2)),$$

we obtain

$$H_T(t,y) = H_T^*(t,y,\nabla f) + \mathcal{O}\left(\frac{1}{\sqrt{T-t}}\right), \tag{3.10}$$

where K_C is the 2-dimensional heat kernel

$$K_C(s,v) \equiv \frac{\sqrt{C}}{2\pi s}\cdot e^{-\frac{\mathcal{A}(v)}{2s}}.$$

In $\hat{\mathcal{Q}}_T^{(2)}(f|\xi)$ the contribution for $t_1 \le T_1$ is given by

$$\bar{\mathcal{Q}}''(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}}\sum_{t_1=0}^{T_1}\sum_{t_2=t_1}^{t_1+T_*}\sum_{y_1,y_2} P_0^{t_1}(y_1)M_\sharp(t_2-t_1,y_2-y_1 \mid \xi_{(t_2,y_2)})\cdot H_T(t_2,y_2),$$

and it can be neglected since

$$\langle(\bar{\mathcal{Q}}''_T(f \mid \xi))^2\rangle \le const\,\frac{(\,\|\,\nabla(f)\,\|_\infty\,)^2}{\sqrt{T}\cdot\ln T}\cdot\sum_{t_1=1}^{T_1}\frac{1}{t_1} \overset{T\to\infty}{\longrightarrow} 0,$$

then by the approximation result (3.10) we are left with the time asymptotic of the quantity

$$(3.11)\quad \left(\frac{1}{\ln T}\right)\sum_{t_1=T_1+1}^{T-T_1}\sum_{t_2=t_1}^{t_1+T_*}\sum_{y_1,y_2}(P_0^{t_1}(y_1))^2\langle\big(M_\sharp(t_2-t_1,y_2-y_1\mid\xi_{(t_2,y_2)})\cdot H_T^*(t_2,y_2,\nabla f)\big)^2\rangle\,.$$

We can start taking into account the diagonal component of index $(1,1)$. The short range condition implies

$$\mid H_T^*(t_2,y_2,f_1)-H_T^*(t_1,y_1,f_2)\mid\le const(\parallel f\parallel_\infty+\parallel\nabla(f)\parallel_\infty)\cdot\frac{T_*}{\sqrt{T}}\,,$$

hence we can replace $H_T^*(t_2,y_2,f_1)$ with $H_T^*(t_1,y_1,f_1)$ and sum over (t_1,y_1). Then, using the definition of the functionals $\mathfrak{C}_{ij}^{(T)}$ given in (3.6), the asymptotic of (3.11) in the first spatial coordinate of H_T^* is the same as

$$(3.12)\qquad \frac{\langle(\mathfrak{C}_{11}^T(\,\cdot\,))^2\rangle}{\ln T}\sum_{t=T_1-1}^{T-T_1}\sum_y(P_0^t(y))^2F_1\left(\frac{t}{T},\frac{y-\boldsymbol{b}t}{\sqrt{T}}\right)\,,$$

where

$$F_1\left(\frac{t}{T},\frac{y-\boldsymbol{b}t}{\sqrt{T}}\right)\equiv\left(\int K_c\left(1-\frac{t}{T},v\right)f_1\left(\frac{y-\boldsymbol{b}t}{\sqrt{T}}+v\right)dv\right)^2.$$

Taking the first order Taylor expansion of F_1 in the space variable, we can rewrite (3.12), for an appropriate point $y^\star=y^\star(y)$, as

$$(3.13)\quad \frac{1}{\ln T}\left\{\sum_{t=T_1}^{T-T_1}\sum_{y\in\mathbb{Z}^2}(P_0^t(y))^2\left[F_1\left(\frac{t}{T},0\right)+\nabla_y F_1\left(\frac{t}{T},y^\star(y)\right)\cdot\left(\frac{y-\boldsymbol{b}t}{\sqrt{T}}\right)\right]\right\}.$$

By the first inequality in Lemma A.1 of [8], we have:

$$\begin{aligned}\frac{1}{\sqrt{T}\ln T}\sum_{t\ge1}\sum_{y\in\mathbb{Z}^2}(P_0^t(y))^2\parallel y-\boldsymbol{b}t\parallel &\le\frac{const}{\sqrt{T}\ln T}\sum_{t\ge1}\frac{1}{t}\sum_{y\in\mathbb{Z}^2}P_0^t(y)\parallel y-\boldsymbol{b}t\parallel\\ &\asymp\frac{const}{\sqrt{T}\ln T}\sum_{t\ge1}\frac{\sqrt{t}}{t}\asymp\frac{const}{\ln T}\overset{T\to\infty}{\longrightarrow}0\,,\end{aligned}$$

then in (3.13) it is sufficient to control the behaviour of the first addendum, i.e. we have to study the asymptotic of a quantity of the following type

$$(3.14)\qquad I_T(f)\equiv\frac{1}{\ln T}\sum_{t=0}^{T-1}f\left(\frac{t}{T}\right)\sum_{y\in\mathbb{Z}^2}(P_0^t(y))^2\,,$$

where f is a sufficiently smooth function in $[0,1]$. First we will find the asymptotic for

$$J(T) \equiv \sum_{t=0}^{T-1} \sum_{y\in\mathbb{Z}^2} \left(P_0^t(y)\right)^2 . \tag{3.15}$$

Let us denote by $\tilde{p}_0(\lambda)$ the characteristic function of P_0, then its centered version is $\hat{p}(\lambda) = e^{-i(\lambda,\boldsymbol{b})}\tilde{p}_0(\lambda)$ and (3.15) can be rewritten as follows

$$J(T) = \sum_{t=0}^{T-1} \int_{\Gamma^2} |\ \hat{p}(\lambda)\ |^{2t}\ dm(\lambda) = \int_{T^2} \frac{1-|\ \hat{p}(\lambda)\ |^{2T}}{1-|\ \hat{p}(\lambda)\ |^2} dm(\lambda) .$$

Splitting the above integral in two parts we have

$$J^{'}(T) \equiv \int\limits_{1-|\hat{p}(\lambda)|^2<\delta} \frac{1-|\ \hat{p}(\lambda)\ |^{2T}}{1-|\ \hat{p}(\lambda)\ |^2} dm(\lambda) ,$$

$$J^{''}(T) \equiv \int\limits_{1-|\hat{p}(\lambda)|^2\geq\delta} \frac{1-|\ \hat{p}(\lambda)\ |^{2T}}{1-|\ \hat{p}(\lambda)\ |^2} dm(\lambda) .$$

The $J^{''}(T)$ term remains bounded for $T\to\infty$, hence its asymptotic in (3.14) is equal to zero. In $J^{'}(T)$ we perform the coordinate change $1-|\ \hat{p}(\lambda)\ |^2 = u^2$ and we indicate its Jacobian by $C(u) \equiv \sum_{k=0}^{\infty} c_k(u)$, where $c_k(u)$ are homogeneous function of degree k, obtaining

$$J^{'}(T) = \int\limits_{u^2<\delta} C(u) \frac{1-\left(1-u^2\right)^{2T}}{u^2} du .$$

If we pass to polar coordinates $(u_1,u_2)\to(\rho,\theta)$ then previous Jacobian is equal to $C(\rho,\theta) = \sum_{k\geq 0} \rho^k \hat{c}_k(\theta)$ and if k is odd then we have $\int \hat{c}_k(\theta)d\theta = 0$. Assuming $\tilde{c}_k \equiv \int \hat{c}_k(\theta)d\theta$, we obtain

$$J^{'}(T) = \sum_{k\geq 0} \tilde{c}_{2k} \int_o^\delta \rho^{2k-1}(1-(1-\rho^2)^{2T})d\rho .$$

For $k\geq 1$ our sum gives a constant, hence we are left with '$k=0$'-*case* in the limit for $T\to\infty$. Let be $\rho = \frac{z}{\sqrt{2T}}$, we have

$$\begin{aligned}\tilde{c}_0 \int_0^\delta \frac{1-(1-\rho^2)^{2T}}{\rho} d\rho &= \tilde{c}_0 \int_0^{\sqrt{2T}\delta} \left[1-\left(1-\frac{z^2}{2T}\right)^{2T}\right] \frac{dz}{z} \\ &= \mathscr{O}(1) + \tilde{c}_0 \int_0^{\sqrt{2T}\delta} \frac{1-e^{-z^2}}{z} \\ &= \mathscr{O}(1) + \frac{\tilde{c}_0}{2} \ln T \Rightarrow J(T) = \mathscr{O}(1) + \frac{\tilde{c}_0}{2} \ln T .\end{aligned}$$

Assuming $J(0) = 0$ quantity in (3.14) reads as follows

$$I_T(f) = \frac{1}{\ln T}\left[f\left(\frac{T-1}{T}\right)J(T) - \sum_{t=1}^{T}\left(f\left(\frac{t}{T}\right) - f\left(\frac{t-1}{T}\right)\right)J(t)\right].$$

Since

$$f\left(\frac{T-1}{T}\right)\frac{J(T)}{\ln T} \overset{T\to\infty}{\longrightarrow} f(1)\frac{\tilde{c}_0}{2} ,$$

and by the asymptotic of $J(T)$, it remains to control the quantity

$$\frac{-\tilde{c}_0}{2\ln T}\sum_{t=0}^{T-1}\left(f\left(\frac{t+1}{T}\right) - f\left(\frac{t}{T}\right)\right)\ln(t+1) ,$$

for which we have

$$-\frac{\tilde{c}_0}{2\ln T}\sum_{t=0}^{T-1}\left(f\left(\frac{t+1}{T}\right) - f\left(\frac{t}{T}\right)\right)\ln(t+1) \asymp -\frac{\tilde{c}_0}{2T\ln T}\sum_{t=0}^{T-1} f'\left(\frac{t}{T}\right)\left(\ln\frac{t}{T} + \ln T\right)$$
$$= -\frac{\tilde{c}_0}{2T\ln T}\sum_{t=0}^{T-1}\left[f'\left(\frac{t}{T}\right)\ln\frac{t}{T}\right] - \frac{\tilde{c}_0}{2T}\sum_{t=0}^{T-1} f'\left(\frac{t}{T}\right)$$
$$\asymp -\frac{\tilde{c}_0}{2\ln T}\int_0^1 f'(x)\ln x\, dx + \frac{\tilde{c}_0}{2}\int_0^1 f'(x)dx = \frac{\tilde{c}_0}{2}\left(f(0) - f(1)\right) ,$$

so that for the asymptotic of (3.14) we obtain

$$I_T(f) \overset{T\to\infty}{\longrightarrow} \frac{\tilde{c}_0}{2} f(0) ,$$

which implies that in our case we have

$$f(0) = F_i(0,0) = \left(\int K_C(1,v) f_i(v) dv\right)^2 ,$$

for $i = 1, 2$.

We complete the proof evaluating the mixed terms by analogous arguments.

Using previous result we can conclude proof of Th. 2.1 showing the CLT for the sequence of functionals $\hat{\mathcal{Q}}^2_T(f \mid \xi)$. Let be

$$\mathcal{E}^{(T)}(t_1 \mid \xi) \equiv \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1 y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2, y_2) ,$$

then

$$\hat{\mathcal{Q}}^2_T(f \mid \xi) = \frac{1}{\sqrt{\ln T}} \sum_{t=0}^{T-T_1} \mathcal{E}^{(T)}(t \mid \xi) , \tag{3.16}$$

and since for $t_1 < t_1'$ and $t_1' - t_1 > T_*$ the quantities $\mathcal{E}^{(T)}(t_1 \mid \xi)$ and $\mathcal{E}^{(T)}(t_1' \mid \xi)$ are independent, we can apply the Bernstein method (see e.g. [19]). Let us define

$$0 < \delta < \gamma < 1, \quad r \equiv [T^\gamma], \quad s \equiv [T^\delta], \quad \mathscr{K}(T) \equiv \left[\frac{T}{T^\gamma + T^\delta} \right],$$

the intervals I_k

$$I_k \equiv [(k-1)(r+s), kr + (k-1)s - 1], \quad k = 1, \dots, \mathscr{K}(T),$$

the corridors J_k

$$J_k \equiv [kr + (k-1)s), k(r+s) - 1], \quad k = 1, \dots, \mathscr{K}(T),$$

and

$$R \equiv [\mathscr{K}(r+s), T-1],$$

which may be empty. If we consider the quantity

$$\hat{\mathscr{Q}}''_T(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t \in \cup_{k=1}^{\mathscr{K}} J_k \cup R} \mathcal{E}^{(T)}(t \mid \xi),$$

then the following result holds

LEMMA 3.4. – *If* $\varepsilon \in [0, 1)$ *is small enough then*

$$\left\langle (\hat{\mathscr{Q}}''_T(f \mid \cdot))^2 \right\rangle \stackrel{T \to \infty}{\longrightarrow} 0.$$

PROOF. – The estimates given in the proof of lemma (3.2) imply

$$\langle (\mathcal{E}^{(T)}(t \mid \cdot))^2 \rangle \le const \frac{\| \nabla f \|_\infty^2}{t},$$

so that

$$\begin{aligned}
\left\langle \left(\sum_{t \in J_k} \mathcal{E}^{(T)}(t \mid \cdot) \right)^2 \right\rangle &\le const \| \nabla f \|_\infty^2 \sum_{t = kr + (k-1)s}^{k(r+s)-1} \frac{1}{t} \\
&\le const \| \nabla f \|_\infty^2 \left[\ln \left(\frac{k(r+s) - 1}{kr + (k-1)s} \right) \right] \\
&\le \frac{s \cdot const \| \nabla f \|_\infty^2}{kr}.
\end{aligned}$$

Summing over k from 1 to $\mathscr{K}$ we have that the numerator grows as the logarithm of $\mathscr{K}$ and it can be bounded by $\ln T$. Hence the behaviour of the numerator is compensated by the factor $\frac{1}{\ln T}$ which appears in $\hat{\mathscr{Q}}_T^2$, see equation (3.16). So that the quantity which we are interested in, including the contribute due to summing over the interval R, tends to zero at least as $\frac{1}{T^{\gamma-\delta}}$.

Lemma (3.4) implies that the limiting distribution of $\hat{\mathscr{Q}}_T$ is the same as that of the difference $\hat{\mathscr{Q}}'_T \equiv \hat{\mathscr{Q}}^{(2)}_T - \hat{\mathscr{Q}}''_T$, which can be written as a sum of independent variables

$$\hat{\mathscr{Q}}'_T(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{j=1}^{\mathscr{K}} \mathscr{A}^{(j)}_T(\xi), \quad \mathscr{A}^{(j)}_T \equiv \sum_{t \in I_j} \mathcal{E}^{(T)}(t \mid \xi) .$$

Hence to finish the proof of Th. (2.1) is enough to establish a Lyapunov condition which is implied by a L^4-estimate for functionals of the type

$$\mathscr{A}^{(T)}_{\tau_1,\tau_2}(\xi) \equiv \sum_{t=\tau_1}^{\tau_2} \mathcal{E}^{(T)}(t \mid \xi) \quad , \quad \tau_2 + T_* < T \ ,$$

since the results holds, see e.g. [18], if we show

$$\frac{1}{(\ln T)^2} \sum_{j=1}^{\mathscr{K}(T)} \langle \left(\mathscr{A}^{(j)}_T(\xi) \right)^4 \rangle \overset{T\to\infty}{\longrightarrow} 0 .$$

By the definitions of r, s and $\mathscr{K}$, if $I_j = [\tau_{1_j}, \tau_{2_j}]$ then

$$\mathscr{A}^{(j)}_T(\xi) = \sum_{t=\tau_{1_j}}^{\tau_{2_j}} \mathcal{E}^{(T)}(t \mid \xi) \ ,$$

moreover $\tau_{1_j} = (j-1)(r+s)$ and $\tau_{2_j} = jr + (j-1)s - 1$, so that

$$\tau_{2_j} - \tau_{1_j} + T_* = r - j + 1 + T_* \le c \cdot (r + T_*) \ ,$$

while $\tau_{1_j} = (j-1)(r+s) \ge j \cdot r$. Hence by (5.1) with $n = 2$ and using the Lagrange Theorem, we obtain

$$\frac{1}{(\ln T)^2} \sum_{j=1}^{\mathscr{K}(T)} \langle (\mathscr{A}^{(j)}_T(\,\cdot\,))^4 \rangle \le \frac{C(\varepsilon, 2)}{(\ln T)^2} \sum_{j=1}^{\mathscr{K}(T)} \left(\frac{r + T_*}{j \cdot r} \right)^3 \overset{T\to\infty}{\longrightarrow} 0 \ ,$$

which concludes the proof of Theorem 2.1.

From now on we will work to prove our results about the behaviour of cumulants in dimension $\nu = 1, 2$, i.e. theorems (2.2), (2.3), (2.4), (2.5).

Let us define

$$\mathscr{E}^{(T)}(\xi) \equiv \mathbb{E}(X_T \mid \xi) - \boldsymbol{b}T = \sum_{t=0}^{T-1} \sum_{y \in \mathbb{Z}^1} M(y, t \mid \xi) \mathfrak{b}(\xi_t(y)) \ ,$$

which can be also written as

$$\sum_{0\le t_1\le t_2\le T-1}\sum_{y_1,y_2\in\mathbb{Z}^1} P_0^{t_1}(y_1)M_\sharp(t_2-t_1,y_2-y_1,\xi_{(t_1,y_1)})\,.$$

PROPOSITION 3.3. – *In dimension* $\nu=1$, *for* $\varepsilon\in[0,1)$ *small enough, there exists a constant depending on* ε *such that*

$$\sum_{y\in\mathbb{Z}^1}\langle M_\sharp^2(t,y\mid\cdot)\rangle\le\frac{const\cdot\varepsilon^2}{(t+1)^{\frac{3}{2}}}$$

PROOF. – Let be

$$b\equiv\max_{s\in\mathscr{S}}\mid\mathfrak{b}(s)\mid, k^t(x)\equiv\max_{s\in\mathscr{S}}\mid h^t(x;s)\mid$$

using the orthogonality of $M_B{}^\sharp(\xi)$, the inequalities of Lemma (A.1) in [8] and iterating, for $a>1$ and some constant $K=K(a)$, the following estimate

$$\sum_{t_1=1}^{T-1}[t_1(T-t_1)]^{-a}\le K(a)T^{-a}\ ,$$

we can choose ε small enough such that

$$\sum_{y\in\mathbb{Z}^1}\langle M_\sharp^2(t,y\mid\cdot)\rangle\le(\varepsilon b)^2\sum_{n=1}^{t}\varepsilon^{2n}\sum_{\substack{t_1+\cdots+t_n=t\\ \min\{t_2,\ldots,t_n\}>0}}\sum_{x_1,\ldots,x_n\in\mathbb{Z}^1}\prod_{i=1}^{n}(k^{t_1}(x_i))^2 \tag{3.17}$$
$$\le\frac{const\cdot\varepsilon^2}{(t+1)^{\frac{3}{2}}}$$

where $const=const(\varepsilon)$.

PROPOSITION 3.4. – *In dimension* $\nu=1$, *for* $\varepsilon\in[0,1)$ *small enough, we have*

$$(S_T)^2\equiv\langle(\mathscr{E}^{(T)})^2\rangle\asymp\sqrt{T}$$

PROOF. – By (3.17) and using the estimates for $P_0^t(y)$ given in appendix A of [8], we can write

$$(S_T)^2\le\sum_{t=0}^{T}\sum_{t_1+t_2=t}\frac{1}{t_1^{\frac{1}{2}}t_2^{\frac{3}{2}}}\le\sum_{t=1}^{T}\frac{const}{t^{\frac{1}{2}}}\le const\sqrt{T}\ ,$$

besides, if $B=\{(y,t)\}$ is a certain set of points, by the LLT about P_0 we obtain

$$(S_T)^2\ge const\sum_{t=0}^{T-1}\sum_{y:|y-\boldsymbol{b}t|>o(t^{\frac{2}{3}})}\left(P_0^T(y)\right)^2\asymp\sqrt{T}\,.$$

Following proof given for Prop. 3.4, one can prove that in dimension $\nu = 2$ and for $\varepsilon \in [0,1)$ small enough, the following holds

$$(S_T)^2 \equiv \langle (\mathscr{E}^{(T)})^2 \rangle \asymp \ln(T)\,.$$

Moreover if we define $T_1 \equiv [T^\beta]$, for $\beta \in (0,1)$, and $T_* \equiv [\log_+ T]$, where we have set $\log_+ T \equiv \max\{1, \log T\}$, and consider the functional

$$\hat{\mathscr{E}}^{(T)}(\xi) \equiv \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1 y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)})\,,$$

which is obtained from $\mathscr{E}^{(T)}(\xi)$ removing those terms with large t_1 and large $t_2 - t_1$, then by the same arguments used proving Lemma 3.2 the following result holds

PROPOSITION 3.5. – *In dimension $\nu = 2$, if $\varepsilon \in [0,1)$ is small enough then*

$$\lim_{T\to\infty} \frac{1}{(S_T)^2} \left\langle \left(\mathscr{E}^{(T)}(\xi) - \hat{\mathscr{E}}^{(T)}(\xi)\right)^2 \right\rangle = 0\,.$$

Hence the proof of the Th. (2.2) *is reduced to prove the CLT for* $\dfrac{1}{S_T}\hat{\mathscr{E}}_T(\xi)$. *Using again the Bernstein method, we divide the axis of time in intervals I_k and corridors J_k. Let be*

$$\hat{\mathcal{E}}^{(T)}(t_1 \mid \xi) \equiv \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1, y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)})\,,$$

and set

$$\hat{\mathscr{E}}'_T(\xi) \equiv \frac{1}{S_T} \sum_{t \in \cup_{k=1}^{\mathscr{K}} J_k \cup R} \hat{\mathcal{E}}^{(T)}(t \mid \xi)\,.$$

Lemma 3.4 *implies*

LEMMA 3.5. – *In dimension $\nu = 2$, if $\varepsilon \in [0,1)$ is small enough then*

$$\left\langle \left(\hat{\mathscr{E}}'_T(\xi)\right)^2 \right\rangle \overset{T\to\infty}{\longrightarrow} 0\,.$$

By Lemma (3.5) we deduce that the limit distribution of $\mathscr{E}^{(T)}(\xi)$ is the same as that of $\hat{\mathscr{E}}''_T \equiv \hat{\mathscr{E}}^{(T)} - \hat{\mathscr{E}}'_T$, which can be written as a sum of independent variables

$$\hat{\mathscr{E}}''_T(\xi) \equiv \sum_{j=1}^{\mathscr{K}} \hat{\mathscr{A}}_T^{(j)}(\xi)\,, \quad \hat{\mathscr{A}}_T^{(j)} \equiv \sum_{t \in I_j} \hat{\mathcal{E}}^{(T)}(t \mid \xi)\,.$$

To prove the CLT for the quantity $\hat{\mathscr{E}}''_T$ it is sufficient to establish a Lyapunov condition. Therefore we need an L^4-estimate for quantities of the type

$$\hat{\mathscr{A}}^{(T)}_{t_1,t_2} \equiv \sum_{t=t_1}^{t_2} \hat{\mathcal{E}}^{(T)}(t \mid \xi) \, ,$$

this result is proved in [12] and it implies

PROPOSITION 3.6. – *In dimension* $\nu = 1$ *if* $\varepsilon \in [0,1)$ *is small enough then there exists a positive constant depending on* ε, *such that*

$$\left\langle \left(\hat{\mathscr{A}}^{(T)}_{\tau1,\tau2} \right)^4 \right\rangle \leq const \cdot \varepsilon^4 \left(\sqrt{t_2} - \sqrt{t_1} \right)^2 .$$

Proposition (3.6) implies

$$\frac{1}{(S_T)^4} \sum_{j=1}^{(T)} \left\langle \left(\hat{\mathscr{A}}^{(j)}_T (\xi) \right)^4 \right\rangle \leq \frac{const}{(S_T)^4} \sum_{j=1}^{(T)} \frac{r^2}{j(r+s)} \leq const \frac{T^\gamma \ln T}{(S_T)^4} \overset{T\to\infty}{\longrightarrow} 0 \, ,$$

hence Th. 2.2 *is proved.*

Previous results make easier to prove theorems 2.3, 2.4 *and* (2.5).

PROOF (of Th. 2.3). – Given a vector $\boldsymbol{v} \in \mathbb{R}^2$ let us define

$$\mathscr{E}^{(T)^{\boldsymbol{v}}} \equiv (\mathscr{E}^{(T)}) \cdot \boldsymbol{v} = \sum_{y \in \mathbb{Z}^2} M(y,t \mid \xi_t(y)) \mathfrak{b}(\xi_t(y)) \cdot \boldsymbol{v} \, .$$

By Prop. 3.4 we have

$$(S^T_{\boldsymbol{v}})^2 \equiv \left\langle \left(\mathscr{E}^{(T)^{\boldsymbol{v}}} \right)^2 \right\rangle \asymp \ln T \, .$$

Following proof given for Th. 2.1 and by results contained in the Appendix A, namely Prop. 5.1, if we define the matrix

$$\Sigma \equiv \left\{ \mathfrak{b}_{ij} \right\} = \left\{ \langle \mathfrak{b}_i(\,\cdot\,) \mathfrak{b}_j(\,\cdot\,) \rangle \right\} \, ,$$

where $\mathfrak{b}_i(\,\cdot\,) \equiv \sum_{u \in \mathbb{Z}^2} u_i c(u; \cdot)$, then we have

$$\frac{\mathscr{E}^{(T)^{\boldsymbol{v}}}}{S^T_{\boldsymbol{v}}} \overset{\mathscr{D}}{\longrightarrow} \mathscr{N}(0, \Sigma) \, ,$$

that is

$$\frac{\mathfrak{b} \cdot \boldsymbol{v}}{\sum_{t=0}^{T-1} \sum_{y \in \mathbb{Z}^2} \langle (M(y,t \mid \cdot)^2 \rangle} \mathscr{E}(\xi) \overset{\mathscr{D}}{\longrightarrow} \mathscr{N}(0, \boldsymbol{b} \Sigma \boldsymbol{b}^T) \, ,$$

where the matrix Σ is non degenerate if and only if $\mathfrak{b}_{11} \mathfrak{b}_{22} \neq \mathfrak{b}_{12}^2$.

PROOFS (of Th. 2.4 and 2.5). – In dimension $\nu = 1, 2$ the corrections to covariance matrix are

$$\mathscr{C}_{ij}^{(T)}(\xi) \equiv \hat{\mathscr{C}}_{ij}^{(T)}(\xi) + \mathscr{E}_{ij}^{(T)}(\xi) - \mathscr{E}_{i}^{(T)}(\xi)\mathscr{E}_{j}^{(T)}(\xi)\,,$$

where we have defined

$$\hat{\mathscr{C}}_{ij}^{(T)} \equiv \sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^\nu} M(y,t\mid\xi)\big[(y_i - b_i t)\mathfrak{b}_i(\xi_t(y)) + (y_j - b_j t)\mathfrak{b}_j(\xi_t(y))\big]\,,$$

$$\mathscr{E}_{ij}^{(T)}(\xi) \equiv \sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^\nu} M(y,t\mid\xi)\sum_{u\in\mathbb{Z}^\nu}(u_i - b_i)(u_j - b_j)c(y;\xi_t(y))\,.$$

By the short range condition, following proof given for Th. 2.2, the asymptotic behaviour of $\mathscr{E}_{ij}^{(T)}$ is the same as that of $\mathscr{E}^{(T)}$. Now we want to consider the $\mathscr{C}_{ij}^{(T)}(\xi)$ term. In [8] the following inequality is proved

$$\big\langle\big(M(y,t\mid\cdot)\big)^2\big\rangle \le \frac{const}{t^\nu}e^{-\beta\frac{(y-bt)^2}{2t}}\,,$$

for some positive constants β and for all $(y,t)\in\mathbb{Z}^{\nu+1}$. Let $\mathbf{b}$ be the drift of our model, we have

$$\sum_{y\in\mathbb{Z}^\nu}\langle(M(y,t\mid\cdot))^2\rangle(y-\boldsymbol{b}t)^2 \le \frac{const}{t^{\frac{\nu}{2}-1}}\,,$$

and

$$\sum_{t=1}^{T-1}\frac{1}{t^{\frac{\nu}{2}-1}} \asymp T^{2-\frac{\nu}{2}}\,,$$

hence setting

$$\tilde{S}_{ij}^{(T)} \equiv \big\langle\big(\hat{\mathscr{C}}_{ij}^{(T)}\big)^2\big\rangle^{\frac{1}{2}}\,,$$

by the same arguments used in Prop. 3.4, we obtain

$$\big(\tilde{S}_{ij}^{(T)}\big)^2 \asymp T^{2-\frac{\nu}{2}}\,,$$

so that

$$\frac{\mathscr{E}_{ij}^{(T)}(\xi)}{\tilde{S}_{ij}^{(T)}} \xrightarrow{\mathscr{D}} 0\,, \quad \frac{\mathscr{E}_{i}^{(T)}\mathscr{E}_{j}^{(T)}}{\tilde{S}_{ij}^{(T)}} \xrightarrow{\mathscr{D}} 0\,,$$

hence the only term that significantly contributes to the asymptotic of correction for $\mathscr{C}_{ij}^{(T)}(\xi)$ is $\hat{\mathscr{C}}_{ij}^{(T)}(\xi)$. Nevertheless we already know the asymptotic behaviour of $\hat{\mathscr{C}}_{ij}^{(T)}(\xi)$ in dimension $\nu = 1, 2$. By proofs of theorems 2.2 and 2.3, in dimension $\nu = 1$ we can use the results contained in [12], while in dimension $\nu = 2$ we have Prop. 5.1.

4. – Conclusions.

Results presented in this article are obtained by a small stochasticity condition which was used to ensure that the power series coming from the Cluster Expansion of some moments with respect to the field distribution converge.

Removing previous assumption results on the a.s. validity of the CLT for the quenched model are shown in [10] and in [11] provided that a non-degeneracy condition is met. Nevertheless neither in [10] nor in [11] authors are able to prove our results without a small randomness condition.

Even if we decided to prove our results following Cluster Expansion's methods developed in [8, 9], nevertheless we would like to thank the anonymous referee who suggest that a different approach using martingale theory can avoid several technical calculations proving our theorems in a more elegant way. The latter will be used in a future work.

5. – Appendix A.

Under our assumptions on the model and in dimension $\nu = 2$ we want to prove the following proposition

PROPOSITION 5.1. – *Let be $n \geq 1$, if $\varepsilon \in [0, 1)$ is small enough, then there exists a positive constant $C = C(\varepsilon, n)$ such that*

$$\left\langle \left(\mathscr{A}_{\tau 1, \tau 2}^{(T)} \right)^{2n} \right\rangle \leq C(\varepsilon, n) \cdot (\ln(\tau_2 + T_*) - \ln(\tau_1))^{2n-1}$$

PROOF. – We have that

$$M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1, y_1)}) = \sum_{B:(t_1, y_1) \to (t_2, y_2)} \varepsilon^{|B|} \cdot M_B^\sharp(\xi) \ , \tag{5.1}$$

and setting

$$M_B^\sharp(\xi) \equiv \prod_{i=1}^{|B|} h^{\tau_i}(z_i, s_i) \cdot \mathfrak{b}(\xi_{t_f(B)}(y_f(B))) \ ,$$

moments of the type $\left\langle \prod_{k=1}^{2n} M_{B_k}^\sharp \right\rangle$ are zero, unless the sets $\{B_1, \ldots, B_{2n}\}$ share the so-called *covering* property, i.e. they have to satisfy following relation

$$B_j \in \bigcup_{\substack{i=1,\dots,2n \\ i\neq j}} B_i, \quad 1,\dots,2n\,.$$

Let us define the following class

$$\mathscr{B}_{2n} \equiv \{\mathscr{B} = \{B_1,\dots,B_{2n}\} \mid \mathscr{B} \text{ has the covering property}\}\ ,$$

an element $\mathscr{B} = \{B_1,\dots,B_{2n}\} \in \mathscr{B}_{2n}$ is identified by a finite subset of points in $\mathbb{Z}^{2+1}$, i.e.

$$B \equiv \bigcup_{j=1}^{2n} B_j \subset \mathbb{Z}^{2+1}.$$

Any point $v \in B$ can be equipped with the specification $l_v \equiv \{j \mid v \in B_j\}$, which is a collection of labels representing the set to which v belongs.

We are interested only in those collections of sets which have the *covering* property so it must be $\mid l_v \mid \geq 2$ for all vertex $v \in \mathscr{B}$. If we define $S \equiv \{l_v \mid v \in B\}$, then there is a one-to-one correspondence between elements $\mathscr{B} \subset \mathscr{B}_{2n}$ and the pairs (B,S) obtained by imposing the following conditions

(i) If two distinct points have the same time coordinate then the correspondent sets l_v are disjoint

(ii) Each label must appear at least once , i.e.

$$\bigcup_{v\in B} l_v = \{1,2,\dots,2n\}.$$

We associate to any given element in $(B,S) \in \mathscr{B}_{2n}$ a graph $\mathscr{G} \equiv (B_0,\mathscr{L})$, where $B_0 \equiv B \bigcup \{0\}$ is the set of vertexes, while $\mathscr{L}$ is the set of bonds obtained by the union of two subsets of bonds $\mathscr{L}_*, \mathscr{L}'$ which is determined as follows.

For each vertex $v = (t,x) \in B$ and any given $j \in l_v$, we consider the class $v_j \equiv \{v' = (t',x') \mid j \in l_{v'}, t' > t\}$. If $v_j \neq \emptyset$ we draw a bond from v to the vertex $v_* \in v_j$ with minimal time coordinate (which is unique by condition (i)), this method complete the construction of $\mathscr{L}'$. To construct $\mathscr{L}_*$ we draw a bond from the origin to the initial point of each B_j.

Let us define $\mathscr{B}_{2n}^{\tau_1,\tau_2+T_*}$ as the subset of $\mathscr{B}_{2n}$ containing all and only those trajectories $\mathscr{B} = \{B_1,\dots,B_{2n}\}$ of the type

$$t_f(B_j) \in \{\tau_1,\dots,\tau_2+T_*\}, j = 1,\dots,2n\,.$$

If we define

$$N(\mathscr{B}) \equiv \sum_{j=1}^{2n} \mid B_j \mid, \quad b \equiv \max_{s\in\mathscr{S}} \parallel \sum_{u\in\mathbb{Z}^2} uc(u;s) \parallel\ ,$$

hence by (5.1), we have

$$\left\langle \left(\mathscr{A}^{(T)}_{\tau_1,\tau_2}(\,\cdot\,)\right)^{2n}\right\rangle = \left\langle \left(\sum_{t_1=\tau_1}^{\tau_2} \mathcal{E}(t_1 \mid \xi)\right)^{2n} \right\rangle$$

$$= \left\langle \left(\sum_{t_1=\tau_1}^{\tau_2} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1,y_2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \cdot) H(t_2,y_2)\right)^{2n} \right\rangle \tag{5.2}$$

$$\leq b^{2n} \parallel \nabla f \parallel_\infty^{2n} \sum_{\mathscr{B}\in\mathscr{B}_{2n}^{\tau 1,\tau_2+T_*}} \varepsilon^{N(\mathscr{B})} \cdot S(\mathscr{G}_{\mathscr{B}})\ ,$$

where $\mathscr{G}_{\mathscr{B}}$ is the graph associated to the particular choice of $\mathscr{B} \in \mathscr{B}_{2n}^{\tau_1,\tau_2+T_*}$ and

$$S(\mathscr{G}_{\mathscr{B}}) \equiv \prod_{b\in\mathscr{L}_*} \pi_*(b) \cdot \prod_{b\in\mathscr{L}'} \pi(b)\ ,$$

with weights $\pi_*(b), \pi(b)$ that, for $b = (v, v'), v = (t, x)$ and $v' = (t', x')$, are defined as follows

$$\pi(b) \equiv \max_{s\in\mathscr{S}} \mid h^t(y; s) \mid = \max_{s\in\mathscr{S}} \mid \sum_u c(u; s) \cdot P_0^{t-1}(y-u) \mid\ ,$$

while $\pi_*(b) \equiv P_0^t(x)$, if $b \in \mathscr{L}_*$ with $b = (0, v)$ and $v = (t, x)$.

For every set of points $B = \{(y_1, t_1), \ldots, (y_n, t_n)\}$ we define the following quantity

$$N_0(B) \equiv P_0^{t_1}(y_1) \prod_{i=1}^{n-1} \max_{s\in\mathscr{S}} \mid h^{t_{i+1}-t_i-1}(s, y_{i+1}-y_i) \mid\ ,$$

hence we can rewrite last row in (5.2) as follows

$$b^{2n} \cdot \|\nabla f\|_\infty^{2n} \cdot \sum_{(\mathscr{B}_1,\ldots,\mathscr{B}_{2n})\in\mathscr{B}_{2n}^{\tau 1,\tau_2+T_*}} \prod_{i=1}^{2n} \varepsilon^{|B_i|} N_0(B_i)\,.$$

Using $Lp-$ inequalities results contained in appendix A of [8], we obtain

$$\left\langle \left(\mathscr{A}^{(T)}_{\tau_1,\tau_2}(\,\cdot\,)\right)^{2n}\right\rangle \leq \sum_{k=1}^{2n-1} \sum_{\substack{n_1,\ldots,n_k\geq 1 \\ \max n_j>1, \sum_j n_j = 2n}} c(n_1,\ldots,n_k) \prod_{j=1}^{k} \sum_{t_j=\tau_1}^{\tau_2+T_*} \frac{1}{(t_j+1)^{m_j}}\ ,$$

where $c(n_1, \ldots, n_k)$ is a constant that depends on ε while the exponent m_j is defined as $m_j \equiv \max\{1, n_j - 1\}$, hence we have

$$\left\langle \left(\mathscr{A}^{(T)}_{\tau_1,\tau_2}(\,\cdot\,)\right)^{2n}\right\rangle \leq C(\varepsilon, n) \cdot (\ln(\tau_2 + T_*) - \ln(\tau_1))^{2n-1}$$

which concludes the proof.

Acknowledgments. I owe gratitude to Prof. A. Pellegrinotti for his deep, constant and highly motivating support and to Prof. C. Boldrighini for fruitful discussions and several foundamental hints. This work has been partially financed by Provincia Autonoma di Trento in the framework of the NeSt research project.

REFERENCES

[1] J. Bérard, *The almost sure central limit theorem for one-dimensional nearest-neighbour random walks in a space-time random environment*, J. Appl. Probab., **41**, no. 1 (2004), 83-92.

[2] M. S. Bernabei, *Anomalous behaviour for the random corrections to the cumulants of random walks in fluctuating random media*, Probab. Theory Related Fields, **119**, no. 3 (2001), 410-432.

[3] M. S. Bernabei - C. Boldrighini - R. A. Minlos - A. Pellegrinotti, *Almost-sure central limit theorem for a model of random walk in fluctuating random environment*, Markov Process. Related Fields, **4**, no. 3 (1998), 381-393.

[4] C. Boldrighini - G. Cosimi - S. Frigio - A. Pellegrinotti, *Computer simulations for some one-dimensional models of random walks in fluctuating random environment*, J. Stat. Phys., **121**, no. 3-4 (2005), 361-372.

[5] C. Boldrighini - Yu. G. Kondratiev - R. A. Minlos - A. Pellegrinotti - E. A. Zhizhina, *Random jumps in evolving random environment*, Markov Process. Related Fields, **14**, no. 4 (2008), 543-570.

[6] C. Boldrighini - R. A. Minlos - F. R. Nardi - A. Pellegrinotti, *Asymptotic decay of correlations for a random walk in interaction with a Markov field*, Mosc. Math. J., **5**, no. 3 (2005), 507-522, 742.

[7] C. Boldrighini - R. A. Minlos - F. R. Nardi - A. Pellegrinotti, *Asymptotic decay of correlations for a random walk on the lattice $\mathbb{Z}^d$ in interaction with a Markov field*, Mosc. Math. J., **8**, no. 3 (2008), 419-431, 615.

[8] C. Boldrighini - R. A. Minlos - A. Pellegrinotti, *Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment*, Probab. Theory Related Fields, **109**, no. 2 (1997), 245-273.

[9] C. Boldrighini - R. A. Minlos - A.Pellegrinotti, *Central limit theorem for a random walk in dynamical environment: integral and local*, Proceedings of the Third Ukrainian-Scandinavian Conference in Probability Theory and Mathematical Statistics (Kiev, 1999), vol. **5** (1999), 16-28.

[10] C. Boldrighini - R. A. Minlos - A. Pellegrinotti, *Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive*, Probab. Theory Related Fields, **129**, no. 1 (2004), 133-156.

[11] C. Boldrighini - R. A. Minlos - A. Pellegrinotti, *Random walks in a random (fluctuating) environment*, Uspekhi Mat. Nauk, **62**, no. 4 (2007), 27-76.

[12] C. Boldrighini - A. Pellegrinotti, *$T^{-1/4}$-noise for random walks in dynamic environment on $\mathbb{Z}$*, Mosc. Math. J., **1**, no. 3 (2001), 365-380, 470-471.

[13] J. Brémont, *On some random walks on $\mathbb{Z}$ in random medium*, Ann. Probab., **30**, no. 3 (2002), 1266-1312.

[14] J. Brémont, *Random walks in random medium on $\mathbb{Z}$ and Lyapunov spectrum*, Ann. Inst. H. Poincaré Probab. Statist., **40**, no. 3 (2004), 309-336.

[15] J. Brémont, *One-dimensional finite range random walk in random medium and*

invariant measure equation, Ann. Inst. Henri Poincaré Probab. Stat., **45**, no. 1 (2009), 70-103.
[16] Y. S. CHOW - H. TEICHER, Probability theory, third ed., Springer Texts in Statistics (Springer-Verlag, New York, 1997), Independence, interchangeability, martingales.
[17] D. DOLGOPYAT - G. KELLER - C. LIVERANI, *Random walk in Markovian environment*, Ann. Probab., **36**, no. 5 (2008), 1676-1710.
[18] Ĭ. Ī. GĪHMAN - A. V. SKOROHOD, *The theory of stochastic processes. I*, English ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **210** (Springer-Verlag, Berlin, 1980), Translated from the Russian by Samuel Kotz.
[19] I. A. IBRAGIMOV - YU. V. LINNIK, *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971, With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman.
[20] O. ZEITOUNI, *Random walks in random environments*, J. Phys. A, **39**, no. 40 (2006), R433-R464.

Dipartimento di Matematica
Università di Trento, Via Sommarive, 14. 38050 Povo (Trento)
E-mail: dipersio@science.unitn.it

Received November 6, 2009 and in revised form December 10, 2009