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J. Kinnunen, M. Kotilainen, V. Latvala

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Hardy-Littlewood Type Gradient Estimates for Quasiminimizers

J. Kinnunen - M. Kotilainen - V. Latvala

Abstract. – *We prove Hardy-Littlewood type integral estimates for quasiminimizers in the unit ball of the Euclidean n-space. These extend known results for planar analytic functions to a more general class of functions. Our results can be regarded as weighted Caccioppoli and Poincaré inequalities for quasiminimizers.*

1. – Introduction.

We recall certain inequalities for analytic functions originally studied by Hardy and Littlewood. Indeed, let f be complex analytic in the unit disc $D(0,1) \subset \mathbb{C}$. Then for every $p > 0$ and $q > -1$ the area integrals satisfy

$$\frac{1}{C} \int\limits_{D(0,1)} |f(z)|^p (1-|z|)^q \, dz \leq |f(0)|^p + \int\limits_{D(0,1)} |f'(z)|^p (1-|z|)^{p+q} \, dz \tag{1.1}$$

and

$$|f(0)|^p + \int\limits_{D(0,1)} |f'(z)|^p (1-|z|)^{p+q} \, dz \leq C \int\limits_{D(0,1)} |f(z)|^p (1-|z|)^q \, dz \tag{1.2}$$

with constants C depending only on p and q. In this form, the inequalities are stated in Lemma 2.2 of [18], see also Theorems 6 and 7 in [7] and Remark 3.2 [1]. These inequalities can be equivalently formulated for classical harmonic functions in the unit disc by replacing the complex derivative with the gradient. Several versions of (1.1) and (1.2) appear in the literature related to analytic function spaces. For extensions to several complex variables including invariant harmonic functions (M-harmonic functions), we refer to [1] and [18].

The purpose of this note is to point out that inequalities analogous to (1.1) and (1.2) hold true for *quasiminimizers* of the s-Dirichlet integral

$$\int |\nabla u(x)|^s \, dx,$$

with $s > 1$, in the Euclidean n-space. The precise definition of a quasiminimizer will be given below. Minor additional restrictions for the range of parameters are necessary in this case. The class of quasiminimizers contains many solutions of linear and nonlinear elliptic partial differential equations, see [9]. Quasiminimizers with constant one are precisely weak solutions of the s-Laplace equation

$$-\operatorname{div}(|\nabla u(x)|^{s-2}\nabla u(x)) = 0.$$

Continous weak solutions of the s-Laplace equation are called *s-harmonic functions*. In the special case $s = 2$, we obtain classical harmonic functions.

Our first result gives an analogue of (1.2) for continuous quasiminimizers in the unit ball $B(0,1)$ of $\mathbb{R}^n$ with $n \geq 2$. Continuity is not a serious restriction, since every quasiminimizer has a continuous representative, see [4] and [9].

THEOREM 1.3. – *Let $s > 1$, $0<p \leq s$ and $q \in \mathbb{R}$ and assume that u is a continuous K-quasiminimizer of the s-Dirichlet integral in $B(0,1) \subset \mathbb{R}^n$. Then there is a constant $C = C(n,p,s,K)$ such that*

$$|u(0)|^p + \int_{B(0,1)} |\nabla u(x)|^p(1-|x|)^{p+q}\,dx \leq C \int_{B(0,1)} |u(x)|^p(1-|x|)^q\,dx. \tag{1.4}$$

Moreover, if u is s-harmonic in $B(0,1)$, then (1.4) *holds for all $p > 0$.*

The proof of Theorem 1.3 relies on such standard tools as Caccioppoli inequality, reverse Hölder inequalities and a Whitney type covering result. Indeed, our argument shows that (1.4) is a weighted Caccioppoli type inequality.

According to our second result, an analogue of (1.1) holds true for quasiminimizers if $p \geq 1$.

THEOREM 1.5. – *Let $s > 1$, $p \geq 1$, $-1<q<\infty$, and assume that u is a continuous K-quasiminimizer of the s-Dirichlet integral in $B(0,1) \subset \mathbb{R}^n$. Then there is a constant $C = C(n,s,K,p,q)$ such that*

$$\int_{B(0,1)} |u(x)|^p(1-|x|)^q\,dx \leq C\left(|u(0)|^p + \int_{B(0,1)} |\nabla u(x)|^p(1-|x|)^{p+q}\,dx\right). \tag{1.6}$$

Theorem 1.5 is sharp in the sense that the claim does not hold for $q = -1$. To see this, it is enough to consider $u \equiv 1$. Our proof of (1.6) is based on a classical Hardy's inequality and the absolute continuity of a Sobolev function on almost every direction (Lemma 3.1 below). In fact, we prove the weighted Poincaré type inequality

$$\int_{B(0,1)} |u(x)-u(0)|^p(1-|x|)^q\,dx \leq C \int_{B(0,1)} |\nabla u(x)|^p(1-|x|)^{p+q}\,dx, \tag{1.7}$$

from which the result follows easily. Such inequalities (without weights) have been studied in [2] and [19] for certain solutions of elliptic partial differential equations in Lipschitz domains. By choosing $q = 0$ we notice that our inequality is an improvement of Theorem 2.1 in [19] in a unit ball.

The key novelty in the proof of (1.7) is the use of Lemma 3.1; we feel that Lemma 3.1 offers a somewhat general method to pass from smooth functions to Sobolev functions in balls.

2. – The proof of Theorem 1.3.

Throughout this work, $s > 1$ is fixed and $W^{1,s}_{loc}(B(0,1))$ denotes the space of those locally s-integrable functions in $B(0,1) \subset \mathbb{R}^n$ with $n \geq 2$, whose weak first order derivatives are locally integrable to the power s in $B(0,1)$. The Sobolev space with zero boundary values, denoted by $W^{1,s}_0(B(0,1))$, is the completion of compactly supported smooth functions $C^\infty_0(B(0,1))$ with respect to the Sobolev space norm. If Ω is an open set with $\overline{\Omega} \subset B(0,1)$, we denote $\Omega \Subset B(0,1)$.

DEFINITION 2.1. – *Let $s > 1$. A function $u \in W^{1,s}_{loc}(B(0,1))$ is a quasiminimizer of the s-Dirichlet integral in $B(0,1) \subset \mathbb{R}^n$ if there is a constant $K > 0$ such that for all open sets $\Omega \Subset B(0,1)$ we have*

$$\int_{\Omega} |\nabla u(x)|^s \, dx \leq K \int_{\Omega} |\nabla v(x)|^s \, dx$$

whenever $v \in W^{1,s}(\Omega)$ with $u - v \in W^{1,s}_0(\Omega)$.

For the properties of quasiminimizers, we refer to [9] and [13]. We recall some auxiliary inequalities for quasiminimizers. We begin with the *Caccioppoli* inequality.

LEMMA 2.2. – *Assume that u is a quasiminimizer of the s-Dirichlet integral in $B(0,1) \subset \mathbb{R}^n$. For every $\delta > 1$, there is a constant $C = C(n, s, K, \delta)$ such that*

$$\int_{B(y,r)} |\nabla u(x)|^s \, dx \leq \frac{C}{r^s} \int_{B(y,\delta r)} |u(x)|^s \, dx$$

whenever $B(y, \delta r) \Subset B(0,1)$.

PROOF. – The claim follows easily from the definition, see Theorem 6.5 in [9] or Lemma 3.3 in [14] for a detailed proof. $\square$

We also need the fact that quasiminimizers and their gradients satisfy *reverse Hölder inequalities*. We denote

$$u_{B(y,r)} = \fint_{B(y,r)} u(x)\,dx = \frac{1}{|B(y,r)|}\int_{B(y,r)} u(x)\,dx,$$

where $|B(y,r)|$ is the volume of the ball $B(y,r)$.

LEMMA 2.3. – *Assume that u is a quasiminimizer of the s-Dirichlet integral in $B(0,1)\subset\mathbb{R}^n$. For every $0<p<q$ and $\delta>1$, there is a constant $C=C(n,s,K,p,q,\delta)$ such that*

$$\left(\fint_{B(y,r)} |u(x)|^q\,dx\right)^{1/q} \le C\left(\fint_{B(y,\delta r)} |u(x)|^p\,dx\right)^{1/p},$$

whenever $B(y,\delta r)\Subset B(0,1)$.

PROOF. – By Theorem 7.4 in [9], we have

$$\left(\fint_{B(y,r)} |u(y)|^q\,dx\right)^{1/q} \le C\sup_{x\in B(y,r)}|u(x)| \le C\left(\fint_{B(y,\delta r)} |u(x)|^p\,dx\right)^{1/p}.$$

□

LEMMA 2.4. – *Assume that u is a quasiminimizer of the s-Dirichlet integral in $B(0,1)\subset\mathbb{R}^n$. For $0<p\le s$ and $\delta>1$, there is a constant $C=C(n,s,K,p,\delta)$ such that*

$$\left(\fint_{B(y,r)} |\nabla u(x)|^s\,dx\right)^{1/s} \le C\left(\fint_{B(y,\delta r)} |\nabla u(x)|^p\,dx\right)^{1/p} \tag{2.5}$$

whenever $B(y,\delta r)\Subset B(0,1)$.

PROOF. – See Theorem 6.5 and Remark 6.12 in [9]. □

REMARK 2.6. – By a self-improving property of reverse Hölder inequalities we may also replace the exponent s on the left-hand side of (2.5) with a slightly larger exponent, see [8] and Theorem 6.7 in [9]. This observation implies that even a slightly stronger statement is true in Theorem 1.3. We do not need this refinement here and we leave the details to the interested reader.

The proof of the claim that the exponent $p>0$ on the left-hand side of (1.4) can be made arbitrarily small in the case of the s-Laplace equation relies on the following regularity property of the gradient of a s-harmonic function.

LEMMA 2.7. – *Let u be s-harmonic in $B(0,1) \subset \mathbb{R}^n$. For every $p > s$ and $\delta > 1$, there is a constant $C = C(n,s,p,\delta)$ such that*

$$\left(\fint_{B(y,r)} |\nabla u(x)|^p \, dx \right)^{1/p} \leq C \left(\fint_{B(y,\delta r)} |\nabla u(x)|^s \, dx \right)^{1/s}$$

whenever $B(y,\delta r) \Subset B(0,1)$.

PROOF. – By regularity theory, the gradient of a s-harmonic function is locally Hölder continuous, see for example [3], [5] and [15]. See also Theorem 3.19 in [16]. Moreover, we have the estimate

$$\sup_{x \in B(y,r)} |\nabla u(x)| \leq C \left(\fint_{B(x,\delta r)} |\nabla u(x)|^s \, dx \right)^{1/s}.$$

The claim follows from this. □

THE PROOF OF THEOREM 1.3. – By Theorem 7.4 in [9] we have

$$|u(0)| \leq \sup_{x \in B(0,1/4)} |u(x)| \leq C \left(\fint_{B(0,1/2)} |u(x)|^p \, dx \right)^{1/p}.$$

Hence it is sufficient to prove the inequality (1.4) without the term $|u(0)|^p$. By a Whitney type covering argument, there is a countable covering of $B(0,1)$ by balls $B(x_i, r_i)$, $i = 1, 2, \ldots$, such that $B(x_i, 2r_i) \Subset B(0,1)$,

$$\frac{1}{C}(1 - |x|) \leq 1 - |x_i| \leq C(1 - |x|) \tag{2.8}$$

for every $x \in B(x_i, r_i)$,

$$1 - |x_i| \leq C r_i, \tag{2.9}$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 2r_i)}(x) \leq C \tag{2.10}$$

for every $x \in B(0,1)$. Here C depends only on n. Condition (2.10) means that the balls $B(x_i, 2r_i)$, $i = 1, 2, \ldots$, are of bounded overlap.

First assume that $p<s$. Hölder's inequality together with (2.8) and (2.9) implies that

$$\begin{aligned}\int_{B(0,1)} |\nabla u(x)|^p(1-|x|)^{p+q}\,dx &\le \sum_{i=1}^{\infty} \int_{B(x_i,r_i)} |\nabla u(x)|^p(1-|x|)^{p+q}\,dx\\ &\le C\sum_{i=1}^{\infty}(1-|x_i|)^{p+q}\int_{B(x_i,r_i)}|\nabla u(x)|^p\,dx\\ &\le C\sum_{i=1}^{\infty}(1-|x_i|)^{p+q+n}\fint_{B(x_i,r_i)}|\nabla u(x)|^p\,dx\\ &\le C\sum_{i=1}^{\infty}(1-|x_i|)^{p+q+n}\left(\fint_{B(x_i,r_i)}|\nabla u(x)|^s\,dx\right)^{p/s}.\end{aligned}$$

Next we apply the Caccioppoli estimate of Lemma 2.2 and the reverse Hölder inequality of Lemma 2.3. This implies that

$$\begin{aligned}&\sum_{i=1}^{\infty}(1-|x_i|)^{p+q+n}\left(\fint_{B(x_i,r_i)}|\nabla u(x)|^s\,dx\right)^{p/s}\\ &\le C\sum_{i=1}^{\infty}(1-|x_i|)^{p+q+n}\left((1-|x_i|)^{-s}\fint_{B(x_i,3r_i/2)}|u(x)|^s\,dx\right)^{p/s}\\ &= C\sum_{i=1}^{\infty}(1-|x_i|)^{q+n}\left(\fint_{B(x_i,3r_i/2)}|u(x)|^s\,dx\right)^{p/s}\\ &\le C\sum_{i=1}^{\infty}(1-|x_i|)^{q+n}\fint_{B(x_i,2r_i)}|u(x)|^p\,dx\\ &= C\sum_{i=1}^{\infty}\int_{B(x_i,2r_i)}|u(x)|^p(1-|x|)^q\,dx\\ &\le C\int_{B(0,1)}|u(x)|^p(1-|x|)^q\,dx.\end{aligned}$$

The last inequality is based on the bounded overlap property (2.10) of the Whitney covering.

The case $p=s$ follows easily from the Caccioppoli estimate. Finally, if u is s-harmonic and $p>s$, we apply the reverse Hölder inequality of Lemma 2.7 for the gradient and the Hölder inequality for the function.

REMARK 2.11. – The proof of Theorem 1.3 extends easily to general domains.

3. – The proof of Theorem 1.5.

The proof of our second result relies on the following absolute continuity property which is analogous to the ACL-property of Sobolev functions. The $(n-1)$-dimensional Hausdorff measure is denoted by H^{n-1}.

LEMMA 3.1. – *Let $u \in W^{1,s}_{loc}(B(0,1))$ and $0 < R_1 < R_2 < 1$. Then u has a representative that is absolutely continuous on the segment $[R_1\zeta, R_2\zeta]$ for H^{n-1}-almost every $\zeta \in \partial B(0,1)$.*

PROOF. – The basic idea for the proof is standard. However, since some technical modifications are needed, we give a detailed proof for reader's convenience. Let us first agree that u is a function which coincides with the original u in $B\left(0, \dfrac{1+R_2}{2}\right)$, belongs to $W^{1,s}(B(0,1))$, and whose support is compactly contained in $B(0,1)$. (In other words we multiply the original u by a suitable cut-off function.) Let $u_k := \varphi_{\varepsilon_k} * u$ be the standard convolution mollification with $\varepsilon_k \to 0$ as $k \to \infty$.

Since the mollifications u_k converge to u in the Sobolev norm as $\varepsilon_k \to 0$, we may pick a sequence (ε_k) so that

$$\int_{B(0,R_2)} (|u_k(x) - u(x)| + |\nabla u(x) - \nabla u_k(x)|)\, dx < 2^{-k}$$

for every $k = 1, 2, \ldots$. We denote by G the set of those points in $B(0,1)$ for which the pointwise limit exists and define u^* by setting

$$u^*(x) = \lim_{k\to\infty} u_k(x)$$

for $x \in G$, $u^*(x) = 0$ in $B(0,1) \setminus G$. Since the set of Lebesgue points of u is contained in G, we conclude that u^* coincides almost everywhere with u.

For each $\zeta \in \partial B(0,1)$ denote

$$v_k(\zeta) = \int_{R_1}^{1} r^{n-1}(|u_k(r\zeta) - u(r\zeta)| + |\nabla u(r\zeta) - \nabla u_k(r\zeta)|)\, dr$$

and

$$v(\zeta) = \sum_{k=1}^{\infty} v_k(\zeta).$$

By the monotone convergence theorem and the spherical coordinates, we have

$$\begin{aligned}\int\limits_{\partial B(0,1)} v(\zeta)\,dH^{n-1}\zeta &= \sum_{k=1}^{\infty} \int\limits_{\partial B(0,1)} v_k(\zeta)\,dH^{n-1}\zeta \\ &= \sum_{k=1}^{\infty} \int\limits_{\partial B(0,1)} \int\limits_{R_1}^{1} r^{n-1}(|u_k(r\zeta)-u(r\zeta)| + |\nabla u(r\zeta) - \nabla u_k(r\zeta)|)\,dr\,dH^{n-1}\zeta \\ &\le \sum_{k=1}^{\infty} \int\limits_{B(0,1)} (|u_k(x)-u(x)| + |\nabla u(x) - \nabla u_k(x)|)\,dx < \infty.\end{aligned}$$

Hence $v(\zeta)$ is finite H^{n-1}-almost every $\zeta \in \partial B(0,1)$. Fix ζ satisfying $v(\zeta)<\infty$ and denote $g_k(t) = u_k(t\zeta)$, $t \in [0,1]$.

For k large enough and for any l, we have $g_{k+l}(1) = g_k(1) = 0$ and hence for $t \in [R_1, 1]$

$$\begin{aligned}|g_{k+l}(t) - g_k(t)| &= \left| \int\limits_t^1 (g'_{k+l} - g'_k)(s)\,ds \right| \le \int\limits_t^1 |\nabla u_{k+l}(s\zeta) - \nabla u_k(s\zeta)|\,ds \\ &\le \int\limits_t^1 |\nabla u_{k+l}(s\zeta) - \nabla u(s\zeta)| + |\nabla u_k(s\zeta) - \nabla u(s\zeta)|\,ds \\ &\le R_1^{1-n}(v_{k+l}(\zeta) + v_k(\zeta)).\end{aligned} \tag{3.2}$$

This implies that (g_k) is a Cauchy sequence in $\mathcal{C}([R_1,1])$. We conclude that the function $g(t) = \lim_{k\to\infty} g_k(t)$ is continuous in $[R_1,1]$. The argumentation in (3.2) also implies that (g'_k) is a Cauchy sequence in $L^1([R_1,1])$. By completeness of $L^1([R_1,1])$ there is a function $\widetilde{g} \in L^1([R_1,1])$ such that $g'_k \to \widetilde{g}$ in $L^1([R_1,1])$. Hence for any $t \in [R_1, R_2]$

$$g(t) = \lim_{k\to\infty} g_k(t) = -\lim_{k\to\infty} \int\limits_t^1 g'_k(s)\,ds = -\int\limits_t^1 \widetilde{g}(s)\,ds.$$

Since $g(1) = 0$, this implies that g is absolutely continuous on $[R_1,1]$. The claim follows since $g(t) = u^*(t\zeta)$ for $t \in [R_1, R_2]$. □

REMARK 3.3. – Notice that Lemma 3.1 is not true for $R_1 = 0$ since the function u may be singular at origin. In this case the argument in (3.2) is useless, since R_1^{1-n} appears on the right-hand-side.

We also recall a useful oscillation estimate.

LEMMA 3.4. – *Assume that u is a quasiminimizer of the s-Dirichlet integral in $B(0,1) \subset \mathbb{R}^n$. For every $t > 0$ and $\delta > 1$, there is a constant $C = C(n,s,K,t,\delta)$*

such that

$$\operatorname*{osc}_{x\in B(y,r)} u(x) \le Cr\Bigg(\fint_{B(y,\delta r)} |\nabla u(x)|^t \, dx \Bigg)^{1/t}$$

whenever $B(y,\delta r) \Subset B(0,1)$.

PROOF. – By Theorem 7.4 in [9] we have

$$\sup_{x\in B(y,r)} |u(x)| \le C\Bigg(\fint_{B(y,(1+\delta)r/2)} |u(x)|^s \, dx \Bigg)^{1/s}.$$

The Poincaré inequality implies that

$$\begin{aligned}\operatorname*{osc}_{x\in B(y,r)} u(x) &\le 2 \sup_{x\in B(y,r)} |u(x) - u_{B(y,(1+\delta)r/2)}| \\ &\le C\Bigg(\fint_{B(y,(1+\delta)r/2)} |u - u_{B(y,(1+\delta)r/2)}|^s \, dx \Bigg)^{1/s} \\ &\le Cr\Bigg(\fint_{B(y,(1+\delta)r/2)} |\nabla u(x)|^s \, dx \Bigg)^{1/s}.\end{aligned}$$

By Lemma 2.4, we may replace s with t on the right hand side of the previous estimate. □

THE PROOF OF THEOREM 1.5. – Our proof is based on the following classical one-dimensional Hardy's inequality in [12] (see also [1], p. 493): If

$$G(r) = \int_0^r g(t)\, dt,$$

where g is integrable on $[0,r]$ for every $0<r<1$, then

$$\int_0^1 |G(r)|^p (1-r)^q \, dr \le C \int_0^1 |g(r)|^p (1-r)^{p+q} \, dr. \tag{3.5}$$

To prove the claim of theorem, it suffices to prove the weighted Poincaré type inequality (1.7). To do this, we divide the integral on the left-hand side of (1.7) in two parts. We first apply Lemma 3.4 and obtain

$$\begin{aligned}\int_{B(0,1/2)} |u(x) - u(0)|^p (1-|x|)^q \, dx &\le C\big(\operatorname*{osc}_{x\in B(0,1/2)} u(x)\big)^p \\ &\le C \int_{B(0,2/3)} |\nabla u(x)|^p \, dx \le C \int_{B(0,1)} |\nabla u(x)|^p (1-|x|)^{p+q} \, dx.\end{aligned} \tag{3.6}$$

Let $\frac{1}{2} < R < 1$. Choose $\zeta \in \partial B(0,1)$ such that u is absolutely continuous in $\left[\frac{1}{2}\zeta, R\zeta\right]$. By Lemma 3.1, this holds for H^{n-1}-almost every $\zeta \in \partial B(0,1)$. For such ζ, we have

$$\left|u(r\zeta) - u\left(\frac{1}{2}\zeta\right)\right| \le \int_{\frac{1}{2}}^{r} |\nabla u(t\zeta)|\, dt. \tag{3.7}$$

We first write the estimate

$$\int_{B(0,R)\setminus B(0,1/2)} |u(x)-u(0)|^p (1-|x|)^q\, dx = \int_{\partial B(0,1)} \int_{\frac{1}{2}}^{R} r^{n-1} |u(r\zeta)-u(0)|^p (1-r)^q\, dr\, d\zeta$$

$$= \int_{\partial B(0,1)} \int_{\frac{1}{2}}^{R} 2^p r^{n-1} \left(\left|u(r\zeta) - u\left(\frac{1}{2}\zeta\right)\right|^p + \left|u\left(\frac{1}{2}\zeta\right) - u(0)\right|^p \right) (1-r)^q\, dr\, d\zeta =: I_1 + I_2.$$

Here the latter integral I_2 can be estimated as in (3.6) by the inequality $\left|u\left(\frac{1}{2}\zeta\right) - u(0)\right| \le \operatorname*{osc}_{x\in B(0,1/2)} u(x)$ and the fact that the weight $(1-|x|)^q$ is integrable.

To estimate the integral I_1 we use (3.7) together with (3.5) and obtain

$$\begin{aligned}
&\int_{\partial B(0,1)} \int_{\frac{1}{2}}^{R} r^{n-1} \left|u(r\zeta) - u\left(\frac{1}{2}\zeta\right)\right|^p (1-r)^q\, dr\, d\zeta \\
&\le \int_{\partial B(0,1)} \int_{\frac{1}{2}}^{R} r^{n-1} \left(\int_{\frac{1}{2}}^{r} |\nabla u(t\zeta)|\, dt \right)^p (1-r)^q\, dr\, d\zeta \\
&\le C \int_{\partial B(0,1)} \int_{\frac{1}{2}}^{R} \left(\int_{0}^{r} t^{(n-1)/p} |\nabla u(t\zeta)|\, dt \right)^p (1-r)^q\, dr\, d\zeta \\
&\le C \int_{\partial B(0,1)} \int_{0}^{1} r^{n-1} |\nabla u(r\zeta)|^p (1-r)^{p+q}\, dr\, d\zeta \\
&\le C \int_{B(0,1)} |\nabla u(x)|^p (1-|x|)^{p+q}\, dx.
\end{aligned}$$

The claim follows by letting $R \to 1$.

REMARK 3.8. – We like to mention that the proofs of theorems 1.3 and 1.5 apply also to those functions u in $B(0,1)$ which are K-quasiminimizers in every ball $B(x,(1-|x|)/2)$ with K independent of $x \in B(0,1)$. An example of such a class of functions is given by *hyperbolic harmonic functions*, see [6], Section 2. These are defined as functions $u \in C^2(B(0,1))$ satisfying the *invariant Laplace-equation*

$$\widetilde{\Delta} u := (1-|x|^2)^2 \Delta u + 2(n-2)(1-|x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = 0$$

on $B(0,1)$. Hyperbolic harmonic functions u also satisfy the analogue of Lemma 2.7. This follows from [10], Proposition 3.4 which may be written in the form: For any $0<\delta<1$ and $0<p<\infty$ there is a constant C such that

$$|\nabla u(y)|^p \leq C \fint_{B(y,\delta(1-|y|))} |\nabla u(x)|^p \, dx.$$

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J. Kinnunen, Institute of Mathematics, Helsinki University of Technology,
P.O. Box 1100, FI-02015 TKK, Finland
E-mail: juha.kinnunen@tkk.fi

M. Kotilainen, Department of Physics and Mathematics, University of Joensuu,
P.O. Box 111, FI-80101 Joensuu, Finland
E-mail: marko.kotilainen@joensuu.fi

V. Latvala, Department of Physics and Mathematics, University of Joensuu,
P.O. Box 111, FI-80101 Joensuu, Finland
E-mail: visa.latvala@joensuu.fi

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