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Representations of Numbers as Sums and Differences of Unlike Powers

ENRICO JABARA

Abstract. – *In this paper we prove that every $n \in \mathbb{Z}$ can be written as*

$$n = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^5$$

and as

$$n = \varepsilon_1 x_1^3 + \varepsilon_2 x_2^4 + \varepsilon_3 x_3^5 + \varepsilon_4 x_4^6 + \varepsilon_5 x_5^7 + \varepsilon_6 x_6^8 + \varepsilon_7 x_7^9 + \varepsilon_8 x_8^{10}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$. We also prove some other results on numbers expressible as sums or differences of unlike powers.

1. – Introduction.

A classical problem in number theory is that of the representation of a given (natural, integer or rational) number as a sum of suitable powers. The best known example of this type is Waring's problem, that is the problem of representing any given positive integer n as a sum of s kth powers (k fixed, s depending on k):

$$n = \sum_{i=1}^{s} x_i^k \tag{1}$$

with $x_i \in \mathbb{N}$ for $1 \le i \le s$ (as usual, we mean $0 \in \mathbb{N}$).

A variation on this theme is the problem of representing an integer as a sum of increasing powers. Given an integer $r \ge 2$, we denote by $H(r)$ the smallest positive integer s such that every *sufficiently large* integer n can be represented in the form

$$n = x_1^r + x_2^{r+1} + \cdots + x_s^{r+s-1} \tag{2}$$

with $x_i \in \mathbb{N}$ ($1 \le i \le s$). Moreover, we denote by $\widehat{H}(r)$ the smallest integer s such that *almost all* (in the sense of asymptotic density) natural numbers can be expressed in the form (2).

Roth in [6] proves that $\widehat{H}(2) = 3$ and, in [7], that $H(2) \le 50$. The latter result has been improved by Ford, who shows that $H(2) \le 14$ ([1], Theorem 1). In the same paper, Ford proves that $H(3) \leq 72$ ([1], Theorem 2) and that for sufficiently large r one has $H(r) \ll r^2 \log(r)$ ([1], Theorem 3). Finally, Laporta and Wooley ([5], Theorem 1) prove that $\widehat{H}(3) \leq 8$.

Given the elementary character of this exposition, we give a simple proof of the following result, which turns out to be useful for fully understanding Theorem 2 and Remarks 4 and 5.

REMARK 1. – Let $s, \mu_1, \mu_2, \ldots, \mu_s$ be positive integers, $s \geq 2$. Then every sufficiently large natural number n can be represented in the form

$$n = \sum_{i=1}^{s} x_i^{\mu_i} \tag{3}$$

with $x_i \in \mathbb{N}$ only if

$$\sum_{i=1}^{s} \frac{1}{\mu_i} > 1. \tag{4}$$

PROOF. – Let $\mu_1, \mu_2, \ldots, \mu_s \in \mathbb{N}$ with $\sum\limits_{i=1}^{s} \mu_i^{-1} = \rho \leq 1$ and assume, working by contradiction, that there exists a $K \in \mathbb{N}$ such that every $n \geq K$ can be represented in the form (3). Clearly, we can assume that $2 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_s$ and that $K \geq 2$.

To get a contradiction, it is enough to produce an $R \in \mathbb{N}, R > K$, such that not all $R - K$ integers contained in the interval $[K, R-1]$ have a representation in the form (3).

Choose $r \in \mathbb{N}$ such that if $R = r^{\mu_1 \mu_2 \cdots \mu_s} - 1$ then $R > (2\mu_s K)^{\mu_s}$. It is easily checked that this inequality implies the following:

$$\mu_1 K R^{(-1/\mu_1)} + \mu_s K R^{(-1/\mu_s)} < 1. \tag{5}$$

For every $i \in \{1, 2, \ldots, s\}$, define $M_i = \{x^{\mu_i} \mid x \in \mathbb{N},\ x^{\mu_i} < R\}$. Every summand in (3) must clearly belong to a suitable M_i. Moreover, as $|M_i| = R^{1/\mu_i}$, one sees that

$$|M_1 \times M_2 \times \ldots \times M_s| = \prod_{i=1}^{s} R^{1/\mu_i} = R^{\rho} \leq R.$$

For every $\lambda_1, \lambda_s \in [1, K]$, taking (5) into account, one gets

$$\begin{aligned}\left(R^{1/\mu_1} - \lambda_1\right)^{\mu_1} + \left(R^{1/\mu_s} - \lambda_s\right)^{\mu_s} &\geq \left(R^{1/\mu_1} - K\right)^{\mu_1} + \left(R^{1/\mu_1} - K\right)^{\mu_s} \\ &> R\left(2 - \mu_1 K R^{-1/\mu_1} - \mu_s K R^{-1/\mu_s}\right) > R,\end{aligned}$$

and hence at least K^2 of the R^{ρ} sums (3) turn out to be bigger than R. Hence, it is possible to represent at most $R^\rho - K^2$ elements of the set $[0, R-1]$ in the form (3), while the set $[K, R-1]$ has cardinality $R - K$. □

A natural generalization of Waring's problem is the so-called "easier Waring's problem" introduced in [9] (see also [2] and §§21.7 and 21.8 in [3]), which consists in representing every integer n as a sum or difference of a suitable number t of kth powers (k fixed, t depending on k):

$$n = \sum_{i=1}^{t} \varepsilon_i x_i^k \tag{6}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$.

In this paper we consider the "easier" version of the problem of representing an integer as a sum of increasing powers.

We denote by $H_\pm(r)$ the smallest positive integer s such that every integer is representable in the form

$$n = \varepsilon_1 x_1^r + \varepsilon_2 x_2^{r+1} + \cdots + \varepsilon_s x_s^{r+s-1} \tag{7}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$.

The main result in this paper is the following:

THEOREM 1. – *Every $n \in \mathbb{Z}$ can be represented (in infinitely many ways) in the form*

$$n = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^5$$

and in the form

$$n = \varepsilon_1 x_1^3 + \varepsilon_2 x_2^4 + \varepsilon_3 x_3^5 + \varepsilon_4 x_4^6 + \varepsilon_5 x_5^7 + \varepsilon_6 x_6^8 + \varepsilon_7 x_7^9 + \varepsilon_8 x_8^{10}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$.

An equivalent way of expressing Theorem 1 is that

$$H_\pm(2) \le 4 \qquad \text{and} \qquad H_\pm(3) \le 8\,.$$

It is worth observing that a general conjecture concerning Waring's problem (see the introduction of chapter 8 in [8]) would imply that $H(2) = 3$ and that $H(3) = 5$ since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 \quad \text{and} \quad \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} > 1.$$

The statement $H_\pm(2) \le 4$ is a consequence of the following more general result:

PROPOSITION 1. – *Let $\nu \in \mathbb{N}$ be odd and coprime to 3. Then every $n \in \mathbb{Z}$ can be represented (in infinitely many ways) as*

$$n = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^\nu \qquad x_i \in \mathbb{Z} \quad \varepsilon_i \in \{-1, 1\}.$$

Similarly, the statement $H_\pm(3) \leq 8$ is a consequence of the following:

PROPOSITION 2. – *Every $n \in \mathbb{Z}$ can be represented as*

$$n = \varepsilon_1 x_1^3 + \varepsilon_2 x_2^5 + \varepsilon_3 x_3^7 + \varepsilon_4 x_4^8 + \varepsilon_5 x_5^8 + \varepsilon_6 x_6^9 + \varepsilon_7 x_7^{10}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$.

We stress here that in the "easier" case, where we allow differences as well as sums, a statement corresponding to that of Remark 1 does not hold. Namely, we have the following:

THEOREM 2. – *Let $\nu_1, \nu_2 \in \mathbb{N}$ with ν_1 odd. Then every $n \in \mathbb{Z}$ can be represented (in infinitely many ways) as*

$$n = x_1^3 + y_1^4 - y_2^4 + z_1^{\nu_1} + z_2^{\nu_2} \qquad x_1, y_1, y_2, z_1, z_2 \in \mathbb{Z}. \tag{8}$$

In particular, if $\nu_1 = \nu_2 = \nu > 12$ is odd, then in (8) one has

$$\sum_{i=1}^{5} \frac{1}{\mu_i} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{\nu} + \frac{1}{\nu} < 1.$$

All the above mentioned results have an elementary proof, which is suggested by the polynomial identity

$$(T + 1)^4 - (T - 1)^4 - (2\,T)^3 = 8T. \tag{9}$$

We finish this introductory part by recalling that while for Waring's problem powerful analytic methods has been applied (see [8]), every approach to the "easier problem" is based, up to now, just on polynomial identities and some elementary arithmetic (e.g. congruences).

2. – Proofs.

We first observe that every in every representation of type (6) or (7) we can omit the factor ε_i in front of every summand raised to an odd power.

In the proofs, we use the same strategy used in [2] and [9]. First, one proves that every element in a suitable coset $a\mathbb{Z} + b$ are representable as sums or differences of h powers. Then, one proves that every element of the ring $\mathbb{Z}/a\mathbb{Z}$ can be represented as sums or differences of k powers. So, it follows that every element of $\mathbb{Z}$ is representable as sums or differences of at most $h+k$ powers.

The proof of Theorem 2 is very easy. In fact, if $\lambda \in \mathbb{N}$ is an odd number, then, for every odd number $y \in \mathbb{Z}$, one has

$$y^\lambda \equiv y \mod 8. \tag{10}$$

Since every $n \in \mathbb{Z}$ can be written as $n = 8t + d_1 + d_2$ with $d_1 \in \{0, 1, 3, 5, 7\}$ and $d_2 \in \{0, 1\}$, the statement follows as an immediate consequence of (9) and (10).

In order to prove Proposition 1, we first observe that the preceding argument implies that the claim of Proposition 1 is true when $n \equiv 0, 1, 3, 5, 7 \mod 8$. It is hence enough to show that every even number $n \in \mathbb{Z}$ can be written in the form (8). We will make use of following lemma (where φ denote Euler's totient function).

LEMMA 1. – *Let $\nu, \tau \in \mathbb{Z}$ be odd and such that $(\nu, \varphi(\tau)) = 1$. Let r be a positive integer and let*

$$\Xi = \{\xi \in \mathbb{Z}/2^r\tau\mathbb{Z} \mid \xi \not\equiv 0 \mod 2\}.$$

Then the map $\Xi \longrightarrow \Xi \quad \xi \mapsto \xi^\nu$ is a bijection.

PROOF. – It is enough to recall that $\mathbb{Z}/2^r\tau\mathbb{Z} \simeq \mathbb{Z}/2^r\mathbb{Z} \times \mathbb{Z}/\tau\mathbb{Z}$ and that the map $\mathbb{Z}/\tau\mathbb{Z} \longrightarrow \mathbb{Z}/\tau\mathbb{Z} \quad \eta \mapsto \eta^\nu$ is injective. □

In order to complete the proof of Proposition 1, we observe that in $\mathbb{Z}[T]$ one has

$$(T+3)^4 - (2T+3)^3 - (T^2+2T+7)^2 = 26T + 5. \tag{11}$$

If $\nu \in \mathbb{N}$ is odd and coprime to 3, then $(\nu, 12) = 1$ and Lemma 1 applied to $\tau = 13$ yields that the map $\Xi \longrightarrow \Xi \quad \xi \mapsto \xi^\nu$ is bijective (where $\Xi = \{\xi \in \mathbb{Z}/26\mathbb{Z} \mid \xi \not\equiv 0 \text{ mod } 2\}$).

If $n \in \mathbb{Z}$ is even, then there exist (infinitely many) $T_0, \eta \in \mathbb{Z}$ such that

$$n = 26\,T_0 + 5 + \eta$$

with η (necessarily) odd. Hence there exists $\xi \in \mathbb{Z}$ with $\xi^\nu \equiv \eta \text{ mod } 26$ and one can determine $T_1 \in \mathbb{Z}$ such that $\eta = 26T_1 + \xi^\nu$. So, one gets

$$n = 26[T_0 + T_1] + 5 + \xi^\nu. \tag{12}$$

Now, recalling the polynomial identity (8), it follows that every integer n can be represented in the form $n = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^\nu$ with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$. Moreover, by (12) we see that there are infinitely many such representations for a given integer n, since there are infinitely many choices for an integer ξ such that $\xi^{\nu} \equiv \eta \bmod 26$.

To prove Proposition 2 we use the identity

$$\begin{aligned}(13)\qquad &(T+2^{52}\cdot 7^{28})^8-(T-2^{52}\cdot 7^{28})^8\\ &-(2^8\cdot 7^4\cdot T)^7-(2^{32}\cdot 7^{17}\cdot T)^5-(2^{88}\cdot 7^{47}\cdot T)^3\\ &=2^{368}\cdot 7^{196}\cdot T\end{aligned}$$

and the following

LEMMA 2. – *If $m = 2^r7^s$ $(r, s \geq 1)$, then every element of the ring $A = \mathbb{Z}/m\mathbb{Z}$ can be written in the form $\pm a_1^9 \pm a_2^{10}$, for suitable $a_1, a_2 \in A$.*

PROOF. – We first introduce some useful notation. If R is a commutative ring with unity, we denote by $\mathcal{I}(R)$ the multiplicative group of the invertible elements of R and, for $k \in \mathbb{N}$, we denote by π_k^R the endomorphism $\pi_k^R : \mathcal{I}(R) \longrightarrow \mathcal{I}(R)\ x \mapsto x^k$. Given two subsets X, Y of R, we write $-X = \{-x \mid x \in X\}$, $\pm X = X \cup -X$, $X + Y = \{x + y \mid x \in X, y \in Y\}$, $X - Y = X + (-Y)$ and $X \pm Y = X + (\pm Y)$. Finally, if $n \in \mathbb{N}$, we define $R^{[n]} = \{r^n \mid r \in R\}$.

We observe that $A = B \times C$ with $B = \mathbb{Z}/2^r\mathbb{Z}$ and $C = \mathbb{Z}/7^s\mathbb{Z}$. The endomorphism π_9^B is in fact an automorphism, since $\ker(\pi_9^B) = \{1\}$ (and B is a finite set). Hence $\mathcal{I}(B) \subseteq B^{[9]}$. Now, $B = \{0, 1\} + \mathcal{I}(B)$ and hence every $b \in B$ can be written as $j + x^9$ and as $h + y^9$ with $j \in \{0, 1\}$, $h \in \{0, -1\}$ (and $x, y \in B$).

We consider now π_{10}^C: one sees that $\ker(\pi_{10}^C) = \ker(\pi_2^C) = \{-1, 1\}$ (as $(5, \varphi(7)) = 1$). Since -1 is not a quadratic residue modulo 7, it follows that $\pi_{10}(\mathcal{I}(C)) \cap [-\pi_{10}(\mathcal{I}(C))] = \emptyset$, and hence

$$\mathcal{I}(C) = \pi_{10}(\mathcal{I}(C)) \cup [-\pi_{10}(\mathcal{I}(C))].$$

and $\mathcal{I}(C) \subseteq \pm C^{[10]}$. Since $\{0, 1\} \subseteq C^{[9]}$, it follows that $C = C^{[9]} \pm C^{[10]}$ and that every $c \in C$ can be written as $k + \ell z^{10}$ with $k \in \{0, 1\}$, $\ell \in \{1, -1\}$ (and $z \in C$).

Therefore, for every $(b, c) \in A$

$$(b, c) = \begin{cases} (x, k)^9 + (j, z)^{10} & \text{if } \ell = 1 \\ (y, k)^9 - (h, z)^{10} & \text{if } \ell = -1 \end{cases}$$

which yields the required decomposition. □

Theorem 1 is an easy consequence of Proposition 1 and Proposition 2. In fact, $H_{\pm}(2) \leq 4$ follows by considering $\nu = 5$ in Proposition 1. On the other hand, by using Proposition 2 and by observing that every 8th power is also 4th power, one gets that every integer n can be represented in the form

$$n = \varepsilon_1 x_1^3 + \varepsilon_2 x_2^4 + \varepsilon_3 x_3^5 + \varepsilon_4 x_4^7 + \varepsilon_5 x_5^8 + \varepsilon_6 x_6^9 + \varepsilon_7 x_7^{10} \tag{14}$$

with $x_i \in \mathbb{Z}$ and $\varepsilon_i \in \{-1, 1\}$. Since in (14) the term corresponding to 6th powers is missing, it easily follows that there are infinitely many representations of n in the second form given in Theorem 1.

3. – Further remarks.

REMARK 2. – Proposition 1 remains true also for $\nu = 3$. In this case, we can prove it by using the identity

$$(T^2 - T + 1)^2 + (T - 1)^3 + T^3 - T^4 = T$$

(for further identities concerning the sum of a square and two cubes see [4]).

REMARK 3. – The polynomial identity (11) is a (slightly changed) particular instance of the more general identity

$$T^4 + (2T + \gamma + 4)^3 - (T^2 + 4T + 6\gamma + 16)^2 = 2f(\gamma)T + g(\gamma) \tag{15}$$

where $f(\gamma) = 3\gamma^2 - 16$ and $g(\gamma) = \gamma^3 - 24\gamma^2 - 144\gamma - 192$. Observe that if $\gamma \in \mathbb{Z}$ is an odd number, then $f(\gamma)$ and $g(\gamma)$ are odd, as well. Using those facts, one can check that Proposition 1 remains true in many cases even if the restriction $(\nu, 3) = 1$ is dropped. In fact, many numbers of the form $f(\gamma)$ are primes of the type $2p + 1$ with p prime. In the following table, γ_i denotes the ith natural number such that $f(\gamma_i)$ is a prime number of the form $2p_i + 1$, with p_i prime.

i	1	2	3	4	5	...	31	...	1.000	...
γ_i	3	5	9	11	21	...	1.001	...	102.455	...
p_i	5	29	113	173	653	...	1.502.993	...	15.745.540.529	...

We remark that Proposition 1 remains valid for every positive integer ν not divisible by any prime p_i such that $2p_i + 1 = f(\gamma_i)$ is prime. In particular, Proposition 1 holds true for every odd ν divisible by at most 10000 distinct primes or such that $\nu \leq 10^{10^6}$ (this follows just by extending the table above).

Not only Proposition 1 is *almost* certainly true for every odd ν, but it is quite likely that the term containing the νth power is not going to be necessary. We state the following conjecture, which is based on the results in [6] and on "experimental" evidence:

CONJECTURE 1. – $H_{\pm}(2) = 3$, i.e. every $n \in \mathbb{Z}$ can be written in the form

$$n = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4$$

with $x_1, x_2, x_3 \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$.

On the other hand, it looks not possible to represent every integer as sums or differences of two 4th powers and a cube. Presumably, the smallest counterexample is 4, but no proof of this is known. Taking a considerable amount of computer-based experiments into account, it looks reasonable the following:

CONJECTURE 2. – Every $n \in \mathbb{Z}$ can be written (in infinitely many ways) in the form

$$n = \varepsilon_1 x_1^3 + \varepsilon_2 x_2^3 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^4$$

with $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$.

REMARK 4. – With the same methods used in the proof of Theorem 2 and using the identity

$$(T^3 - 16\,T^2 + 192\,T + 512)^2 + (2\,T - 8)^5 - T^6 = 2^{13} \cdot 29 \cdot T + 2^{15} \cdot 7 \tag{16}$$

one can show that, given any $\nu_1, \nu_2 \in \mathbb{N}$ with ν_1 odd and coprime to 7, every $n \in \mathbb{Z}$ can be represented in the form

$$n = x_1^2 + x_2^5 - x_3^6 + x_4^{\nu_1} + x_5^{\nu_2}$$

with $x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}$. To see this, it is enough to apply Lemma 1, recalling that $\varphi(2^{13} \cdot 29) = 2^{14} \cdot 7$.

Also in this case, if $\nu_1 = \nu_2 = \nu > 15$ is odd and coprime to 7, one gets that

$$\sum_{i=1}^{5} \frac{1}{\mu_i} = \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{\nu} + \frac{1}{\nu} < 1.$$

REMARK 5. – Let A be an algebra over a field $\mathbb{F}$ of characteristic $\neq 2$ (in particular, one can consider $A = \mathbb{F} = \mathbb{Q}$). Then the identity (9) shows that every element $a \in A$ can be written as

$$a = a_1^3 + a_2^4 - a_3^4$$

with suitable $a_1, a_2, a_3 \in A$. Moreover, if the characteristic of $\mathbb{F}$ is not 2 and not 7, by the identity (13) we can write every $a \in A$ as

$$a = \beta_1^3 + \beta_2^5 + \beta_3^7 + \beta_4^8 - \beta_5^8$$

with suitable $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in A$. Finally, if the characteristic of $\mathbb{F}$ is not 2 and not

29, by the identity (16) we can write every $a \in A$ as

$$a = \gamma_1^2 + \gamma_2^5 - \gamma_3^6$$

with suitable $\gamma_1, \gamma_2, \gamma_3 \in A$.

We note that the numbers

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} \approx 0.833, \quad \frac{1}{2} + \frac{1}{5} + \frac{1}{6} \approx 0.866, \quad \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{8} \approx 0.926$$

are both less than 1.

The polynomial identities (13), (15) and (16), though rather elementary, do not appear in any of the paper available to the author of this note.

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