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Approximation of Anisotropic Perimeter Functionals by Homogenization

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Abstract. – We show that all anisotropic perimeter functionals of the form \( \int_{\partial E \cap \Omega} \varphi(v_E) \, d\mathcal{H}^{n-1} \) (\( \varphi \) convex and positively homogeneous of degree one) can be approximated in the sense of \( \Gamma \)-convergence by (limits of) isotropic but inhomogeneous perimeter functionals of the form \( \int_{\partial E \cap \Omega} a(x/z) \, d\mathcal{H}^{n-1} \) (a periodic).

1. – Introduction.

Object of this paper is the approximation for anisotropic and crystalline energies of the form

\[ \mathcal{F}(E) = \int_{\partial E \cap \Omega} \varphi(v_E) \, d\mathcal{H}^{n-1} \]

defined on sets \( E \) with finite perimeter on an open set \( \Omega \subset \mathbb{R}^n \). Here and henceforth \( \partial E \) and \( v_E \) are the boundary and the inner normal of \( E \) in the usual measure theoretic sense and \( \varphi \) is convex, even, and positively homogeneous of degree one. In other words, \( \varphi \) is a norm on \( \mathbb{R}^n \). We do not assume that \( \varphi \) is smooth or isotropic. More precisely, we address the problem of approximating anisotropic functionals of the form (1.1) by locally isotropic but inhomogeneous perimeter functionals of the form

\[ \mathcal{G}_\varepsilon(E) = \int_{\partial E \cap \Omega} a(x_E) \, d\mathcal{H}^{n-1} , \]

with \( a \) a 1-periodic function.

Functionals of the form (1.1) are object of active research, especially in connection with crystalline motion by curvature (see Almgren and Taylor [2], Taylor [22]-[25] and the works by Bellettini, Gogline and Novaga [7], Bellettini and Novaga [8]).

Our approximation suggests an indirect way to deal with crystalline problems where anisotropy is replaced by inhomogeneity and a passage to the limit.
In A. Braides, M. Maslennikov, L. Sigalotti [14] it has been shown that energies of the form (1.2) converge to energies of the form (1.1) (see also Ambrosio-Braides [4]). Here we show that, conversely, all anisotropic energies can be approximated by (limits of) energies of the form (1.2) in the sense of $\Gamma$-convergence.

In this paper we suggest two way to approximate $\varphi$. In Section 3 given a target $\varphi$, $0 < a \leq \varphi \leq \beta < +\infty$, we define $a$ as

$$a(x) = \begin{cases} 
\varphi(v_j) & \text{if } x \in A_j \setminus \left( \bigcup_{k \notin j} A_k \right), \ j \in \mathbb{N} \\
\beta & \text{otherwise in } \mathbb{R}^n,
\end{cases}$$

for $\mathcal{H}^{n-1}$ a.e. $x$, where $\{ v_j \}$ is a dense sequence in $S^{n-1}$ such that $v_h \neq \pm v_j$ for $h \neq j$, $A_j = \mathbb{Z}^n + \Sigma_j$ and $\Sigma_j$ is the hyperplane through the origin and orthogonal to $v_j$. The idea behind the construction of the function $a$ is that the optimal sequences of sets $E_{\varepsilon} \to E$ will have boundaries that avoid the sets where the coefficient of $a$ is $\beta$; on the contrary these boundaries will lie on hyperplanes $A_j$, on which $a(x/\varepsilon) = \varphi(v_j) = \varphi(v_{E_{\varepsilon}})$, so that indeed $G_\varepsilon(E_{\varepsilon}) = \mathcal{F}(E_{\varepsilon}) \to \mathcal{F}(E)$.

In Section 4 in order to improve the regularity of $a$ a number of technical difficulties must be overcome. First we need to split our construction by considering a finite set $\{ v_1, \ldots, v_k \}$ of rational directions before letting $k \to +\infty$, and at the same time regularize our function $a$ to obtain a continuous integrand. In this way we obtain a $\Gamma$-limit depending on $k$ that is a candidate for an approximation of $\mathcal{F}$. The identification of the energy density of this $\Gamma$-limit requires the introduction of some carefully constructed piecewise-constant comparison energy densities on which to use the representation formulas for the homogenization of perimeters in [14].

Our result has some connections with a paper by Braides, Buttazzo and Fragalà [11] where (smooth) isotropic Riemannian metrics are shown to be dense in (lower semicontinuous) Finsler metrics in the sense similar to that stated above. Previously Acerbi and Buttazzo [1] proved that the class of Riemannian metrics is not closed in the class of all Finsler metrics with respect to the $\Gamma$-convergence of energy integrals. The result in [11] has been generalized to Borel Finsler metrics by Davini [16] (see also [17]).

A possible application of our result is the approximation of perimeter functionals by elliptic energies as in Modica-Mortola [20] (see also [10]) using a double-scale procedure as in Ansini, Braides and Chiodo Piat [6]. In fact, upon identifying a set $E$ with its characteristic functions $u = \chi_E$, the results in [6] show that energies (1.2) can be substituted by energies

$$\mathcal{J}_{\varepsilon,\delta}(u) = \int_{\Omega} \frac{W(u)}{\delta} + \delta \alpha^2 \left( \frac{x}{\varepsilon} \right) |Du|^2 \, dx$$

defined on $H^1(\Omega)$ where $W$ is a ‘double-well energy’ and $\alpha$ is periodic.
2. – Notation and preliminaries.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We denote the Lebesgue $n$-dimensional measure and the Hausdorff $(n-1)$-dimensional measure of a set $E \subset \mathbb{R}^n$ by $|E|$ and $\mathcal{H}^{n-1}(E)$, respectively, and we set

$$S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}.$$ 

We say that a sequence $\{E_j\}$ of measurable sets of $\Omega$ converges to a measurable set $E \subset \Omega$, and we write $E_j \to E$, if $|E_j \Delta E| \to 0$. Let $E$ be a Lesbegue measurable subset of $\mathbb{R}^n$. We denote the essential boundary of $E$ by $\partial^* E$ i.e.

$$\partial^* E = \left\{ x \in \mathbb{R}^n : \lim_{\rho \to 0^+} \frac{|B_{\rho}(x) \setminus E|}{\rho^n} > 0 \text{ and } \lim_{\rho \to 0^+} \frac{|B_{\rho}(x) \cap E|}{\rho^n} > 0 \right\}.$$ 

We say that $E$ is a set of finite perimeter in $\Omega$, or a Caccioppoli set, if it is measurable and

$$P(E, \Omega) := \sup \left\{ \int_E \text{div} \, g \, dx : g \in C_0^1(\Omega; \mathbb{R}^n), |g| \leq 1 \right\} < +\infty;$$

the number $P(E, \Omega)$ is called perimeter of $E$ in $\Omega$. We denote the class of sets with finite perimeter in $\Omega$ by $\mathcal{P}(\Omega)$ and the class of sets of locally finite perimeter in $\mathbb{R}^n$ by

$$\mathcal{P}_{loc}(\mathbb{R}^n) = \{ F \subset \mathbb{R}^n : F \in \mathcal{P}(\Omega), \text{ for any open set } \Omega \subset \subset \mathbb{R}^n \}.$$ 

Let $\chi_E$ be the characteristic function of $E$. For any set $E \in \mathcal{P}(\Omega)$ the essential boundary of $E$, $\partial^* E$, is $\mathcal{H}^{n-1}$-rectifiable i.e. there exists a countable family $(\Gamma_i)_{i=1}^\infty$ of graphs of Lipschitz functions of $(n-1)$ variables such that $\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$ and $\mathcal{H}^{n-1}(\partial^* E \cap \Omega) < +\infty$. Moreover, the distributional derivative $D\chi_E$ is an $\mathbb{R}^n$-valued finite Radon measure in $\Omega$, $P(E, \Omega) = |D\chi_E|(\Omega)$ and a generalized Gauss-Green formula holds

$$\int_E \text{div} \, g \, dx = -\int_{\Omega} \langle v_E, g \rangle \, d|D\chi_E|, \quad g \in C_0^1(\Omega; \mathbb{R}^n),$$

where $D\chi_E = v_E \cdot |D\chi_E|$ is the polar decomposition of $D\chi_E$ (see Theorem 3.36 in [5]). If $E$ has smooth boundary, the Gauss-Green theorem implies that $D\chi_E = v_E \mathcal{H}^{n-1} \llcorner \partial^* E$, where $v_E$ is the inner normal to $E$. This representation of the distributional derivative was generalized by De Giorgi and Federer as follows:

$$\exists \ v_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_{\rho}(x))}{|D\chi_E|(B_{\rho}(x))} \in S^{n-1} \quad \mathcal{H}^{n-1} \text{- a.e. } x \in \partial^* E$$
and
\[ D_{\mathcal{H}} = v_E \mathcal{H}^{n-1} \cap \partial^* E. \]

In particular, for every set \( E \in \mathcal{P}(\Omega) \), we have that \( P(E, \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega). \)

We refer to the books by Ambrosio, Fusco and Pallara [5] and Federer [19] for the complete exposition of the theory of sets with finite perimeter.

Let \( v \in S^{n-1} \), let \( Q^v \) be an open cube of \( \mathbb{R}^n \) centered at the origin having side length 1 and one face orthogonal to \( v \), and let \( \Pi^v_+ = \{ x \in \mathbb{R}^n : \langle x, v \rangle > 0 \} \). \( \partial^* Q^v \)
\( \partial_{Q^v} \) denote the side of \( \partial Q^v \) orthogonal to \( v \) and included in \( \Pi^v_+ \), respectively, while \( \partial_{Q^v} = \partial Q^v \setminus (\partial_{Q^v} \cup \partial_{Q^v}^v) \) is the lateral part of the boundary \( i.e. \) the union of the sides of \( Q^v \) that are parallel to \( v \).

2.1 – Preliminary results.

In this section we recall some results that we will use in the sequel.

**Theorem 2.1.** – Let \( \varphi : S^{n-1} \to [0, +\infty) \) be a bounded Borel function and
\[ \mathcal{F}(E) = \int_{\partial E \cap \Omega} \varphi(v_E) \, d\mathcal{H}^{n-1} \]
for every \( E \in \mathcal{P}(\Omega) \). Then the functional \( \mathcal{F} \) is lower semicontinuous, in the sense that for every sequence \( \{E_h\} \in \mathcal{P}(\Omega) \) and \( E \in \mathcal{P}(\Omega) \)
\[ \lim_{h \to +\infty} |(E_h \triangle E) \cap \Omega| = 0 \implies \mathcal{F}(E) \leq \liminf_{h \to +\infty} \mathcal{F}(E_h), \]
if and only if the positively one-homogeneous extension of \( \varphi \) from \( S^{n-1} \) to \( \mathbb{R}^n \) is convex.

The proof of the necessity of Theorem 2.1 is due to Ambrosio-Braides [3] while for the sufficiency we recall the Reshetnyak’s theorem (see e.g. [10]).

For simplicity in the following we will say that a real valued function defined on \( S^{n-1} \) is convex if its positively one-homogeneous extension from \( S^{n-1} \) to \( \mathbb{R}^n \)
\[ p \mapsto \varphi \left( \frac{p}{|p|} \right) |p| \]
is convex.

**Definition 2.2.** – Let \( A \) be an open set with bounded Lipschitz boundary and let \( F \) and \( G \) be sets with finite perimeter in \( A \). Let \( \omega \subseteq \partial A \), we say that
\[ G = F \quad \text{on} \quad \omega \]
if and only if the trace (in the usual sense of BV functions) of \( \chi_F \) and \( \chi_G \) coincide for \( \mathcal{H}^{n-1} \)-almost every \( x \in \omega \).

**Remark 2.3.** – By Theorem 2.1 and by a simple rescaling argument, for every convex function \( \varphi : S^{n-1} \to [0, \infty) \) we have that

\[
T^{n-1} \varphi(v) \leq \int_{\partial^v E \cap TQ^v} \varphi(v_E) \, d\mathcal{H}^{n-1},
\]

for every \( T > 0 \) and every \( E \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n) \) such that \( E = \Pi^v_+ \) on \( T\partial Q^v \).

Similarly, for every convex function \( \varphi : S^{n-1} \to [0, \infty) \) we have also that

\[
T^{n-1} \varphi(v) \leq \int_{\partial^v E \cap TQ^v} \varphi(v_E) \, d\mathcal{H}^{n-1},
\]

for every \( E \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n) \) such that \( E + T \eta_i = E, \ i = 1, \ldots, n - 1, \) and \( E = \Pi^v_+ \) on \( T\partial \pm Q^v \).

**Theorem 2.4** (Homogenization of perimeters [14]). – Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with Lipschitz boundary and let \( f : \mathbb{R}^n \to [a, \beta], \) with \( 0 < a < \beta < +\infty, \) be a 1-periodic Borel function. Then, there exists the limit

\[
\Gamma\text{-lim}_{\varepsilon \to 0} \int_{\partial^v E \cap \Omega} f \left( \frac{x}{\varepsilon} \right) \, d\mathcal{H}^{n-1} = \int_{\partial^v \mathcal{E} \cap \Omega} f_{\text{hom}}(v_E) \, d\mathcal{H}^{n-1},
\]

for every \( E \in \mathcal{P}(\Omega). \) Moreover, there exists the limit

\[
\lim_{\varepsilon \to 0} \inf \left\{ \int_{\partial^v F \cap Q^v} f \left( \frac{x}{\varepsilon} \right) \, d\mathcal{H}^{n-1} : F \in \mathcal{P}(Q^v), \ F = \Pi^v_+ \text{ on } \partial Q^v \right\}
\]

for every \( v \in S^{n-1}, \) the function \( f_{\text{hom}} \) is convex and satisfies the asymptotic formula

\[
f_{\text{hom}}(v) = \lim_{\varepsilon \to 0} \inf \left\{ \int_{\partial^v F \cap Q^v} f \left( \frac{x}{\varepsilon} \right) \, d\mathcal{H}^{n-1} : F \in \mathcal{P}(Q^v), \ F = \Pi^v_+ \text{ on } \partial Q^v \right\},
\]

for every \( v \in S^{n-1}. \)

(See also [4]).

**Proposition 2.5** (Periodic boundary conditions). – Let \( f \) be as in Theorem 2.4. Then

\[
\]
\begin{equation}
(2.3) \quad f_{\text{hom}}(v) = \lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial F \cap Q^v} f(x) \, dH^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_+^v \text{ on } T\partial_\pm Q^v, \ F + T\eta_i = F \quad i = 1, \ldots, n - 1 \right\}
\end{equation}

for every \( v \in S^{n-1} \) where \( (\eta_1, \ldots, \eta_{n-1}) \) are linearly independent vectors orthogonal to the faces of \( Q^v \) other than \( v \).

**Proof.** - Let us define

\begin{equation}
(2.4) \quad g_T^p(v) = \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial_+ F \cap Q^v} f(x) \, dH^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_+^v \text{ on } T\partial_\pm Q^v, \ F + T\eta_i = F \quad i = 1, \ldots, n - 1 \right\}
\end{equation}

\begin{align*}
&= \inf \left\{ \int_{\partial_+ F \cap Q^v} f(Tx) \, dH^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), \frac{1}{T}F = \Pi_+^v \text{ on } \partial_\pm Q^v, \right. \\
&\quad \left. \frac{1}{T}F + \eta_i = \frac{1}{T}F \quad i = 1, \ldots, n - 1 \right\}
\end{align*}

and

\( g_T(v) = \inf \left\{ \int_{\partial E \cap Q^v} f(Tx) \, dH^{n-1} : E \in \mathcal{P}(Q^v), \quad E = \Pi_+^v \text{ on } \partial Q^v \right\} \)

for every \( T \in \mathbb{N} \). Note that the limit of (2.4), as \( T \to +\infty \), exists since it is an infimum on \( \mathbb{N} \).

Let \( F_T \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n) \) be such that \( F_T + T\eta_i = F_T, i = 1, \ldots, n - 1, F_T = \Pi_+^v \text{ on } T\partial_\pm Q^v \) and

\begin{equation}
(2.5) \quad \int_{\partial_+ F_T \cap Q^v} f(Tx) \, dH^{n-1} \leq g_T^p(v) + o(1)
\end{equation}

as \( T \to +\infty \). Hence, if we denote by \( E_T := (1/T)F_T \) then the sequence \( \{E_T\} \) converges to \( \Pi_+^v \) as \( T \) tends to \( \infty \). Reasoning as in [14] Lemma 3.2 we may construct a new sequence \( \{\tilde{E}_T\} \) still converging to \( \Pi_+^v \) such that \( \tilde{E}_T = \Pi_+^v \text{ in a neighborhood of } \partial Q^v \) and

\begin{equation}
(2.6) \quad \lim_{T \to +\infty} \mathcal{H}^{n-1}(\partial^* \tilde{E}_T \triangle \partial^* E_T \cap Q^v) = 0.
\end{equation}
In fact, let us define
\[
E_T = \begin{cases} E_T & \text{on } Q^c \setminus Q^c_{\delta} \\
\Pi^c_{\delta} & \text{on } Q^c_{\delta} \end{cases}
\]
where \( Q^c_{\delta} = \{ x \in Q^c : d(x) := \text{dist}(x, \mathbb{R}^n \setminus Q^c) < \delta \} \), for all \( \delta > 0 \). Note that \( E_T \) is a test set for \( g_T \). Moreover,
\[
(\partial^* E_T \triangle \partial^* E_T) \cap Q^c = ((\partial^* E_T \triangle \partial^* E_T) \cap Q^c \setminus Q^c_{\delta}) \cup ((\partial^* E_T \triangle \partial^* E_T) \cap Q^c_{\delta})
\]
\[
= (E_T \triangle \Pi^c_{\delta}) \cap \{ d(x) = \delta \} \cup ((\partial^* E_T \triangle \Pi^c) \cap Q^c_{\delta}).
\]
Since \( E_T \to \Pi^c_{\delta} \), by Coarea formula we have that
\[
0 = \lim_{j \to +\infty} |Q^c_{\delta} \cap (E_T \triangle \Pi^c_{\delta})|
\]
\[
= \lim_{j \to +\infty} \int_{Q^c_{\gamma} \cap (E_T \triangle \Pi^c_{\delta})} |\nabla d| \, dx
\]
\[
= \lim_{j \to +\infty} \int_{0}^{\delta} \mathcal{H}^{n-1}(\{d(x) = t\} \cap (E_T \triangle \Pi^c_{\delta})) \, dt.
\]
By a suitable choice of \( \delta = \delta_T \to 0 \) there exists \( t_T \in (0, \delta_T) \) such that
\[
\lim_{t \to +\infty} \mathcal{H}^{n-1}(\{d(x) = t_T\} \cap (E_T \triangle \Pi^c_{\delta})) = 0.
\]
Hence, by replacing \( \delta \) with \( t_T \) in (2.7) and (2.8) we get (2.6).

Now we can compare \( g^p_T \) with \( g_T \). In fact,
\[
\int_{\partial^* E_T \cap Q^c} f(Tx) \, d\mathcal{H}^{n-1} = \int_{\partial^* E_T \cap Q^c} f(Tx) \, d\mathcal{H}^{n-1} - \int_{(\partial^* E_T \triangle \partial^* E_T) \cap Q^c} f(Tx) \, d\mathcal{H}^{n-1}
\]
\[
+ \int_{(\partial^* E_T \triangle \partial^* E_T) \cap Q^c} f(Tx) \, d\mathcal{H}^{n-1}.
\]
By (2.2), (2.5) and (2.6) we have that \( f_{\text{hom}}(v) \leq \lim_{T \to +\infty} g^p_T \). The other inequality easily follows by definition of \( g_T \) and \( g^p_T \). \( \square \)

3. – Approximation of \( \varphi \).

In this section we prove that given a target \( \varphi \) we can construct a suitable function \( a \) such that
\[
\Gamma^* \lim_{\varepsilon \to 0} \int_{\partial^* E \cap \Omega} a(\frac{x}{\varepsilon}) \, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} \varphi(y_E) \, d\mathcal{H}^{n-1}
\]
for every \( E \in \mathcal{P}(\Omega) \).
THEOREM 3.1. – Let $\varphi : S^{n-1} \to [a, \beta]$, with $0 < a < \beta$, be a Borel function such that the positively one-homogeneous extension of $\varphi$ to $\mathbb{R}^n$

$$p \mapsto \varphi \left( \frac{p}{|p|} \right) |p|$$

is convex and even. Let $\{v_j\}$ be a dense sequence in $S^{n-1}$ such that $v_h \neq \pm v_j$ for $h \neq j$ and let $A_j = \mathbb{Z}^n + \Sigma_j$ where $\Sigma_j$ is the hyperplane through the origin and orthogonal to $v_j$, for every $j \in \mathbb{N}$. Let $a : \mathbb{R}^n \to [a, \beta]$ be a Borel function defined by

$$a(x) = \begin{cases} 
\varphi(v_j) & \text{if } x \in A_j \setminus \left( \bigcup_{h \in \mathbb{N} \setminus \{j\}} A_h \right), \ j \in \mathbb{N} \\
\alpha & \text{if } x \in \bigcup_{h \in \mathbb{N} \setminus \{j\}} (A_j \cap A_h), \ j \in \mathbb{N} \\
\beta & \text{otherwise in } \mathbb{R}^n.
\end{cases} \tag{3.1}$$

Then,

$$I^- \lim_{\varepsilon \to 0} \int_{\partial^* E \cap \Omega} a \left( \frac{x}{\varepsilon} \right) d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} \varphi(v_E) d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$.

PROOF. – We recall that by Theorem 2.4 we have that

$$I^- \lim_{\varepsilon \to 0} \int_{\partial^* E \cap \Omega} a \left( \frac{x}{\varepsilon} \right) d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} a_{\text{hom}}(v_E) d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$, where $a_{\text{hom}}$ is convex and

$$a_{\text{hom}}(v) = \lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial^* F \cap T\mathcal{Q}^y} a(x) d\mathcal{H}^{n-1} : F \in \mathcal{P}(T\mathcal{Q}^y), F = \Pi^+_x \text{ on } T\partial\mathcal{Q}^y \right\}.$$ 

Hence, it remain to prove then that $\varphi = a_{\text{hom}}$.

We first deal with the inequality: $a_{\text{hom}}(v) \geq \varphi(v)$. Let $F \in \mathcal{P}(T\mathcal{Q}^y)$ such that $F = \Pi^+_x$ on $T\partial\mathcal{Q}^y$. The essential boundary $\partial^* F$ may intersect $A_j$ in a set of positive $\mathcal{H}^{n-1}$ measure, which means that a relevant part of $\partial^* F$ coincides with a part of $A_j$; hence,

$$v_j = \pm v_F \quad \mathcal{H}^{n-1} \text{- a.e. in } \partial^* F \cap A_j.$$
Since \( \varphi(v) = \varphi(-v) \in [a, b] \), by (3.1) we get then
\[
\int_{\partial F \cap T Q'} a(x) \, d\mathcal{H}^{n-1}
= \sum_{j \in \mathbb{N}} \int_{\partial F \cap \mathcal{A}_j \cap T Q'} \varphi(v_j) \, d\mathcal{H}^{n-1} + \int_{(\partial F \cup \mathcal{A}) \cap T Q'} \beta \, d\mathcal{H}^{n-1}
\geq \sum_{j \in \mathbb{N}} \int_{\partial F \cap \mathcal{A}_j \cap T Q'} \varphi(v_F) \, d\mathcal{H}^{n-1} + \int_{(\partial F \cup \mathcal{A}) \cap T Q'} \varphi(v_F) \, d\mathcal{H}^{n-1}
= \int_{\partial F \cap T Q'} \varphi(v_F) \, d\mathcal{H}^{n-1}.
\]
(3.2)

By (3.2) and Remark 2.3, we can conclude that
\[
\int_{\partial F \cap T Q'} a(x) \, d\mathcal{H}^{n-1} \geq T^{n-1} \varphi(v)
\]
and by definition of \( a_{\text{hom}} \)
\[
a_{\text{hom}}(v) \geq \varphi(v) .
\]
(3.3)

By (3.1), we have that
\[
\int_{\sum_j \cap T Q'} a(x) \, d\mathcal{H}^{n-1} = T^{n-1} \varphi(v_j);
\]
hence, \( a_{\text{hom}}(v_j) = \varphi(v_j) \) for every \( j \in \mathbb{N} \). To conclude the proof of the theorem it remains to show that \( a_{\text{hom}}(v) = \lim_{j \to \infty} a_{\text{hom}}(v_j) \), and this is an easy consequence of the convexity of \( a_{\text{hom}} \).

\[\square\]

The following proposition allows us to describe \( \varphi \) also by a homogenization formula with periodic boundary conditions.

**Proposition 3.2.**
\[
\varphi(v) = \lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial F \cap T Q'} a(x) \, d\mathcal{H}^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_{+}^{v} \right. \\
\left. \text{on } T \partial Q', F + T \eta_i = F \quad i = 1, \ldots, n - 1 \right\}
\]
(3.4) for every \( v \in S^{n-1} \) where \( \eta_1, \ldots, \eta_{n-1} \) are linearly independent vectors orthogonal to the faces of \( Q' \) other than \( v \).

**Proof.** – By Theorem 3.1 we have that \( a_{\text{hom}}(v) = \varphi(v) \); hence, by Proposition 2.5 we get (3.4).

\[\square\]
4. – Approximation scheme for \( \varphi \) by regularization.

In this section we suggest another way to approximate \( \varphi \) by regularizing \( a \).

**Theorem 4.1.** – Let \( \varphi : S^{n-1} \to [a, \beta] \), with \( 0 < a < \beta \), be a Borel function such that the positively homogeneous of degree one extension of \( \varphi \) to \( \mathbb{R}^n \)

\[ p \mapsto \varphi\left( \frac{p}{|p|} \right) |p| \]

is convex and even. Then there exists a family of functions \( a_{k,\lambda} : \mathbb{R}^n \to [a, \beta] \), \( 1 \)-periodic and \( \lambda \)-Lipschitz, such that

\[ \Gamma_{\mathcal{E}} \lim_{\varepsilon \to 0} \int_{\partial^* E \cap \Omega} a_{k,\lambda} \left( \frac{x}{\varepsilon} \right) d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} \varphi_{k,\lambda}(v_E) d\mathcal{H}^{n-1} \]

for every \( \lambda \in \mathbb{R}^+ \), \( k \in \mathbb{N} \) and \( E \in \mathcal{P}(\Omega) \). Moreover,

\[ \lim_{k \to +\infty} \lim_{\lambda \to +\infty} \varphi_{k,\lambda}(v) = \varphi(v) \]

for every \( v \in S^{n-1} \) and

\[ \Gamma_{\mathcal{E}} \lim_{k \to +\infty} \left( \Gamma_{\mathcal{E}} \lim_{\lambda \to +\infty} \int_{\partial^* E \cap \Omega} \varphi_{k,\lambda}(v_E) d\mathcal{H}^{n-1} \right) = \int_{\partial^* E \cap \Omega} \varphi(v_E) d\mathcal{H}^{n-1} \]

for every \( E \in \mathcal{P}(\Omega) \).

**Proof.** – Let \( \Xi \) be the set of unit rational directions i.e. \( \Xi = \left\{ v = \frac{x}{|x|} \in S^{n-1} : x \in \mathbb{Z}^n \setminus \{0\} \right\} \). Since \( \Xi \) is dense and countable in \( S^{n-1} \) we consider a dense sequence \( \{v_j\}_{j \in \mathbb{N}} \in \Xi \) such that \( v_h \neq \pm v_j \) for \( h \neq j \). We define \( A_j = \mathbb{Z}^n + \Sigma_j \) where \( \Sigma_j \) is the hyperplane through the origin and orthogonal to \( v_j \), for every \( j \in \mathbb{N} \). The set \( A_j \) is closed and \( 1 \)-periodic with respect to the canonical basis \( (e_1, \ldots, e_n) \) of \( \mathbb{R}^n \). We fix \( k \in \mathbb{N} \) and we consider the first \( k \) directions \( (v_1, \ldots, v_k) \subset \{v_j\}_{j \in \mathbb{N}} \). We define

\[
\begin{align*}
    a_{k}(x) = \begin{cases} 
        \varphi(v_j) & \text{if } x \in A_j \setminus \left( \bigcup_{h \neq j} A_h \right), \quad j = 1, \ldots, k \\
        a & \text{if } x \in \bigcup_{h \neq j} (A_j \cap A_h), \quad j = 1, \ldots, k \\
        \beta & \text{otherwise in } \mathbb{R}^n
    \end{cases}
\end{align*}
\]

(4.1)

and we denote by \( a_{k,\lambda} \) the Yosida transform of \( a_k \) i.e.,

\[ a_{k,\lambda}(x) = \inf_{y \in \mathbb{R}^n} \{ a_k(y) + \lambda |x - y| \}, \quad \lambda \in \mathbb{R}^+ \.]
Hence, \(a_{k, \lambda}\) is \(\lambda\)-Lipschitz. Moreover, since \(a_k\) is lower semicontinuous and 1-periodic, we have that \(a_{k, \lambda}\) is also 1-periodic and the sequence \(\{a_{k, \lambda}\}_\lambda\) converges increasingly to \(a_k\) as \(\lambda \to +\infty\), i.e.

\[
a_k(x) = \sup_{\lambda \geq 0} a_{k, \lambda}(x)
\]

(see e.g. [13] Remark 1.6 and Proposition 1.7). For any fixed \(k \in \mathbb{N}\) and \(\lambda \in \mathbb{R}^+\) the function \(a_{k, \lambda}\) is continuous, bounded and 1-periodic. By Theorem 2.4 we have that

\[
\Gamma^\varepsilon \lim_{\varepsilon \to 0} \int_{\partial E \cap \Omega} a_{k, \lambda}(\frac{x}{\varepsilon}) \, d\mathcal{H}^{n-1} = \int_{\partial E \cap \Omega} \varphi_{k, \lambda}(v_E) \, d\mathcal{H}^{n-1}
\]

and

\[
\Gamma^\varepsilon \lim_{\varepsilon \to 0} \int_{\partial E \cap \Omega} a_k(\frac{x}{\varepsilon}) \, d\mathcal{H}^{n-1} = \int_{\partial E \cap \Omega} \varphi_k(v_E) \, d\mathcal{H}^{n-1}
\]

for every \(E \in \mathcal{P}(\Omega)\) where \(\varphi_{k, \lambda}\) and \(\varphi_k\) are convex functions. By Proposition 2.5, \(\varphi_{k, \lambda}\) and \(\varphi_k\) can be also described by the following formulas

\[
\varphi_{k, \lambda}(v) = \lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial F \cap \Omega} a_{k, \lambda}(x) \, d\mathcal{H}^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_v^\perp \text{ on } T\partial_\pm Q^v, F + T\eta_i = F \quad i = 1, \ldots , n - 1 \right\}
\]

and

\[
\varphi_k(v) = \lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial F \cap \Omega} a_k(x) \, d\mathcal{H}^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_v^\perp \text{ on } T\partial_\pm Q^v, F + T\eta_i = F \quad i = 1, \ldots , n - 1 \right\},
\]

for every \(v \in S^{n-1}\). Our aim is to study the pointwise convergence of \(\{\varphi_{k, \lambda}\}_{k, \lambda}\) letting first \(\lambda\) and then \(k\) go to \(+\infty\). In the following we first prove that \(\{\varphi_{k, \lambda}\}_\lambda\) pointwise converges to \(\varphi_k\) as \(\lambda\) tends to \(+\infty\); then, we show that \(\{\varphi_k\}_k\) pointwise converges to \(\varphi\) as \(k\) tends to \(+\infty\). Therefore to conclude the proof of the theorem it remains to observe that the pointwise convergence of the convex integrands \(\{\varphi_{k, \lambda}\}_\lambda\) and \(\{\varphi_k\}_k\) implies the \(\Gamma\)-convergence of the corresponding families of functionals. In fact, the pointwise convergence of convex functions implies the uniform convergence on \(S^{n-1}\). Hence, we have that

\[
\Gamma^\lambda \lim_{\lambda \to +\infty} \int_{\partial E \cap \Omega} \varphi_{k, \lambda}(v_E) \, d\mathcal{H}^{n-1} = \int_{\partial E \cap \Omega} \varphi_k(v_E) \, d\mathcal{H}^{n-1}
\]
and

$$\Gamma - \lim_{k \to +\infty} \int_{\partial E \cap \Omega} \varphi_k(v_E) \, d\mathcal{H}^{n-1} = \int_{\partial E \cap \Omega} \varphi(v_E) \, d\mathcal{H}^{n-1}$$

which concludes the proof of the theorem.

Let us deal with the pointwise convergence of $\{\varphi_{k,\lambda}\}$ to $\varphi_k$. By (4.2) we have that $\varphi_k(v) \geq \varphi_{k,\lambda}(v)$. Hence, if we prove that

$$\liminf_{\lambda \to +\infty} \varphi_{k,\lambda}(v) \geq \varphi_k(v)$$

then we can conclude that there exists the limit, as $\lambda$ tends to $+\infty$, and

$$\lim_{\lambda \to +\infty} \varphi_{k,\lambda}(v) = \varphi_k(v)$$

for every $v \in S^{n-1}$. To obtain (4.5) we need to introduce some auxiliary functions $\tilde{a}_{k,\lambda}$.

**Definition of the auxiliary functions.** – Let $k \geq n$. We define

$$S(k) = \{ s = (s_1, \ldots, s_k) : \quad s_j = 0 \text{ or } j = 1, \ldots, k \quad \text{and} \quad \text{at least two of } (s_1, \ldots, s_k) \text{ are different from 0} \}.$$ 

For every fixed $s \in S(k)$ we define then

$$H^d_s = \bigcap_{j=1,\ldots,k} A_{s_j}$$

where $d = 0, \ldots, n - 2$ denotes the dimension of $H^d_s$. Note that for any fixed $d$ the sets $H^d_s$, $s \in S(k)$, may be not disjoints. Moreover, the intersection between $\{H^d_s\}_{d,k}$ and $TQ^r$ gives rise to a finite number of sets.

Around any $H^d_s$ and $A_j$ we construct suitable neighborhoods and we define the following sets

$$U = \bigcup_{s \in S(k)} \big\{ x \in \mathbb{R}^n : \text{dist} (x, H^d_s) < \lambda^{-1/n-d} \big\}$$

and

$$U_j = \big\{ x \in \mathbb{R}^n : \text{dist} (x, A_j) < \lambda^{-\gamma} \big\} \setminus U$$

with $1/2 < \gamma < 1$ and $j = 1, \ldots, k$. Since the sets $A_j$, $j = 1, \ldots, k$ are closed and pairwise disjoint, the sets $U_j$ are pairwise disjoint for $\lambda$ large enough. Note that $\lambda^{-\gamma} < \lambda^{-1/2} < \cdots < \lambda^{-1/n-d} < \cdots < \lambda^{-1/n}$. Finally, we define for $\lambda$ large
enough

\[ (4.7) \quad \tilde{a}_{k,\lambda}(x) = \begin{cases} 
\varphi(v_j) & \text{if } x \in U_j, \quad j = 1, \ldots, k \\
\alpha & \text{if } x \in U \\
\beta & \text{otherwise in } \mathbb{R}^n; 
\end{cases} \]

where \( U_j \) and \( U \) are defined as above.

The choice of the radii in the definition of \( U \) and \( U_j \) allows to compare easily \( a_{k,\lambda} \) and \( \tilde{a}_{k,\lambda} \) and prove that

\[ (4.8) \quad a_{k,\lambda}(x) \geq \tilde{a}_{k,\lambda}(x) \]

for every \( x \in \mathbb{R}^n \) and for \( \lambda \) big enough. In fact, if \( x \in U \) the inequality is trivial since \( \tilde{a}_{k,\lambda}(x) = \alpha \) while, by definition, \( a_{k,\lambda}(y) \geq \alpha \) for every \( y \in \mathbb{R}^n \). If \( x \in U_j \) then \( \tilde{a}_{k,\lambda}(x) = \varphi(v_j) \) and there exists \( H^d_s \) such that

\[ a_{k,\lambda}(x) = \min\{\alpha + \lambda|x - P^d_s x|, \varphi(v_j) + \lambda|x - P_j x|, \beta\} \]

where \( P^d_s \) denotes the orthogonal projection on \( H^d_s \) and \( P_j \) is the orthogonal projection on \( A_j \). The sets \( A_j \) are not convex. The orthogonal projection makes no sense globally. By the way, in most formulas only \( |x - P_j x| \) is used. This is the distance from \( A_j \), which is always well defined. In order to prove the inequality (4.8) it is sufficient to exclude that \( a + \lambda|x - P^d_s x| \) can be the minimum, for example, showing that

\[ (4.9) \quad a + \lambda|x - P^d_s x| > \beta. \]

In fact,

\[ a + \lambda|x - P^d_s x| \geq a + \lambda^{(1-1/n-d)} > \beta \]

for \( \lambda \) big enough. Similarly, if \( x \in \mathbb{R}^n \setminus \left( U \cup \bigcup_{j=1}^k U_j \right) \), then \( \tilde{a}_{k,\lambda}(x) = \beta \) and \( a_{k,\lambda}(x) \) is essentially reduced to be

\[ a_{k,\lambda}(x) = \min\{\alpha + \lambda|x - P^d_s x|, \varphi(v_h) + \lambda|x - P_h x|, \beta\} \]

for a suitable choice of \( A_k \) and \( H^d_s \). But also in this case (4.9) is satisfied and

\[ \varphi(v_h) + \lambda|x - P_h x| \geq \varphi(v_h) + \lambda^{1-\gamma} > \beta \]

for \( \lambda \) big enough, which implies (4.8).

In order to prove (4.5) we now have to compare \( \tilde{a}_{k,\lambda} \) with \( a_k \). To this end in the following Steps we introduce the functions \( \varphi^d_s : \mathbb{R}^n \rightarrow \mathbb{R}^n \), for \( d = 0, \ldots, n-1 \), such that if we compose them then we get a function \( \varphi = \varphi^{n-1} \circ \cdots \circ \varphi^0 \) that “projects” the sets \( U, U_j \) into \( H^d_s, A_j \), respectively, for every \( j = 1, \ldots, k \), \( d = 0, \ldots, n-1 \) and \( s \in S(k) \).
Step 0 ($d = 0$). – For every $s \in S(k)$ the set $H^s$ is a sequence of points $(x^s_p)_p$.
We denote by $B^s_p(r) = B(x^s_p, r)$ and we define $\phi^0_\lambda$ as follows:
(a) $\phi^0_\lambda$ projects $B^s_p(\lambda^{1/n})$ into the center $x^s_p$, i.e.,
$$\phi^0_\lambda(x) = x^s_p, \quad x \in B^s_p(\lambda^{1/n}), \quad \forall \ p \text{ and } s \in S(k);$$
(b) $\phi^0_\lambda$ dilates $B^s_p(\lambda^{1/n+1}) \setminus B^s_p(\lambda^{1/n})$ into $B^s_p(\lambda^{1/n+1})$ keeping fixed $\partial B^s_p(\lambda^{1/n+1})$; i.e.,
$$\phi^0_\lambda(x) = \frac{1}{1 - \lambda^{1/(n(n+1))}} \left( |x - x^s_p| - \lambda^{1/n} \right) + \frac{x - x^s_p}{|x - x^s_p|} + x^s_p$$
for every $x \in B^s_p(\lambda^{1/n+1}) \setminus B^s_p(\lambda^{1/n})$, $p$ and $s \in S(k)$;
(c) $\phi^0_\lambda$ is the identity on $\mathbb{R}^n \setminus \bigcup_{s \in S(k), p} B^s_p(\lambda^{1/n+1})$; i.e.,
$$\phi^0_\lambda(x) = x, \quad x \notin \bigcup_{s \in S(k), p} B^s_p(\lambda^{1/n+1}).$$
Note that $\phi^0_\lambda$ is a Lipschitz function with constant
$$\text{Lip}(\phi^0_\lambda) = \frac{1}{1 - \lambda^{1/(n(n+1))}}.$$
In dimension \( d \geq 1 \), \( \varphi^d_\lambda \) is the natural generalization of \( \varphi^0_\lambda \) but we have also to take into account that \( \varphi^d_\lambda \) is applied to sets already modified by \( \varphi^{d-1}_\lambda \circ \cdots \circ \varphi^0_\lambda \).

**Step \( d \) (\( d = 1, \ldots, n - 2 \)).** We denote by

\[
I^d_s = \varphi^{d-1}_\lambda \circ \cdots \circ \varphi^0_\lambda \left( \{ x \in \mathbb{R}^n : \text{dist}(x, H^d_s) < \lambda^{-1/(n-d)} \} \right)
\]

and

\[
N^d_s = \left\{ x \in \mathbb{R}^n : \text{dist}(x, H^d_s) < \lambda^{-1/(n-d+1)} \right\}.
\]

Let \( \tilde{H}^d_s \) be the orthogonal projection on \( \partial I^d_s \); \( \tilde{H}^d_s \) is defined only locally. We recall that \( H^d_s \) is the orthogonal projection on \( H^d_s \). Then we define \( \varphi^d_\lambda \) as follows:

(a) \( \varphi^d_\lambda \) projects \( I^d_s \) into \( H^d_s \); i.e.,

\[
\varphi^d_\lambda(x) = \tilde{H}^d_s(x), \quad x \in I^d_s, \quad \forall s;
\]

(b) \( \varphi^d_\lambda \) dilates \( N^d_s \setminus I^d_s \) into \( N^d_s \) keeping fixed \( \partial N^d_s \); i.e.,

\[
\varphi^d_\lambda(x) = \frac{\lambda^{-1/(n-d+1)}}{\lambda^{-1/(n-d+1)} - |\tilde{H}^d_s(x) - \tilde{H}^d_s(x)|} \cdot \left( \frac{|x - \tilde{H}^d_s(x)| - |\tilde{H}^d_s(x) - \tilde{H}^d_s(x)|}{|x - \tilde{H}^d_s(x)|} + \frac{x - \tilde{H}^d_s(x)}{|x - \tilde{H}^d_s(x)|} + \tilde{H}^d_s(x) \right)
\]

for every \( x \in N^d_s \setminus I^d_s \) and \( s \in S(k) \);

(c) \( \varphi^d_\lambda \) is the identity on \( \mathbb{R}^n \setminus \bigcup_{s \in S(k)} N^d_s \); i.e.,

\[
\varphi^d_\lambda(x) = x, \quad x \notin \bigcup_{s \in S(k)} N^d_s.
\]

Note that

\[
I^d_s = \varphi^{d-1}_\lambda \circ \cdots \circ \varphi^0_\lambda \left( \{ x \in \mathbb{R}^n : \text{dist}(x, H^d_s) < \lambda^{-1/(n-d)} \} \right)
\]

then

\[
\lambda^{-1/(n-d+1)} - |\tilde{H}^d_s(x) - \tilde{H}^d_s(x)| \leq \lambda^{-1/(n-d+1)} - \lambda^{-1/(n-d)} = \frac{1}{1 - \lambda^{-1/(n-d)(n-d+1)}}.
\]

Therefore, \( \varphi^d_\lambda \) is a Lipschitz function with constant

\[
\text{Lip}(\varphi^d_\lambda) = \frac{1}{1 - \lambda^{-1/(n-d)(n-d+1)}}.
\]

Moreover, by definition, any function \( \varphi^d_\lambda \) maps \( H^d_s \) and \( A_j \) in themselves for every \( d = 0, \ldots, n - 2 \), \( s \in S(k) \) and \( j = 1, \ldots, k \).
Step $(n - 1)$. – Finally, we define $\phi^{n-1}_\lambda$ as the function that dilates

$$\mathbb{R}^n \setminus \left( \phi^{n-2}_\lambda \circ \cdots \circ \phi^0_\lambda \left( \bigcup_{j=1}^k \bigcup_{j=1}^k U_j \cup U \right) \right)$$

such that

$$\begin{cases}
\phi^{n-1}_\lambda \circ \cdots \circ \phi^0_\lambda \left( \bigcup_{j=1}^k U_j \cup U \right) = \bigcup_{j=1}^k A_j, \\
\phi^{n-1}_\lambda(A_j) = A_j,
\end{cases} \quad j = 1, \ldots, k$$

$$\text{Lip} \phi^{n-1}_\lambda = 1 + \left( c_k / \lambda^n \right).$$

We define then

$$\phi_\lambda = \phi^{n-1}_\lambda \circ \cdots \circ \phi^0_\lambda,$$

that is a Lipschitz function with constant

$$\text{Lip}(\phi_\lambda) = \left( 1 + \frac{c_k}{\lambda^n} \right) \prod_{d=0}^{n-2} \frac{1}{1 - \lambda^{-1/(n-d(n-d+1))}}.
$$

and

$$\lim_{\lambda \to +\infty} \text{Lip}(\phi_\lambda) = 1.$$

Now we use the construction of $\phi_\lambda$ and of the auxiliaries functions $\tilde{a}_{k, \lambda}$ to prove the inequality (4.5). Let $G \in \mathcal{P}_{loc}(\mathbb{R}^n)$ such that $G = \Pi^{x}_{\epsilon}$, on $T \partial \chi Q^\nu$, $G + T \eta_i = G$ for $i = 1, \cdots, (n - 1)$; by (4.7) we have that

$$\int_{\partial G \cap \pi Q^\nu} \tilde{a}_{k, \lambda}(x) \, d\mathcal{H}^{n-1}$$

$$= \sum_{j=1}^k \phi(v_j) \mathcal{H}^{n-1}(\partial^* G \cap \pi Q^\nu \cap U_j) + a \mathcal{H}^{n-1}(\partial^* G \cap \pi Q^\nu \cap U) + \beta \mathcal{H}^{n-1}(\partial^* G \cap V_j),$$

where $V_j = \pi Q^\nu \setminus (U_j \cup U)$. Note that by definition of $\phi_\lambda$ we have that $\mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap \pi Q^\nu \cap U)) = 0$. Hence, by (4.12), (4.10) and the property of the Hausdorff measure with respect to a Lipschitz function (see [5] Proposition 2.49 (iv)) we get that

$$\text{Lip}(\phi_\lambda)^{n-1} \int_{\partial G \cap \pi Q^\nu} a_{k, \lambda}(x) \, d\mathcal{H}^{n-1}$$

$$\geq \sum_{j=1}^k \phi(v_j) \mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap \pi Q^\nu \cap U_j)) + \beta \mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap V_j)) \geq \int_{\partial G \cap \pi Q^\nu} a_{k}(x) \, d\mathcal{H}^{n-1},$$
where $\tilde{G} \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n)$ such that

\begin{equation}
\partial^* G \cap TQ' \subseteq \phi_\lambda(\partial^* G \cap TQ').
\end{equation}

Note that, since $A_j$ is 1-periodic with respect to the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, we have that, in general, $A_j + T\eta_i \neq A_j$ for every $i = 1, \ldots, n - 1$ and $j = 1, \ldots, k$. Hence, when we apply $\phi_\lambda$, we may have that $\tilde{G} + T\eta_i \neq \tilde{G}$ for some $i = 1, \ldots, n - 1$. By definition, any change due to $\phi_\lambda$ remains in a neighborhood of $\partial^* G$ with radius smaller than $\lambda^{-1/n + 1}$ (see Steps $d = 0, \ldots, n - 1$). Hence, we may slightly modify $\tilde{G}$ close to $T\partial_\lambda Q'$ to match the periodic boundary conditions. We denote by $S \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n)$ this new set such that $S = \Pi_+^v$ on $T\partial_\lambda Q'$, $S + T\eta_i = S$, for every $i = 1, \ldots, n - 1$ and such that

\begin{equation}
|\mathcal{H}^{n-1}(\partial^* S \cap TQ') - \mathcal{H}^{n-1}(\partial^* G \cap TQ')| = O(\lambda^{-1/n + 1} T^{n-2}).
\end{equation}

By (4.13) and (4.15), since $a_k$ is bounded, we get then

\begin{equation}
\frac{\text{Lip}(\phi_\lambda)^{n-1}}{T^{n-1}} \int_{\partial^* G \cap TQ'} \tilde{a}_{k,\lambda}(x) d\mathcal{H}^{n-1}
\geq \frac{1}{T^{n-1}} \int_{\partial^* S \cap TQ'} a_k(x) d\mathcal{H}^{n-1} + O(\lambda^{-1/n + 1} T^{-1})
\geq \frac{1}{T^{n-1}} \inf \left\{ \int_{\partial^* F \cap TQ'} a_k(x) d\mathcal{H}^{n-1} : F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n), F = \Pi_+^v \text{ on } T\partial_\lambda Q',
\quad F + T\eta_i = F \quad i = 1, \ldots, n - 1 \right\}
+ O(\lambda^{-1/n + 1} T^{-1}).
\end{equation}

Hence, by (4.8), (4.16) and the definition of $\phi_{k,\lambda}$ and $\phi_k$ (see (4.3) and (4.4)) passing to the limit as $T$ tends to $+\infty$ we have that

$$
\phi_{k,\lambda}(v) \geq \frac{1}{\text{Lip}(\phi_\lambda)^{n-1}} \phi_k(v)
$$

for every $v \in S^{n-1}$. Finally, by (4.11) passing to the limit as $\lambda$ tends to $+\infty$ we get (4.5) which implies, as already observed, the pointwise convergence of $\{\phi_{k,\lambda}\}_{\lambda}$ to $\phi_k$.

It remains to study the pointwise convergence of $\{\phi_k\}$ to $\phi$ as $k$ tends to $+\infty$. By Remark 2.3 and (4.1) we have that

$$
\phi(v) \leq \frac{1}{T^{n-1}} \int_{\partial^* F \cap TQ'} \phi(v_f) d\mathcal{H}^{n-1} \leq \frac{1}{T^{n-1}} \int_{\partial^* F \cap TQ'} a_k(x) d\mathcal{H}^{n-1},
$$
for every $F \in \mathcal{P}_{\text{loc}}(\mathbb{R}^n)$ such that $F + T\eta_i = F$, $i = 1, \ldots, n - 1$, and $F = \Pi_x^r$ on $T\partial_x Q^r$. By definition of $\varphi_k$ we have then

\begin{equation}
\varphi(v) \leq \varphi_k(v)
\end{equation}

while

\begin{equation}
\varphi(v_j) = \varphi_k(v_j), \quad j = 1, \ldots, k.
\end{equation}

We define

$$
\psi_k(z) = \begin{cases} 
\varphi(z) & \text{if } z \in \bigcup_{j=1}^k \mathbb{R} v_j \\
\beta |z| & \text{otherwise,}
\end{cases}
$$

then, $\varphi(v) \leq \psi_k(v)$. Let $\text{co} \psi_k$ be the convex envelope of $\psi_k$. Since $\varphi$ is convex we have $\varphi(v) \leq \text{co} \psi_k(v)$. Moreover, $\varphi(v_j) \leq \text{co} \psi_k(v_j) \leq \psi_k(v_j) = \varphi(v_j)$; hence,

\begin{equation}
\varphi(v_j) = \text{co} \psi_k(v_j)
\end{equation}

for $j = 1, \ldots, k$. The functions $\text{co} \psi_k$ are equi-lipschitz on compact sets of $\mathbb{R}^n$ (see Section 5.1, Chapter 5 in [13]); hence, by (4.19) and the density of $\{v_j\}$ we get that

\begin{equation}
\lim_{k \to +\infty} \text{co} \psi_k(v) = \varphi(v)
\end{equation}

for every $v \in S^{n-1}$. By (4.18) and the definition of $\psi_k$ we have then

\begin{equation}
\varphi_k(v) \leq \psi_k(v).
\end{equation}

We recall that by Theorem 2.4 the functions $\varphi_k$ are convex for every $k \in \mathbb{N}$. By (4.21) it follows then

\begin{equation}
\varphi_k(v) \leq \text{co} \psi_k(v)
\end{equation}

for every $v \in S^{n-1}$.

By (4.17), (4.22) and (4.20) we can prove that there exists the limit as $k$ tends to $+ \infty$ and

$$
\lim_{k \to +\infty} \varphi_k(v) = \varphi(v)
$$

for every $v \in S^{n-1}$ which concludes the proof. \hfill \Box

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