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Abstract. – *We present a q-analogue of the Rhin-Viola method for the analysis of Φ-adic valuations of the q-gamma factors occurring in the basic Euler-Pochhammer integral representation of the Heine series ${}_2\phi_1$. Moreover, we show that this approach yields the best known irrationality measures for* $\log_q(z)$, $\log_q 2$ *and* $\zeta_q(1)$.

1. – Introduction.

In the diophantine approximation of the values of ordinary logarithm at rational points, good irrationality measures derive from the analysis of p-adic valuation of the gamma factors occurring in the Euler-Pochhammer integral representation

$$(1)\qquad {}_2F_1(a,b;c;y)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{x^{b-1}(1-x)^{c-b-1}}{(1-xy)^a}dx,\quad \Re(c)>\Re(b)>0$$

of the Gauss hypergeometric function. The scope of this paper is to present a q-analogue of this approach to the study of $\log_q(z)$, a q-analogue of the logarithmic function defined by

$$\log_q(z)=\sum_{n=1}^{\infty}\frac{zq^n}{1-zq^n},\quad |z|\le 1,\quad q=1/p,\quad p\in\mathbb{Z}\setminus\{0,\pm1\},$$

and to point out that good approximations come from the application of the q-analogue of Euler-Pochhammer integral representation for the Heine series ${}_2\phi_1$ (a q-analogue of ${}_2F_1$) and from the study of the related q-gamma factors.

In the recent articles [4], [8] and [9] sharp irrationality measures were obtained through Φ-adic analysis of suitable q-binomial coefficients. Indeed, the method corresponds to the well-known arithmetic approach introduced by Hata in [3], based on the properties of Legendre type polynomials. In [7] Viola proposed a technique, making use of the equation (1), and showed that it gave the best-known irrationality measures for a set of values of the logarithm at rational points. In the following we present the q-analogue of [7] and show that this also yields the best-known irrationality measures for $\log_q(z)$, $\log_q 2$ and $\zeta_q(1)$.

We recall that μ is an irrationality measure of the irrational number $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a constant $q_0 = q_0(\varepsilon) > 0$ such that

$$\left|\xi - \frac{p}{q}\right| > q^{-\mu-\varepsilon}$$

for all integers p and $q > q_0$. If q_0 can be effectively computed, then we say that μ is an effective irrationality measure. Furthermore, we denote by $\mu(\xi)$ the least irrationality measure of ξ.

THEOREM 1. – *Let* $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, *then for* $z \in \mathbb{Q}$, $0 < |z| \leq 1$

$$\mu(\log_q(z)) \leq 3{,}7633\ldots. \tag{2}$$

In the particular cases corresponding to $z = -1$ *and* $z = 1$, *respectively, we get the sharper estimates*

$$\mu(\log_q 2) \leq 2{,}9383\ldots, \quad \mu(\zeta_q(1)) \leq 2{,}4649\ldots, \tag{3}$$

where we denote

$$\log_q 2 = \sum_{n=1}^{\infty} \frac{-q^n}{1+q^n} = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1-q^n} \quad \textit{and} \quad \zeta_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

2. – Preliminaries.

We shall only briefly recapitulate some notations and lemmas used in the presentation of this paper. Firstly, the q-analogue of an ordinary Riemann integral on the interval $(0, a)$ is a "discrete integral"

$$\int_0^a f(x) d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad 0 < |q| < 1,$$

(see [1], p. 486) i.e. Riemann sum, when f is a Riemann-integrable function. By Φ_s we denote the cyclotomic polynomials

$$\Phi_s(q) = \prod_{k=1,(k,s)=1}^{s} (q - e^{2\pi i k/s}) \in \mathbb{Z}[q], \quad s = 1, 2, \ldots,$$

irreducible in the ring $\mathbb{Z}[q]$ with the property

$$q^n - 1 = \prod_{s|n} \Phi_s(q), \quad n \in \mathbb{Z}^+. \tag{4}$$

In addition, we need the q-Pochhammer product notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad \lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k) = (a;q)_\infty, \quad |q|<1,$$

as well as the notation for the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \in \mathbb{Z}[q], \quad \deg_q \begin{bmatrix} n \\ k \end{bmatrix}_q = k(n-k). \tag{5}$$

Let

$${}_2\phi_1(q^a,q^b;q^c;q;y) = \sum_{n=0}^{\infty}\frac{(q^a;q)_n(q^b;q)_n}{(q^c;q)_n(q;q)_n}y^n, \quad |y|<1,$$

denote the Heine series. Then the basic Euler-Pochhammer integral representation for ${}_2\phi_1$ (see [1], p. 521) yields the identities

$${}_2\phi_1(q^a,q^b;q^c;q;y) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 x^{b-1}\frac{(xyq^a;q)_\infty(xq;q)_\infty}{(xy;q)_\infty(xq^{c-b};q)_\infty}d_qx \tag{6}$$

$$= \frac{(q^b;q)_\infty(q^ay;q)_\infty}{(q^c;q)_\infty(y;q)_\infty}\,{}_2\phi_1(q^{c-b},y;q^ay;q;q^b) \tag{7}$$

for $0<|q|,|y|<1$, with the real part of b, $\Re(b)>0$, $c-b\neq 0,-1,\ldots$, and where

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}$$

is the q-gamma function. The equation (7) is commonly known as Heine's transformation.

The following two standard lemmas will also be needed.

LEMMA 1. – *For*

$$D_n(p) = lcm\{p-1,p^2-1,\ldots,p^n-1\} = \prod_{j=1}^{n}\Phi_j(p), \quad p\in\mathbb{Z}\setminus\{0,\pm 1\}$$

we have the asymptotic bound

$$\lim_{n\to\infty}\frac{\log|D_n(p)|}{n^2} = \frac{3}{\pi^2}\log|p|.$$

Furthermore, if $[u,v)\subset(0,1)$ *and by* $\{n/s\}$ *we denote the fractional part of* n/s, *i.e.* $n/s=[n/s]+\{n/s\}$, *then*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{s:\{n/s\}\in[u,v)}\log|\Phi_s(p)| = -\frac{3\log|p|}{\pi^2}\int_u^v d\psi'(x),$$

where

$$\psi'(x) = \frac{d^2}{dx^2} \log \Gamma(x). \quad (\textit{Note that } \Gamma\ (x) \textit{ is the ordinary gamma function.})$$

LEMMA 2. – *Let* $a \in \mathbb{R}$ *and* (p_n) *and* (q_n) *be two sequences of integers satisfying*

$$\lim_{n\to\infty} \frac{\log |p_n - q_n a|}{n^2 \log |p|} = -R,$$

and

$$\limsup_{n\to\infty} \frac{\log |q_n|}{n^2 \log |p|} = T,$$

where R, T *are positive numbers, then*

$$\mu(a) \leq \frac{T}{R} + 1$$

holds.

For more details concerning Lemma 1, one is referred to [8], whereas Lemma 2 is an application of Lemma 3.5 in [2].

3. – The q-hypergeometric transformation and Φ-adic valuations.

THEOREM 2. – *Let*

$$I_q(h,j,l;y) = \int_0^1 x^h \frac{(xyq^{l+1};q)_\infty (xq;q)_\infty}{(xy;q)_\infty (xq^{j+1};q)_\infty} d_q x, \ \ |y|<1, \tag{8}$$

where $h,j,l,j+h-l \in \mathbb{Z}^+$ *and* $q = 1/p,\ p \in \mathbb{Z} \setminus \{0, \pm 1\}$. *Suppose we have a sequence of approximations*

$$\mathcal{I}_q(hn,jn,ln;y_0) = \mathcal{Q}_q(hn,jn,ln;y_0)\xi_q - \mathcal{P}_q(hn,jn,ln;y_0), \quad n \in \mathbb{Z}^+$$

to an irrational number ξ_q *at* $y = y_0$ *with* $\mathcal{Q}_q(hn,jn,ln,y_0)$, $\mathcal{P}_q(hn,jn,ln;y_0) \in \mathbb{Z}[p]$ *and*

$$\mathcal{I}_q(hn,jn,ln;y_0) = C_n \cdot I_q(hn,jn,ln;y_0) \quad \textit{for every} \quad h,j,l, \tag{9}$$

where the constant $C_n > 0$ *is chosen to be minimal satisfying* (9). *Then*

$$\Delta_n | \mathcal{P}_q(hn,jn,ln;y_0), \quad \Delta_n = \prod_{\omega=\left\{\frac{n}{s}\right\}\in\Omega} \Phi_s(p),$$

where $\{n/s\}$ *is the fractional part of* n/s *and* $\omega \in \Omega \subseteq [0,1)$ *if and only if*

$$[l\omega] + [(j+h-l)\omega] < [j\omega] + [h\omega].$$

PROOF. – By the symmetry ${}_2\phi_1(q^a, q^b; q^c; q; y) = {}_2\phi_1(q^b, q^a; q^c; q; y)$, we get

$$\int_0^1 x^{b-1} \frac{(xyq^a;q)_\infty (xq;q)_\infty}{(xy;q)_\infty (xq^{c-b};q)_\infty} d_q x = \frac{\Gamma_q(b)\Gamma_q(c-b)}{\Gamma_q(a)\Gamma_q(c-a)} \int_0^1 x^{a-1} \frac{(xyq^b;q)_\infty (xq;q)_\infty}{(xy;q)_\infty (xq^{c-a};q)_\infty} d_q x \tag{10}$$

when $\Re(a), \Re(b) > 0$ and $c-a, c-b \neq 0, -1, -2, \ldots$, $|y| < 1$. Set $a = l+1$, $b = h+1, c = j+h+2$, then (10) holds when $h, j, l, j+h-l \geq 0$. Moreover, we obtain

$$\mathcal{I}_q(h,j,l;y_0) = \frac{(p;p)_h (p;p)_j}{(p;p)_l (p;p)_{j+h-l}} \mathcal{I}_q(l, j+h-l, h; y_0) \tag{11}$$

and by the irrationality of ξ_q, it follows that

$$(p;p)_l (p;p)_{j+h-l} \mathcal{P}_q(h,j,l;y_0) = (p;p)_h (p;p)_j \mathcal{P}_q(l, j+h-l, h; y_0). \tag{12}$$

Let $A, B, C, D \in \mathbb{Z}^+$ such that $A+B = C+D$, and denote by

$$a_{\Phi_s(p)} = v_{\Phi_s(p)}((p;p)_{An}(p;p)_{Bn}), \quad \beta_{\Phi_s(p)} = v_{\Phi_s(p)}((p;p)_{Cn}(p;p)_{Dn})$$

the Φ-adic valuations of the p-products. By (4), we know that

$$v_{\Phi_s(p)}(p;p)_H = \left[\frac{H}{s}\right],$$

hence

$$\begin{aligned} a_{\Phi_s(p)} &= v_{\Phi_s(p)}((p;p)_{An}) + v_{\Phi_s(p)}((p;p)_{Bn}) \\ &= \left[\frac{An}{s}\right] + \left[\frac{Bn}{s}\right], \end{aligned}$$

and

$$\begin{aligned} \beta_{\Phi_s(p)} &= v_{\Phi_s(p)}((p;p)_{Cn}) + v_{\Phi_s(p)}((p;p)_{Dn}) \\ &= \left[\frac{Cn}{s}\right] + \left[\frac{Dn}{s}\right]. \end{aligned}$$

By denoting the fractional part of n/s by ω, i.e.

$$\omega = \left\{\frac{n}{s}\right\} = \frac{n}{s} - \left[\frac{n}{s}\right],$$

we get

$$\left[\frac{Hn}{s}\right] = H\left[\frac{n}{s}\right] + [H\omega],$$

and therefore

$$a_{\Phi_s(p)} - \beta_{\Phi_s(p)} = [A\omega] + [B\omega] - [C\omega] - [D\omega].$$

Similarly to [6] (Lemma 4.1), we deduce that if $A+B=C+D$, then

$$-1 \le a_{\Phi_s(p)} - \beta_{\Phi_s(p)} \le 1.$$

This is promptly seen, because

$$[A]+[B]-[C]-[D] = \{C\}+\{D\}-\{A\}-\{B\},$$

whence

$$-2 < -\{A\}-\{B\} \le [A]+[B]-[C]-[D] \le \{C\}+\{D\} < 2.$$

Now, set $A=l$, $B=j+h-l$, $C=j$, $D=h$, then if

$$[l\omega] + [(j+h-l)\omega] < [j\omega] + [h\omega],$$

we have

$$\Phi_s(p) | \mathcal{P}_q(hn, jn, ln; y_0)$$

by equation (12). □

3.1 – *The case of* $\log_q(1)$.

Let us consider the case when $y_0 = q^{k+1}$, $k \in \mathbb{N}$ and denote

$$I_q(h,j,l,k) = \int_0^1 x^h \frac{(xq;q)_j}{(xq^{k+1};q)_{l+1}} d_q x.$$

When $k \le j \le l$ (the choice $k > j$ does not provide good approximations) the partial fraction decomposition yields

$$\frac{(xq;q)_j}{(xq^{k+1};q)_{l+1}} = \frac{(xq;q)_k}{(xq^{j+1};q)_{l+k-j+1}} = \frac{a_{j+1}}{1-xq^{j+1}} + \ldots + \frac{a_{l+k+1}}{1-xq^{l+k+1}},$$

where

$$\begin{aligned}
a_s &= \lim_{x \to q^{-s}} (1-xq^s) \frac{(xq;q)_k}{(xq^{j+1};q)_{l+k-j+1}} \\
&= \frac{(q^{1-s};q)_k}{(q^{j+1-s};q)_{s-j-1}(q;q)_{l+k-s+1}} \qquad (j+1 \le s \le l+k+1) \\
&= \frac{(-1)^k q^{\binom{k+1}{2}-sk}(q^{s-k};q)_k}{(-1)^{s-j-1} q^{-\binom{s-j}{2}}(q;q)_{s-j-1}(q;q)_{l+k-s+1}} \\
&= (-1)^{s+j+k+1} q^{\binom{s-j}{2}+\binom{k+1}{2}-sk} \frac{(q;q)_k}{(q;q)_{l+k-j}} \begin{bmatrix} s-1 \\ k \end{bmatrix}_q \begin{bmatrix} l+k-j \\ s-j-1 \end{bmatrix}_q.
\end{aligned}$$

Thus,

$$\int_0^c \frac{x^h(xq;q)_k}{(xq^{j+1};q)_{l+k-j+1}} d_qx = \sum_{s=j+1}^{l+k+1} \int_0^c \frac{x^h a_s}{1-xq^s} d_qx.$$

Since, $(q^i q;q)_k = 0$ for $i=-1,\dots,-k$, we have for $c=q^{-k}$ that

$$\int_0^1 \frac{x^h(xq;q)_k}{(xq^{j+1};q)_{l+k-j+1}} d_qx = \int_0^{q^{-k}} \frac{x^h(xq;q)_k}{(xq^{j+1};q)_{l+k-j+1}} d_qx$$

and

$$\begin{aligned}
I_q(h,j,l,k) &= \int_0^{q^{-k}} \sum_{s=j+1}^{l+k+1} \frac{a_s x^h}{1-xq^s} d_qx = \sum_{s=j+1}^{l+k+1} a_s \int_0^{q^{-k}} \frac{x^h}{1-xq^s} d_qx \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s \sum_{n=-k}^{\infty} \frac{q^{(h+1)n}}{1-q^{n+s}} = (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \sum_{n=-k}^{\infty} \frac{q^{(h+1)(n+s)}}{1-q^{n+s}} \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \sum_{n=s-k}^{\infty} \frac{q^{(h+1)n}}{1-q^n} \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \left(\sum_{n=s-k}^{\infty} \frac{q^n - q^n + q^{(h+1)n}}{1-q^n} \right) \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \left(\log_q(1) - \sum_{n=1}^{s-k-1} \frac{q^n}{1-q^n} - \sum_{n=s-k}^{\infty} \frac{q^n - q^{(h+1)n}}{1-q^n} \right) \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \left(\log_q(1) - \sum_{n=1}^{s-k-1} \frac{q^n}{1-q^n} - \sum_{n=s-k}^{\infty} \sum_{m=1}^{h} q^{mn} \right) \\
&= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \left(\log_q(1) - \sum_{n=1}^{s-k-1} \frac{q^n}{1-q^n} - \sum_{m=1}^{h} \frac{q^{m(s-k)}}{1-q^m} \right) \\
&= Q_q(h,j,l,k)\log_q(1) - P_q(h,j,l,k)
\end{aligned}$$

where

$$\begin{aligned}
Q_q(h,j,l,k) &= (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s} \\
&= (1-q)\frac{(q;q)_k}{(q;q)_{l+k-j}} \sum_{s=j+1}^{l+k+1} (-1)^{s+j+k+1} q^{\binom{s-j}{2}+\binom{k+1}{2}-s(k+h+1)} \begin{bmatrix} s-1 \\ k \end{bmatrix}_q \begin{bmatrix} l+k-j \\ s-j-1 \end{bmatrix}_q,
\end{aligned}$$

and

$$P_q(h,j,l,k) = (1-q)\sum_{s=j+1}^{l+k+1} a_s q^{-(h+1)s}\left(\sum_{n=1}^{s-k-1}\frac{q^n}{1-q^n}+\sum_{m=1}^{h}\frac{q^{m(s-k)}}{1-q^m}\right)$$

$$=(1-q)\frac{(q;q)_k}{(q;q)_{l+k-j}}\sum_{s=j+1}^{l+k+1}(-1)^{s+j+k+1}q^{\binom{s-j}{2}+\binom{k+1}{2}-s(k+h+1)}\begin{bmatrix}s-1\\ k\end{bmatrix}_q\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_q\cdot$$

$$\left(\sum_{n=1}^{s-k-1}\frac{q^n}{1-q^n}+\sum_{m=1}^{h}\frac{q^{m(s-k)}}{1-q^m}\right).$$

(See [4], [8] and [9] for similar computations.)

When $q = 1/p$, we denote

$$Q(h,j,l,k;p) = \sum_{s=j+1}^{l+k+1}(-1)^{s+j+k+1}p^{\phi(s)}\begin{bmatrix}s-1\\ k\end{bmatrix}_p\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_p, \tag{13}$$

and compute

$$P(h,j,l,k;p) = \sum_{s=j+1}^{l+k+1}(-1)^{s+j+k+1}p^{\phi(s)-h(s-k-1)}\cdot$$

$$\cdot\begin{bmatrix}s-1\\ k\end{bmatrix}_p\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_p\left(\sum_{n=1}^{s-k-1}\frac{p^{(s-k-1)h}}{p^n-1}+\sum_{m=1}^{h}\frac{p^{(s-k-1)(h-m)}}{p^m-1}\right)$$

$$=\sum_{s=j+1}^{l+k+1}(-1)^{s+j+k+1}p^{\phi(s)-h(s-k-1)}\begin{bmatrix}s-1\\ k\end{bmatrix}_p\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_p\sum_{n=1}^{s-k-1}\frac{p^{(s-k-1)h}}{p^n-1}$$

$$+p^{\gamma}\sum_{m=1}^{h}\sum_{s=0}^{j}\frac{p^{(k+1-s)(k-s)/2}}{p^m-1}(-1)^s\begin{bmatrix}l+k+s-j\\ k\end{bmatrix}_p\begin{bmatrix}j\\ s\end{bmatrix}_p(p^{h-m};p^{h-m+1})_{l-j+s},$$

where $\gamma = k(k+1)/2+(k+1)(h+1)+(l+k-j+1)(j-k)$ (in the last equation, Lemma 3 from [8] was used), and

$$p^{s(k+h+1)-\binom{s-j}{2}-\binom{k+1}{2}}\begin{bmatrix}s-1\\ k\end{bmatrix}_{1/p}\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_{1/p}=p^{\phi(s)}\begin{bmatrix}s-1\\ k\end{bmatrix}_p\begin{bmatrix}l+k-j\\ s-j-1\end{bmatrix}_p.$$

Therefore,

$$\deg_p Q(h,j,l,k;p)\le(l+k)(h+j+k)-\frac{1}{2}\left((l+k)^2+j^2+k^2\right)+\mathcal{O}(\max\{l,h\}), \tag{14}$$

since by (5) and (13) $\deg_p Q(h,j,l,k;p)\le\max_s\eta(s)$, where

$$\begin{aligned}\eta(s) &:= \phi(s) + k(s-k-1) + (s-j-1)(l+k-s+1)\\ &= s(k+h+1) - \binom{s-j}{2} - \binom{k+1}{2}\\ &\le \eta(l+k+1).\end{aligned}$$

Moreover,

$$\begin{aligned}\phi(s) &= \binom{s+1}{2} - s(l+k+1-h) + \binom{k+1}{2} - \binom{j+1}{2} + (j+1)(l+k+1)\\ &\ge \phi(j+1) \quad \text{if} \quad l+k \le j+h\end{aligned}$$

and

$$\begin{aligned}K(h,j,l,k) &:= \min\{\phi(j+1), \gamma\}\\ &= \binom{k+1}{2} + \min\{(j+1)(h+1), (k+1)(h+1) + (l+k-j+1)(j-k)\}\\ &= \binom{k+1}{2} + (k+1)(h+1) + (l+k-j+1)(j-k).\end{aligned}$$

By denoting

$$\mathcal{Q}(h,j,l,k) := p^{-K(h,j,l,k)} D_{\max\{l,h\}}(p) \frac{(q;q)_{l+k-j}}{(1-q)(q;q)_k} Q(h,j,l,k;p) \in \mathbb{Z}[p]$$

and

$$\mathcal{P}(h,j,l,k) := p^{-K(h,j,l,k)} D_{\max\{l,h\}}(p) \frac{(q;q)_{l+k-j}}{(1-q)(q;q)_k} P(h,j,l,k;p) \in \mathbb{Z}[p],$$

we have

$$\mathcal{I}(h,j,l,k) = \mathcal{Q}(h,j,l,k)\log_q(1) - \mathcal{P}(h,j,l,k) \in \mathbb{Z}\cdot\log_q(1) + \mathbb{Z}, \tag{15}$$

where

$$\mathcal{I}(h,j,l,k) = p^{-K(h,j,l,k)} D_{\max\{l,h\}}(p) \frac{(q;q)_{l+k-j}}{(1-q)(q;q)_k} I_q(h,j,l,k).$$

Note that

$$\frac{(q;q)_{l+k-j}}{(1-q)(q;q)_k} = (-1)^{l+j+1} p^{\binom{k+1}{2} - \binom{l+k-j+1}{2} + 1} \frac{(p;p)_{l+k-j}}{(1-p)(p;p)_k}.$$

For $h,l,j \in \mathbb{Z}^+$, $j \le l \le j+h$, the q-hypergeometric transformation (10) yields

$$p^{\delta_1}\mathcal{I}(h,j,l,k) = p^{\varepsilon_1} \frac{(p;p)_h (p;p)_j}{(p;p)_l (p;p)_{j+h-l}} \mathcal{I}(l, j+h-l, h, k), \quad \delta_1, \varepsilon_1 \in \mathbb{N},$$

after multiplication by a suitable power of p, similarly to equation (11). Further, when $l+k \leq j+h$ a repeated application of Heine's transformation and q-hypergeometric transformation yields

$$p^{\delta_2}\mathcal{I}(h,j,l,k) = p^{\varepsilon_2}\frac{(p;p)_j(p;p)_{l+k-j}}{(p;p)_k(p;p)_l}\mathcal{I}(l,k,l+k-j,j+h-l), \quad \delta_2,\varepsilon_2 \in \mathbb{N}.$$

Again, by the irrationality of $\log_q(1)$, we get the identities

$$p^{\delta_1}(p;p)_l(p;p)_{j+h-l}\mathcal{P}(h,j,l,k) = p^{\varepsilon_1}(p;p)_h(p;p)_j\mathcal{P}(l,j+h-l,h,k), \tag{16}$$

and

$$p^{\delta_2}(p;p)_k(p;p)_l\mathcal{P}(h,j,l,k) = p^{\varepsilon_2}(p;p)_{l+k-j}(p;p)_j\mathcal{P}(l,k,l+k-j,j+h-l). \tag{17}$$

Since $gcd(p,p-1)=1$, we know by (16), (17) and Theorem 2 that

$$\Phi_s(p)|\mathcal{P}(hn,jn,ln,kn), \quad n \in \mathbb{Z}^+ \tag{18}$$

when $\omega = \{n/s\} \in \Omega$ is such that

$$[l\omega] + [(j+h-l)\omega] < [j\omega] + [h\omega]$$

or

$$[l\omega] + [k\omega] < [j\omega] + [(l+k-j)\omega],$$

that is

$$[l\omega] - [j\omega] < \max\{[h\omega] - [(j+h-l)\omega], [(l+k-j)\omega] - [k\omega]\}.$$

If $\omega < 1/c$, where $c = \max\{h,l\}$, then $[j\omega] = [h\omega] = [(l+k-j)\omega] = 0$, and thus $\omega \notin \Omega$. Therefore, if $\Phi_s(p)|\Delta_n$, then $\omega = \{n/s\} \in \Omega$, and $n/s \geq \omega \geq 1/c$. This implies that $s \leq cn = \max\{h,l\}n$, which in turn yields $\Phi_s(p)|D_{\max\{l,h\}n}(p)$, and so

$$\frac{D_{\max\{ln,hn\}}}{\Delta_n} \in \mathbb{Z}[p].$$

Whence,

$$\Delta_n^{-1}\mathcal{Q}(hn,jn,ln,kn) \in \mathbb{Z}[p]. \tag{19}$$

Let us denote $K_n = K(hn,jn,ln,kn)$, i.e.

$$\begin{aligned} K_n &= \binom{kn+1}{2} + (kn+1)(hn+1) + (ln+kn-jn+1)(jn-kn) \\ &= \left(\frac{k^2}{2} + kh + (l+k-j)(j-k)\right)n^2 + \mathcal{O}(n) \\ &= An^2 + \mathcal{O}(n). \end{aligned} \tag{20}$$

From (15), (18) and (19) , we obtain

$$\varDelta_n^{-1}\mathcal{I}(hn, jn, ln, kn) \in \mathbb{Z}\log_q(1) + \mathbb{Z}$$

and by Lemma 1 and (20)

$$\lim_{n\to\infty}\frac{\log|\varDelta_n^{-1}\mathcal{I}(hn, jn, ln, kn)|}{n^2\log|p|} = -A + \frac{3}{\pi^2}\max\{l,h\}^2 - \left(-\frac{3}{\pi^2}\int_\Omega d\psi'(x)\right)$$

$$= \frac{3}{\pi^2}\left(\max\{l,h\}^2 + \int_\Omega d\psi'(x)\right) - \frac{k^2}{2} - (kh + (l+k-j)(j-k))$$

$$=: R(h,j,l,k).$$

On the other hand, by the estimate (14)

$$\lim_{n\to\infty}\frac{\log|\varDelta_n^{-1}\mathcal{Q}(hn, jn, ln, kn)|}{n^2\log|p|} \le (l+k)(h+j+k)$$

$$-\frac{1}{2}\left((l+k)^2 + j^2 + k^2\right) + R(h,j,l,k) =: T(h,j,l,k) + R(h,j,l,k).$$

Lemma 2 with the choice $h = j = 14$, $l = 15$, $k = 12$ gives

$$\mu(\log_q(1)) \le 2,4649\ldots.$$

Usually, we denote $\log_q(1) = \zeta_q(1)$ and call the value a q-analogue of zeta function at point one or the q-harmonic series (see, for example [5] for further discussions).

OBSERVATION 1. – *For* $\log_q(y)$, *it is more advantageous to employ the connection* (6) *when computing the approximation form, instead of using the expression coming from the direct computation of the q-integral* (8). *In fact, in* [4] *the authors derive an approximation*

$$z^{n_1+1}\frac{(q;q)_{n_1}(q;q)_{n_2}}{(q;q)_{n_1+n_2+1}}\,{}_2\phi_1(q^{n_0+1}, q^{n_1+1}; q^{n_1+n_2+2}; q; q^{m+1}z) = Q(z)\log_q(z) + P(z)$$

with $n_1 \ge n_0, n_2 \ge n_0$ *and* $n_2 - n_0 \le m \le n_2$. *By denoting* $n_1 = h$, $n_0 = l$, $n_2 = j$, $m = k$ *and* $y = zq^{k+1}$, *we have by* (6)

$$z^{h+1}(1-q)^{-1}I(h,j,l;zq^{k+1}) = Q(z)\log_q(z) + P(z)$$

and Theorem 2 *yields the irrationality measures*

$$\mu(\log_q(z)) \le 3,7633\ldots, \quad \mu(\log_q(-1)) \le 2,9383\ldots \ (q\text{-analogue of } \log 2)$$

as in [4], *given the same analytic estimates for the approximation polynomials.*

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