Bollettino Unione Matematica Italiana

Giovanni Vidossich

Smooth Dependence on Initial Data of Mild Solutions to Evolution Equations

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **2** (2009), n.3, p. 731–754.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2009_9_2_3_731_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2009.

Bollettino U. M. I.
(9) **II** (2009), 731-754

Smooth Dependence on Initial Data of Mild Solutions to Evolution Equations

GIOVANNI VIDOSSICH

A Guido Stampacchia,
mio Maestro,
che ricordo con affetto e gratitudine.

Abstract. – *We prove two general theorems related to the smooth dependence on data of mild solutions to evolution Cauchy problems and provide some of their applications to the Faedo-Galerkin method for approximating solutions as well as to the existence and uniqueness of periodic solutions.*

1. – Introduction.

This paper is devoted to two general theorems related to the smooth dependence on data of mild solutions to evolution Cauchy problems

$$u' = Au + f(t, u), \ \ u(a) = u_0$$

as well as to some of their applications, A being the generator of a C_0-semigroup on a given Banach space.

One of the theorems provides sufficient conditions for the continuous dependence of u on f, a and u_0. This theorem generalizes the main results of VIDOSSICH [8] because the convergence $f_n \to f_0$ is now only pointwise. Exactly this improvement is useful in the approximation of mild solutions (as shown in § 4 below for the Faedo-Galerkin method) as well as in the existence of periodic solutions (as shown in § 5 below).

The other theorem states that u depends in a C^1-way on u_0 when f is C^1 (in constrast to the fact that $\frac{\partial}{\partial t} u$ does not exist always), generalizing and unifying in a single statement various results proved in ch.6 of TEMAM [7] (e.g., in his § 6.8 Temam assumes A positive and symmetric on a Hilbert space). The partial derivative $\frac{\partial}{\partial u_0} u$ turns out to solve a Volterra integral equation in the space of bounded linear operators. The proof is adapted from the original (but apparently not well-known) approach to the differentiability of solutions to ODEs due to SOTOMAYOR [6], based on the Fiber Contraction Theorem of HIRSCH-PUGH [4] (the traditional argument used in ODEs does not seem to work for evolution equations). Applications to the existence and uniqueness of periodic solutions are given in § 5.

2. – Notations, terminology and preliminaries.

We shall denote by:

- X a Banach space;
- $U \subseteq X$ an open set;
- $J := [a, b[$ with $a \geqslant 0$;
- $\mathcal{L}(X, Y)$ the space of bounded linear operators $X \to Y$;
- $\mathcal{L}(X) := \mathcal{L}(X, X)$;
- I the identity mapping of X.

We say that

• $f : D \subseteq \mathbb{R} \times X \to X$ *satisfies locally the Caratheodory hypotheses* when $f(\cdot, x)$ is measurable for all x, $f(t, \cdot)$ is continuous for a.e. t and for every point $z \in D$ there exist a neighborhood N_z of z and $h_z \in L^1_{\mathrm{loc}}$ such that

$$\|f(t, x)\| \leqslant h_z(t)$$

in $N_z \cap D$ for a.e. t;

• $u_n \to u_0$ *uniformly on compacta* when for every compact subset K of the domanin of u_0 there is n_K such that u_n is defined on K for $n \geqslant n_K$ and $\lim_n u_n = u_0$ uniformly on K.

We use repeatedly the following results:

LEMMA 0.1. (PAZY). – *Let $T(t)$ be a compact semigroup on X, $a < c < b$, $h \in L^1([a, c], \mathbb{R}^+)$ and*

$$F(u)(t) := \int_a^t T(t - s)u(s)\, ds$$

for $u \in L^1([a, c], X)$. The set

$$\{F(u)(s) : u \in L^1([a, c], X),\ \|u(\,\cdot\,)\| \leqslant h \ \text{ a.e.}\}$$

has compact closure in X for avery $a \leqslant s \leqslant c$, and the set

$$\{F(u) : u \in L^1([a, c], X),\ \|u(\,\cdot\,)\| \leqslant h \ \text{ a.e.}\}$$

has compact closure in the space of continuous functions $[a, c] \to X$ endowed with the L^∞-norm.

LEMMA 0.2. (WARD) – *When $f(\cdot, u)$ is p-periodic and satisfies the Caratheodory hypotheses, the evolution equation*

$$u' = Au + f(t, u)$$

has a p-periodic mild solution if and only if there exists a mild solution u such that $u(0) = u(p)$. Then the mild solution with initial value $u(0)$ is p-periodic.

Proofs of these lemmas are omitted by the following reasons: Lemma 01 is substantially what is proved on p. 25 (lines 1-9 from bottom) and p. 26 (lines 1-11 from top) of PAZY [5], while Lemma 02 is proved on p. 596 (lines 6-10 from top) of WARD [10].

3. – The two general results.

The first result is devoted to the continuity of mild solutions as functions of data. It shows that the pointwise convergence $f_n \to f_0$ suffices when a uniform local Lipschitz condition holds on the f_n's. This fact is needed in the proofs of Theorems 2, 3 and 5 below.

THEOREM 1. – *For every $n \geqslant 0$ let u_n be the maximally defined mild solution of*

$$u' = Au + f_n(t, u), \;\; u(a_n) = u_0^n$$

where A is the generator of a C_0-semigroup $T(t)$ on X, $f_n : J \times U \to X$ satisfies locally the Caratheodory assumptions, $u_0^n \in U$ and $a_n \in J$. If

(i) *for every $(t_0, x_0) \in J \times U$ there exist a neighborhood W of (t_0, x_0) in $J \times U$ and $h, L \in L^1_{\text{loc}}$ such that*

$$\|f_n(t,x) - f_n(t,y)\| \leqslant L(t) \cdot \|x - y\| \qquad \text{and} \qquad \|f_n(t,u)\| \leqslant h(t)$$

in W for a.e. t and all $n \geqslant 0$;

(ii) *$f_n \to f_0$ pointwise on $J' \times U$ with $J \setminus J'$ having measure zero;*

(iii) *$u_0^n \to u_0^0 \in U$;*

(iv) *$a_n \leqslant a_0$ and $a_n \to a_0$;*

then $u_n \to u_0$ uniformly on compacta.

Note that the uniqueness of u_n follows from (i).

PROOF. – The proof is based on a local version of the theorem, namely on the following claim:

(∗) *Let A and f_n be as in the theorem. Let $c_n \in J$ and $v_n \in U$ satisfy $c_n \leqslant c_0$,*

$c_n \to c_0$ and $v_n \to v_0$. Let w_n be the maximally defined mild solution of

$$w' = Aw + f_n(t,w),\ w(c_n) = v_n$$

for $n \geqslant 0$. Then there exists $\delta > 0$ such that w_n is defined on $J_\delta := [c_0, c_0 + \delta]$ for n large and:

- *$w_n \to w_0$ uniformly on J_δ;*
- *$\sup_{c_n \leqslant t \leqslant c_0} \|w_n(t) - v_0\| \to 0$ as $n \uparrow \infty$.*

To prove $(*)$, at first we fix $M > 0$ such that

$$\|T(t)\| \leqslant M \qquad (0 \leqslant t \leqslant c_0 + 1)$$

and next $\varepsilon > 0$ and $h, L \in L^1_{\mathrm{loc}}$ such that the closed ball B in X with center v_0 and radius ε is contained in U and moreover

$$\|f_n(t,x) - f_n(t,y)\| \leqslant L(t) \cdot \|x - y\| \qquad \text{and} \qquad \|f_n(t,u)\| \leqslant h(t)$$

for a.e. $t \in J \cap [c_0 - \varepsilon, c_0 + \varepsilon]$, $n \geqslant 0$ and $x, y \in B$ [as allowed by (i)]. Fix $\delta \in\,]0, \min\{1, \varepsilon\}]$ such that $J_\delta := [c_0, c_0 + \delta] \subseteq J$ and

$$M \int_{J \cap [c_0 - \delta, c_0 + \delta]} h(s)\, ds < \varepsilon \quad , \quad M \int_{J \cap [c_0 - \delta, c_0 + \delta]} L(s)\, ds < 1.$$

In the last side of the inequalities

$$\|T(t - c_n)v_n - v_0\| \leqslant \|T(t - c_n)v_n - T(t - c_n)v_0\| + \|T(t - c_n)v_0 - v_0\|$$

$$\leqslant M\|v_n - v_0\| + \|T(t - c_n)v_0 - v_0\|$$

all addenda become small for n large and $c_n \leqslant t \leqslant c_0$: the first because $v_n \to v_0$, the second by the continuity of $T(\,\cdot\,)v_0$. This and

$$\left\| \int_{c_n}^{t} T(t - s) f_n\big(s, w(s)\big)\, ds \right\| \leqslant M \int_{c_n}^{t} h(s)\, ds < \varepsilon$$

when $w(s) \in B$ for $c_0 - \delta \leqslant c_n \leqslant s \leqslant t \leqslant c_0 + \delta$, imply that:

- the functions

$$w_n(t) = T(t - c_n)v_n + \int_{c_n}^{t} T(t - s) f_n\big(s, w_n(s)\big)\, ds$$

are defined on $[c_n, c_0 + \delta]$ and take values in B for n large by virtue of the classical argument based on fixed points of contractions;

- $\sup_{c_n \leqslant t \leqslant c_0} \|w_n(t) - v_0\| \to 0$ as $n \uparrow \infty$.

From the integral representation of mild solutions, in J_δ we have:

$$\begin{aligned}\|w_n(t)-w_0(t)\| &\leqslant \|T(t-c_0)w_n(c_0)-T(t-c_0)v_0\| \\ &\quad+\int_{c_0}^{t}\|T(t-s)\|\cdot\|f_n\big(s,w_n(s)\big)-f_0\big(s,w_0(s)\big)\pm f_n\big(s,w_0(s)\big)\|\,ds \\ &\leqslant M\|w_n(c_0)-v_0\|+\int_{c_0}^{c_0+\delta} M\cdot L(s)\cdot\max_{\xi\in J_\delta}\|w_n(\xi)-w_0(\xi)\|\,ds \\ &\quad+\int_{c_0}^{c_0+\delta} M\|f_n\big(s,w_0(s)\big)-f_0\big(s,w_0(s)\big)\|\,ds\end{aligned}$$

so that from the Lebesgue dominated convergence theorem, from $w_n(c_0)\to v_0$ and from $\int_{c_0}^{c_0+\delta} M\cdot L(s)\,ds<1$ we get

$$\max_{\xi\in J_\delta}\|w_n(\xi)-w_0(\xi)\|\to 0 \text{ as } n\uparrow\infty.$$

This means that $(*)$ holds true.

Now we proceed to the proof of the theorem. To start with, we note that it suffices to show the following: given any closed interval of the type $[a_0,\beta]$ which is contained in the maximal domain J_0 of u_0, every subsequence of $(u_n)_n$ has a subsequence whose elements are defined in $[a_0,\beta]$ and converge there uniformly to u_0. So fix $[a_0,\beta]\subseteq J_0$ and any subsequence $(u_{n_k})_k$ of $(u_n)_n$. Define

$$A:=\big\{t\geqslant a_0 : \text{there exists } k_{i,t}\to\infty \text{ such that } \lim_i u_{n_{k_{i,t}}}=u_0 \text{ uniformly on } [a_0,t]\big\},$$
$$c:=\sup A.$$

Note that A is not empty and that $c>a_0$: this follows from $(*)$ with $c_n:=a_n$ and $v_n:=u_0^n$. If we show that $c>\beta$, then we are done. So assume $c\leqslant\beta$, and let us find a contradiction. There are $t_i\in A$ such that $t_i\uparrow c$. By definition of A, for every i there is $k_i>i$ such that

$$\|u_{n_{k_i}}(t)-u_0(t)\|<1/i \qquad (a_0\leqslant t\leqslant t_i).$$

Applying $(*)$ with $c_i:=t_i$, $c_0:=c$ and $v_i:=u_{n_{k_i}}(t_i)$, $v_0:=u_0(c)$ we see that $\sup A\geqslant c+\delta$ for a suitable $\delta>0$ [as J_0 is a right-open interval]. This is the desired contradiction. □

Now we establish the differentiability of mild solutions as functions of initial data and parameters.

THEOREM 2. – *Let X, Y be Banach spaces, A the generator of a C_0-semigroup $T(t)$ on X, $U\subseteq X$ and $V\subseteq Y$ open sets and let $f:J\times U\times V\to X$ satisfy locally* *the Caratheodory hypotheses. Under these assumptions, the mild solution* $u := u(t,x,\lambda)$ *of the evolution equation in* X

$$u' = Au + f(t,u,\lambda),\ \ u(a) = x$$

has the following properties as a function of (t,x,λ)*:*

- *if* $f_u := \dfrac{\partial}{\partial u} f$ *exists and satisfies locally the Caratheodory hypotheses, then* u *has continuous partial derivative* $v := \dfrac{\partial}{\partial x} u(\cdot,x,\lambda)$ *and it is the solution of the Volterra equation*

$$v(t) = T(t-a) + \int_a^t T(t-s)\circ f_u\big(s,u(s,x,\lambda),\lambda\big)\circ v(s)\,ds$$

in $\mathcal{L}(X)$*;*

- *if* $f_u := \dfrac{\partial}{\partial u} f$ *and* $f_\lambda := \dfrac{\partial}{\partial \lambda} f$ *exist and satisfy locally the Caratheodory hypotheses, then* u *has continuous partial derivative* $z := \dfrac{\partial}{\partial \lambda} u(\cdot,x,\lambda)$ *and it is the solution of the Volterra equation*

$$z(t) = \int_a^t T(t-s)\circ \big\{f_u\big(s,u(s,x,\lambda),\lambda\big)\circ z + f_\lambda\big(s,u(s,x,\lambda),\lambda\big)\big\}\,ds$$

in $\mathcal{L}(Y,X)$*.*

The solutions v *and* z *to the above Volterra equations have the same domain of* $u(\cdot,x,\lambda)$*. Moreover,* $\dfrac{\partial}{\partial x} u(\cdot,x,\lambda)\cdot y$ *is the mild solution of the variational Cauchy problem*

$$v' = Av + f_u\big(s,u(s,x,\lambda),\lambda\big)\cdot v,\ \ v(a) = y.$$

In other words, $\dfrac{\partial}{\partial x} u(\cdot,x,\lambda)$ acts like the principal matrix of variational equations for ODEs in $\mathbb{R}^N$.

PROOF. – We treat only the case of u_x since the argument for u_λ is similar. We shall use freely the fact that $u(t,x,\lambda)$ is a continuous function of (t,x,λ). This follows by applying Theorem 1 to

$$f_n(t,x) := f(t,x,\lambda_n)$$

whenever $\lambda_n \to \lambda_0$ [as allowed by the fact that the inequality

$$\|f(t,x,\lambda_n) - f(t,y,\lambda_n)\| \leqslant \sup_z \|f_u(t,z,\lambda_n)\|\cdot\|x-y\| \leqslant L(t)\cdot\|x-y\|$$

holds locally for a suitable $L \in L^1_{\text{loc}}$ by virtue of the mean value theorem].

Fix $x_0 \in U$ and $\lambda_0 \in V$. Let $[a, b_0[$ be the maximum domain of $u(\cdot,x_0,\lambda_0)$. To start with, we prove the following claim:

($*$) *If* $t_0 \in [a, b_0[$, $h \in L^1_{\text{loc}}$ *and the positive constants* δ, ε, η, M *fulfil the following conditions:*

(i) *the closed balls* $B_0 := B\big(u(t_0, x_0, \lambda_0), \varepsilon\big)$ *and* $B_1 := B(x_0, \eta)$ *are contained in U; the closed ball* $B_2 := B(\lambda_0, \eta)$ *is contained in V; the interval* $J_\delta := [t_0, t_0 + \delta]$ *is contained in* $[a, b_0[$*;*

(ii) $u_x(t_0, x, \lambda)$ *exists and is continuous and bounded on* $B_1 \times B_2$*;*

(iii) $\|f(t, y, \lambda)\| \leqslant h(t)$ *and* $\|f_u(t, y, \lambda)\| \leqslant h(t)$ *for* $y \in B_0$, $\lambda \in B_2$ *and a.e.* $t \in J_\delta$*;*

(iv) $\|T(t)\| \leqslant M$ *for* $0 \leqslant t \leqslant 1$*;*

(v) $\delta \leqslant 1$ and $M \int_{t_0}^{t_0+\delta} h(s)\, ds < \min\{1/2, \varepsilon/3\}$*;*

(vi) $\|T(t - t_0)\, u(t_0, x_0, \lambda_0) - u(t_0, x_0, \lambda_0)\| < \varepsilon/3$ *for* $t \in J_\delta$*;*

(vii) $M\, \|u(t_0, x, \lambda) - u(t_0, x_0, \lambda_0)\| < \varepsilon/3$ *for* $t \in J_\delta$, $x \in B_1$, $\lambda \in B_2$*;*

then u_x *exists continuous and is bounded on* $J_\delta \times B_1 \times B_2$.

To state ($*$) we shall use the Fiber Contraction Theorem in HIRSCH-PUGH [4]. To this aim, let C^0 and L be the metric spaces of bounded continuous functions $J_\delta \times B_1 \times B_2 \to B_0$ and $J_\delta \times B_1 \times B_2 \to \mathcal{L}(X)$, respectively, with the metrics induced by the L^∞-norms. Obviously C^0 and L are complete metric spaces. For $t \in J_\delta$, $x \in B_1$, $\lambda \in B_2$ and $v \in C^0$ we have

$$\int_{t_0}^{t} \|T(t-s) f\big(s, v(s, x, \lambda), \lambda\big)\|\, ds \leqslant M \int_{t_0}^{t_0+\delta} h(s)\, ds < \varepsilon/3.$$

This, (vi), (vii) and (iii), (ii) show that the functions

$$(t, x, \lambda) \rightsquigarrow T(t - t_0) u(t_0, x, \lambda) + \int_{t_0}^{t} T(t-s) f\big(s, v(s, x, \lambda), \lambda\big)\, ds,$$

$$(t, x, \lambda) \rightsquigarrow T(t - t_0) \circ u_x(t_0, x, \lambda) + \int_{t_0}^{t} T(t-s) \circ f_u\big(s, v(s, x, \lambda), \lambda\big) \circ w(s, x, \lambda)\, ds$$

belong to C^0 and L, respectively, whenever $v \in C^0$ and $w \in L$. Define now the maps

$$F_1 : C^0 \to C^0 \quad , \quad F_2 : C^0 \times L \to L \quad , \quad F : C^0 \times L \to C^0 \times L$$

by the following formulas:

- $F_1(v)(t, x, \lambda) := T(t - t_0) u(t_0, x, \lambda) + \int_{t_0}^{t} T(t-s) f\big(s, v(s, x, \lambda), \lambda\big)\, ds$;

- $F_2(v,w)(t,x,\lambda) := T(t-t_0)\circ u_x(t_0,x,\lambda) + \int_{t_0}^{t} T(t-s)\circ f_u\big(s,v(s,x,\lambda),\lambda\big)\circ w(s,x,\lambda)\,ds$;
- $F := (F_1, F_2)$.

We have:

(a) F_1 is a contraction because

$$\begin{aligned}
&\|F_1(v_1)(t,x,\lambda) - F_1(v_2)(t,x,\lambda)\| \\
&\qquad \leqslant \int_{t_0}^{t} \|T(t-s)\|\cdot\|f\big(s,v_1(s,x,\lambda),\lambda\big) - f\big(s,v_2(s,x,\lambda),\lambda\big)\|\,ds \\
&\qquad \leqslant \int_{t_0}^{t_0+\delta} \|T(t-s)\|\cdot h(s)\cdot\|v_1(s,x,\lambda) - v_2(s,x,\lambda)\|\,ds \\
&\qquad\quad \text{[by the mean value theorem]} \\
&\qquad \leqslant \|v_1 - v_2\|_\infty M \int_{t_0}^{t_0+\delta} h(s)\,ds \\
&\qquad < \min\{1/2, \varepsilon/3\}\cdot\|v_1 - v_2\|_\infty,
\end{aligned}$$

so that

$$\|F_1(v_1) - F_1(v_2)\|_\infty < \frac{1}{2}\|v_1 - v_2\|_\infty;$$

(b) for every w, the map $F_2(\cdot,w) : C^0 \to L$ is continuous, as follows from the Lebesgue dominated convergence theorem;

(c) the map $F_2(v,\cdot) : L \to L$ is a contraction with the same constant for all v since

$$\begin{aligned}
&\|F_2(v,w_1)(t,x,\lambda) - F_2(v,w_2)(t,x,\lambda)\|_\infty \\
&\qquad \leqslant \sup_t \int_{t_0}^{t} \|T(t-s)\|\cdot h(s)\cdot\|w_1(s,x,\lambda) - w_2(s,x,\lambda)\big)\|\,ds \\
&\qquad \leqslant \|w_1 - w_2\|_\infty M \int_{t_0}^{t_0+\delta} h(s)\,ds \\
&\qquad < \min\{1/2, \varepsilon/3\}\cdot\|w_1 - w_2\|_\infty
\end{aligned}$$

so that

$$\|F_2(v,w_1) - F_2(v,w_2)\|_\infty < \frac{1}{2}\|w_1 - w_2\|_\infty.$$

This shows that F fulfils all assumptions of the Fiber Contraction Theorem in

HIRSCH-PUGH [4], so that F has an attractive fixed point (v_∞, w_∞). We have

$$F_1(v_\infty) = v_\infty \qquad \text{and} \qquad F_2(v_\infty, w_\infty) = w_\infty$$

hence in particular $v_\infty(t,x,\lambda) = u(t,x,\lambda)$ on $J_\delta \times B_1 \times B_2$ by the uniqueness of solutions to Cauchy problems for our evolution equation [the definition of F_1 being based on the formula allowing the extension of $u(\cdot,x,\lambda)$ to the right of t_0]. To prove that $\frac{\partial}{\partial x} v_\infty = w_\infty$, we consider the successive approximations of (v_∞, w_∞) defined inductively by

$$\begin{cases} v_0(t,x,\lambda) := u(t_0,x,\lambda), \\ w_0(t,x,\lambda) := \dfrac{\partial}{\partial x}\, u(t_0,x,\lambda) \end{cases} \quad \text{and} \quad (v_{n+1}, w_{n+1}) := F(v_n, w_n) = F^{n+1}(v_0, w_0).$$

By differentiating (using Leibnitz rule) the integral equation corresponding to $v_{n+1} = F_1(v_n)$, it is easily seen by induction that

$$\frac{\partial}{\partial x} v_n = w_n$$

for every n. Consequently from the uniform convergences

$$v_n \to v_\infty \qquad \text{and} \qquad \frac{\partial}{\partial x} v_n \to w_\infty$$

and well-known theorems of Calculus we deduce that v_∞ is continuous and that $\frac{\partial}{\partial x} v_\infty = w_\infty$ and is continuous. As $u = v_\infty$ on $J_\delta \times B_1 \times B_2$, $(*)$ is proved.

After these preliminaries we are ready for the proof of the theorem. Define

$$J_0 := \big\{ t \in [a, b_0[\, : u_x \text{ exists continuous and is bounded in a neighborhood of } (s, x_0, \lambda_0) \text{ for each } s \in [a,t] \big\},$$
$$c := \sup J_0.$$

Since the derivative is a local concept, it suffices to show that $c = b_0$. At first we note that the set J_0 is not empty and that $c > a$: this follows from $(*)$ applied with $t_0 = a$; for, the equality $u(a,x,\lambda) = x$ implies (ii) while the other assumptions of $(*)$ are fulfilled by the continuity of $T(\,\cdot\,)u(t_0,x_0,\lambda_0)$ and $u(\,\cdot\,)$ [the continuity of $u(\,\cdot\,)$ follows from Theorem 1 since f satisfies (i) of Theorem 1] as well as by the properties of f and f_u. Assume $c < b_0$ and argue for a contradiction. In view of the assumptions of the theorem, there are $\varepsilon > 0$ and $h \in L^1_{\mathrm{loc}}$ such that the closed balls $\tilde{B}_0 := B\big(u(c,x_0,\lambda_0), \varepsilon\big)$ and $\tilde{B}_1 := B(x_0, \varepsilon)$ are contained in U, the closed ball $\tilde{B}_2 := B(\lambda_0, \varepsilon)$ is contained in V, the interval $[c-\varepsilon, c+\varepsilon]$ is contained in $[a, b_0[$ and

$$\|f(t,x,\lambda)\| \leqslant h(t) \quad \text{and} \quad \|f_u(t,x,\lambda)\| \leqslant h(t)$$

when $|t-c| \leqslant \varepsilon$ a.e., $x \in \tilde{B}_0$ and $\lambda \in \tilde{B}_2$. Fix $0<\delta \leqslant \min\{\varepsilon, 1\}$, $0<\eta \leqslant \varepsilon$ and $M > 1$ such that

- $\|T(t)\| \leqslant M$ whenever $0 \leqslant t \leqslant 1$;
- $M \int_t^{t+\delta} h(s)\, ds < \min\{1/2, \varepsilon/3\}$ whenever $|t-c| \leqslant \delta$;
- $\|T(t)\, u(c, x_0, \lambda_0) - u(c, x_0, \lambda_0)\|| < \varepsilon/9$ whenever $0 \leqslant t \leqslant \delta$ [as allowed by the continuity of $T(\,\cdot\,)\, u(c, x_0, \lambda_0)$];
- $M\, \|u(t, x, \lambda) - u(c, x_0, \lambda_0)\| < \varepsilon/9$ whenever $\|x - x_0\| \leqslant \eta$, $\|\lambda - \lambda_0\| \leqslant \eta$ and $|t-c| \leqslant \delta$ [as allowed by the continuity of $u(\,\cdot\,)$].

Now we fix any point t_0 in $J_0 \cap [c - \delta/2, c[$. By taking η smaller if necessary, we assume that

- $u_x(t_0, x, \lambda)$ exists, is continuous and uniformly bounded whenever $\|x - x_0\| \leqslant \eta$ and $\|\lambda - \lambda_0\| \leqslant \eta$;

and we plan to apply $(*)$ in t_0. In view of

$$\begin{aligned}
&\|T(t-t_0)\, u(t_0, x_0, \lambda_0) - u(t_0, x_0, \lambda_0)\| \\
&\qquad \leqslant \|T(t-t_0)\, u(t_0, x_0, \lambda_0) - T(t-t_0)\, u(c, x_0, \lambda_0)\| \\
&\qquad\qquad + \|T(t-t_0)\, u(c, x_0, \lambda_0) - u(c, x_0, \lambda_0)\| + \|u(c, x_0, \lambda_0) - u(t_0, x_0, \lambda_0\| \\
&\qquad \leqslant M\, \|u(t_0, x_0, \lambda_0) - u(c, x_0, \lambda_0)\| \\
&\qquad\qquad + \|T(t-t_0)\, u(c, x_0, \lambda_0) - u(c, x_0, \lambda_0)\| + \|u(c, x_0, \lambda_0) - u(t_0, x_0, \lambda_0\| \\
&\qquad < \varepsilon/3 \quad [\text{as } M > 1]
\end{aligned}$$

and of

$$\begin{aligned}
M\, &\|u(t, x, \lambda) - u(t_0, x_0, \lambda_0)\| \\
&\leqslant M\, \|u(t, x, \lambda) - u(c, x_0, \lambda_0)\| + M\, \|u(c, x_0, \lambda_0) - u(t_0, x_0, \lambda_0)\| \\
&< \varepsilon/3
\end{aligned}$$

true for $|t - t_0| \leqslant \delta$, we have (vi) and (vii) of $(*)$, while the other assumptions of $(*)$ are trivially satisfied. Then $(*)$ implies that $t_0 + \delta \in J_0$, hence $c \geqslant t_0 + \delta > c$, a contradiction showing that necessarily $c = b_0$. With this we have stated the existence and continuity of $\frac{\partial}{\partial x} u$. Now we differentiate the identity

$$u(t, x, \lambda_0) = T(t-a)x + \int_a^t T(t-s) f\big(s, u(s, x, \lambda_0), \lambda_0\big)\, ds$$

with respect to x and get the desired formula for $\frac{\partial}{\partial x} u$.

Finally, to state the last assertion of the theorem, fix $y \in X$ and apply $v := \dfrac{\partial}{\partial x} u$ at y. From the Volterra integral representation of the partial derivative just obtained we get

$$v(t) \cdot y = T(t-a)y + \int_a^t T(t-s) f_u\big(s, u_0(s), \lambda_0\big) v(s) \cdot y \, ds.$$

This means that $v(t) \cdot y$ is the mild solution to the variational Cauchy problem. □

4. – Application to the approximation of mild solutions.

The following theorem is a corollary to Theorem 1. It seems to be an abstract framework for the Faedo-Galerkin method as shown by the two subsequent examples. These examples show that Theorem 3 provides an easy way to check the traditional convergence of the Faedo-Galerkin approximations as well as a better convergence in some cases.

Note that all assumptions about the P_n's are trivially fulfilled when X is a Hilbert space, the P_n's are orthogonal projections with $E_n := P_n(X)$ invariant by A, $\dim E_n < \infty$, $E_n \subseteq E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n$ is dense in X.

THEOREM 3. – *Let A be the generator of a C_0-semigroup $T(t)$ on X and let $f : J \times U \to X$ satisfy locally the Caratheodory hypotheses. Assume that:*

(a) *f is locally Lipschitz in u;*
(b) *there exists a sequence of projections $P_n : X \to \mathrm{dom}(A)$ such that*
- *$P_n \to I$ pointwise,*
- *every $E_n := P_n(X)$ is invariant by A,*
- *$AP_n\big|_{E_n}$ is the generator of a C_0-semigroup on E_n.*

If u_n is the maximally defined mild solution of

$$u' = AP_n u + P_n f(t, P_n u), \;\; u(a) = P_n(u_0),$$

then $(u_n)_n$ converges uniformly on compacta to the unique solution of

$$u' = Au + f(t, u), \;\; u(a) = u_0.$$

PROOF. – Since the elements of the C_0-semigroup generated by $AP_n|_{E_n}$ maps $E_n \to E_n$, the range of u_n is contained in E_n. Then we have

$$u_n' = Au_n + P_n f(t, u_n)$$

because $P_n\big|_{E_n}$ is the identity mapping. This suggests to apply Theorem 1 with

$f_n := P_n \circ f$ for $n \geqslant 1$, $f_0 := f$, $u_0^n := P_n(u_0)$, $a_n := a$. By the pointwise convergence $P_n \to I$ and the Banach-Steinhaus theorem there is $N > 0$ such that

$$\|P_n\| \leqslant N \quad (n \geqslant 0).$$

This implies that assumption (i) of Theorem 1 is satisfied because f is locally Lipschitz and fulfills locally the Caratheodory assumptions. Assumptions (ii) and (iii) of Theorem 1 follow from the pointwise convergence $P_n \to I$. Then Theorem 1 provides the conclusion. □

EXAMPLE 1. – Consider the following parabolic initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} + g(t,x,u) & \text{on }]0,1[\, \times\,]0,\pi[\\ u(t,x) = 0 & \text{on }]0,1[\, \times \{0,\pi\} \\ u(0,x) = u_0(x) & \text{on }]0,\pi[\end{cases}$$

where

- $g : [0,1] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $g(t,x,0) = 0$ when $x = 0, \pi$;
- $u_0 \in C_0^2$;

[so that one of the simplest choices for g is a polynomial in u of arbitrary order with 0 and π as roots].

We plan to apply the above theorem to implement the Faedo-Galerkin method in L^2 for the evolution equation corresponding to this parabolic problem and deduce that in reality the approximations converge uniformly on a suitable interval $[0,\delta]$ to the solution not only with respect to the L^2-norm but also in the L^∞-norm.

To reach this goal we fix $R > \|u_0\|_{C_0^2}$ and a twice continuously differentiable function $\rho : \mathbb{R} \to \mathbb{R}$ with compact support such that $\rho(x) = 1$ whenever $|x| \leqslant R$, and define

$$G(t,x,u) := \rho(u) \cdot g(t,x,u).$$

This is a C^2-function with compact support. It is well-known that the parabolic problem

$$\begin{cases} u_t = u'' + G(t,x,u) & \text{on }]0,1[\, \times\,]0,\pi[\\ u(t,x) = 0 & \text{on }]0,1[\, \times \{0,\pi\} \\ u(0,x) = u_0(x) & \text{on }]0,\pi[\end{cases}$$

is equivalent to the evolution equation

$$u' = A_1 u + f(t,u), \; u(0) = u_0 \quad \text{in } C^0 := C^0([0,\pi]) \tag{4.1}$$

as well as to the evolution equation

$$v' = A_2 v + f(t,v), \; v(0) = u_0 \quad \text{in } L^2 := L^2([0,\pi]) \tag{4.2}$$

where

• f is defined by $f(t,u) := G\big(t,\cdot,u(\,\cdot\,)\big)$;

• A_1 is the linear operator corresponding to the restriction of u'' on $\mathrm{dom}(A_1) := C_0^2$;

• A_2 is the linear operator corresponding to the restriction of u'' on $\mathrm{dom}(A_2) := \{u \in H_0^1([0,\pi]) :\, u'' \in L^2\}$].

Let $e_n(x) := \sqrt{\dfrac{2}{\pi}} \cdot \sin(nx)$, $n \geqslant 1$, be the orthonormal sequence in L^2 made of the eigenfunctions of

$$\begin{cases} w'' + \lambda w = 0 \\ w(0) = 0 = w(\pi) \end{cases}, \tag{4.3}$$

let

$$E_n := \mathrm{sp}(e_1, \ldots, e_n)$$

be the vector subspace spanned by the first n eigenfunctions and let P_n be the orthogonal projection $L^2 \to E_n$, i.e. $P_n(u) = \sum_{k=1}^{n} \hat{u}_k\, e_k$ with $\hat{u}_k$ the k^{th} Fourier coefficient of u. Let v_n be the solution to the finite-dimensional ODE

$$v' = A_2 P_n v + P_n f(t, P_n v),\ \ v(0) = P_n(u_0)$$

[which is equivalent (via the canonical identification between finite–dimensional vector spaces) to the system in $\mathbb{R}^n$

$$z' = M_n z + F_n(t,z)$$

where $M_n := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix made of the first n eigenvalues of A_2 and $F_n(t,z) := \big((f(t,z)|e_1)_{L^2}, \ldots, (f(t,z)|e_n)_{L^2}\big)$].

By Theorem 3, the sequence $(v_n)_n$ converges uniformly on compacta in the L^2-norm to the mild solution v of (4.2). We plan to deduce a stronger conclusion: $v_n \to v$ in the L^∞-norm.

We shall show below that

$$\sup_{u \in C_0^2, \|u\|_{C_0^2} \leqslant R, n \geqslant 1} \|P_n(u)\|_\infty < \infty \tag{4.4}$$

Assume for the moment that (4.4) holds true and let us finish with our analysis. As $P_n\big|_{E_n}$ is the identity mapping, each v_n satisfies also the evolution equation

$$v_n' = A_1 v_n + P_n f(t, v_n),\ \ v_n(0) = P_n(u_0) \tag{4.5}$$

in C^0. As the semigroup generated by A_1 is compact and the Fourier series of eigenfunctions of $u_0 \in C_0^2$ converges in the L^∞-norm, (4.4) allows to apply

Lemma 01 to (4.5) in the space C^0 and conclude that $(v_n)_n$ has compact closure in C^0. Consequently every subsequence has a convergent subsequence in C^0. The limit is necessarily v because the topology of C^0 is finer than the topolgy of L^2. This implies that $v_n \to v$ uniformly in the L^∞-norm. Thus the solution v to (4.2) will be continuous with respect to the L^∞-norm, so that $\|v(t)\|_\infty \leqslant R$ for $0 \leqslant t \leqslant \delta$, $\delta > 0$ suitable, hence v is the solution on $]0, \delta[\times]0, \pi[$ of the original parabolic problem. This proves the claimed convergence of the Faedo-Galerkin approximations.

To finish with the proof we have to state (4.4). We shall follow some ideas from the treatment of Fourier series of eigenfunctions in VIDOSSICH [9]. Fix $\mu < 0$. Since μ is not an eigenvalue of (4.3), the following BVP

$$\begin{cases} w'' + \mu \cdot w = -h(x) \\ w(0) = 0 = w(\pi) \end{cases} \tag{4.6}$$

has a unique solution for every $h \in L^2$, denoted hereafter by $S_\mu(h)$. We claim that

(i) $\|h\|_{L^2} \leqslant M$ and $\|S_\mu(h)\|_{L^2} \leqslant N \quad \Rightarrow \quad \|S_\mu(h)\|_\infty \leqslant \text{const} =: K = K(M, N)$;

(ii) the relation between the partial sums of the Fourier series of h and of $S_\mu(h)$ is

$$S_\mu\Big(\sum_{i=1}^n \hat{h}_i \cdot e_i\Big) = \sum_{i=1}^n \big(\widehat{S_\mu(h)}\big)_i \cdot e_i,$$

i.e. $\sum_{i=1}^n \big(\widehat{S_\mu(h)}\big)_i \cdot e_i$ is the unique solution of

$$\begin{cases} w'' + \mu \cdot w = -\displaystyle\sum_{i=1}^n \hat{h}_i \cdot e_i \\ w(0) = 0 = w(\pi) \end{cases} . \tag{4.7}$$

We prove only (i), because (ii) is nothing else than a mere computation (and is the content of Example 4 in § 13.2 of VIDOSSICH[10]). To state (i), assume $\|h\|_{L^2} \leqslant M$ and $\|S_\mu(h)\|_{L^2} \leqslant N$. To simplify notations, set $w := S_\mu(h)$. From (4.6) we get

$$\|w''\|_{L^2} \leqslant -\mu N + M.$$

Since w is C^1 with w' absolutely continuous, there is x_0 such that $w'(x_0) = 0$. Then we have

$$|w'(0)| \leqslant \int_0^{x_0} |w''(y)|\, dy \leqslant \int_0^{\pi} |w''(y)|\, dy \leqslant \|w''\|_{L^2} \cdot \pi^{1/2} \leqslant (-\mu N + M) \cdot \pi^{1/2}$$

by the Cauchy-Schwarz inequality. This bound depends only on M and N. Now we have simply to apply Gronwall lemma in the following inequality

$$\begin{aligned}|w(x)| &= |w'(0)\cdot x + \int_0^x (x-y)\cdot\{-\mu\cdot w(y) - h(y)\}\,dy| \\ &\leqslant \pi|w'(0)| + \|h\|_{L^2}\cdot\pi^{1/2} + \int_0^x (x-y)\cdot\{-\mu\,|w(y)|\}\,dy \\ &\leqslant (-\mu N + M)\cdot\pi^{3/2} + M\pi^{1/2} + \pi\int_0^x \{-\mu\,|w(y)|\}\,dy\end{aligned}$$

and get an a priori bound for $\|w\|_\infty$ depending only on M and N. This establishes (i).

Now we are ready to prove (4.4). Choose any $w \in C_0^2$ with $\|w\|_{C_0^2} \leqslant R$ and set

$$h := -w'' - \mu w. \tag{4.8}$$

Clearly $S_\mu(h) = w$. Consequently $w_n := \sum_{i=1}^n \hat{w}_i \cdot e_i$ is the unique solution of (4.7) by virtue of (ii). Moreover, Bessel inequality implies

$$\|w_n\|_{L^2} \leqslant \|w\|_{L^2} \leqslant \pi^{1/2}\|w\|_\infty \leqslant \pi^{1/2}R,$$

while

$$\|h\|_{L^2} \leqslant \|w''\|_{L^2} - \mu\|w\|_{L^2} \leqslant \pi^{1/2}\|w''\|_\infty - \mu\pi^{1/2}\|w\|_\infty \leqslant \pi^{1/2}(1-\mu)R.$$

Then (i) imply that

$$\|w_n\|_\infty \leqslant \text{const} =: C = C(R) \quad \text{for all } n \geqslant 1$$

and (4.4) follows because $P_n w = \sum_{i=1}^n \hat{w}_i \cdot e_i = w_n$. □

EXAMPLE 2. – Consider the following hyperbolic initial-boundary value problem:

$$\begin{cases} u_{tt} = \Delta u + g(u) & \text{on }]0,b[\, \times \Omega \\ u(t,x) = 0 & \text{on }]0,b[\, \times \partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega \\ u_t(0,x) = v_0(x) & \text{on } \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions

- $g(0) = 0$;
- $|g(x) - g(y)| \leqslant \text{const}\cdot(1 + |x|^a + |y|^a)\cdot|x-y|$ for all $x, y \in \mathbb{R}$ with $0 \leqslant a < \infty$ a constant such that $(N-2)a \leqslant 2$;

and $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$.

As shown in § 6.2 of CAZENAVE-HARAUX [3], the given problem is equivalent to the evolution equation

$$U' = AU + F(U)$$

in the Hilbert space $X := H_0^1(\Omega) \times L^2(\Omega)$ with $U := (u, v)$, $F(u, v) := \big(0, g(u)\big)$ and

$$A := \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with $\operatorname{dom}(A) := \{(u, v) \in X : \Delta u \in L^2(\Omega), v \in H_0^1(\Omega)\}$.

By Proposition 6.1.5 of CAZENAVE-HARAUX [3], F is Lipschitz on bounded sets of X. Let E_n be the vector space spanned by the eigenvectors of the first n eigenvalues of the Laplacian in $H_0^1(\Omega)$. Then $E := \bigcup_n E_n$ is dense in $H_0^1(\Omega)$ as well as in $L^2(\Omega)$, hence $E \times E$ is dense in X. Let $P_n : X \to E_n \times E_n$ be the orthogonal projection. Obviuosly $E_n \times E_n$ is invariant by A and contained in $\operatorname{dom}(A)$. Then we apply Theorem 3 and conclude that the solutions to the finite dimensional Cauchy problems

$$U' = AP_nU + P_nf(t, P_nU), \; U(0) = P_n(u_0, v_0)$$

converge uniformly on compacta in the norm of X to a mild solution of the above evolution equation. □

5. – Applications to the existence and uniqueness of periodic solutions.

In this section we use the previous theorems to determine the existence and uniqueness of periodic mild solutions. Roughly speaking, the existence proofs are based on Theorem 1, while the uniqueness proofs on Theorem 2.

To start with, we prove a perturbation type result in the spirit of the implicit function theorem. Next, we extend to evolution equations Theorems 4 and 5 proved in CASTRO-LAZER [2] for scalar parabolic equations by the method of upper and lower solutions [a tecnique not applicable to systems, contrary to the present one]. This also answers the open problem raised by BECKER [1, § 4] for the case in which the non-linearity is below the first eigenvalue of $-A$ [showing that the position of the eigenvalues of the periodic problem for $u' - A$ has no influence in the present case, contrary to Becker's conjecture].

THEOREM 4. – *Let X, Y be Banach spaces, A the generator of a compact semigroup $T(t)$ on X, $U \subseteq X$ and $V \subseteq Y$ open sets. Let $f : \mathbb{R}^+ \times U \times V \to X$ be continuously differentiable with $f(\cdot, u, \lambda)$ p-periodic for every $u \in U$ and $\lambda \in V$. If u_0 is a p-periodic mild solution to*

$$u' = Au + f(t, u, \lambda_0)$$

such that the variational equation in X

$$z' = Az + f_u\big(t, u_0(t), \lambda_0\big) \cdot z \tag{5.1}$$

has only the trivial solution as p-periodic mild solution, then there exists $\delta > 0$ such that

$$u' = Au + f(t, u, \lambda)$$

has a unique p-periodic mild solution whenever $|\lambda - \lambda_0| \leqslant \delta$.

PROOF. – Let $\varphi(t, x, \lambda)$ be the value at t of the unique mild solution to

$$u' = Au + f(t, u, \lambda), \;\; u(0) = x.$$

By virtue of Lemma 02, we have to show the solvability of the equation

$$\varphi(p, x, \lambda) = x$$

for λ close to λ_0. We plan to apply the implicit function theorem to

$$F(\lambda, x) := x - \varphi(p, x, \lambda)$$

by considering $x = x(\lambda)$ a function of λ. The function F is continuously differentiable in a neighborhood of $\big(\lambda_0, u_0(0)\big)$ because $\varphi(p, \cdot, \cdot)$ is continuously differentiable in a neighborhood of $\big(u_0(0), \lambda_0\big)$ by Theorem 2. For $\lambda = \lambda_0$ we have

$$F\big(\lambda_0, u_0(0)\big) = u_0(0) - \varphi\big(p, u_0(0), \lambda_0\big) = 0$$

in view of the p-periodicity of $u_0 = \varphi(\cdot, u_0(0), \lambda_0)$. Thus to apply the implicit function theorem we need only to show that $\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big)$ is an invertible linear operator. We have

$$\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big) = I - \frac{\partial}{\partial x} \varphi\big(p, u_0(0), \lambda_0\big)$$

and, by Theorem 2, $v(t) := \frac{\partial}{\partial x} \varphi\big(t, u_0(0), \lambda_0\big)$ is the solution of the Volterra equation

$$v(t) = T(t) + \int_0^t T(t-s) \circ f_u\big(s, u_0(s), \lambda_0\big) \circ v(s)\, ds \tag{5.2}$$

in $\mathcal{L}(X)$. Consequently we have the representation

$$v(p) = T(p) + \int_0^p T(p-s) \circ f_u\big(s, u_0(s), \lambda_0\big) \circ v(s)\, ds.$$

We use the argument in the proof of Theorem 3.1 in PAZY [5] to show that $v(p)$ is a compact linear operator on X. For $0<\varepsilon<p$ we define $v_\varepsilon : X \to X$ by

$$\begin{aligned} v_\varepsilon &:= T(p) + \int_0^{p-\varepsilon} T(p-s) \circ f_u\big(s, u_0(s), \lambda_0\big) \circ v(s)\, ds \\ &= T(p) + T(\varepsilon) \circ \int_0^{p-\varepsilon} T(p-s-\varepsilon) \circ f_u\big(s, u_0(s), \lambda_0\big) \circ v(s)\, ds. \end{aligned}$$

Since $T(p)$ and $T(\varepsilon)$ are compact linear operators, v_ε is a compact linear operator $X \to X$. Since $T(\,\cdot\,), f_u\big(\cdot, u_0(\,\cdot\,), \lambda_0\big)$ and $v(\,\cdot\,)$ are bounded on $[0,p]$, there exists a constant $K > 0$ such that

$$\|v(p) - v_\varepsilon\| \leqslant \int_{p-\varepsilon}^{p} \|T(p-s) \circ f_u\big(s, u_0(s), \lambda_0\big) \circ v(s)\|\, ds \leqslant \varepsilon K$$

and so $\lim_{\varepsilon \downarrow 0} v_\varepsilon = v(p)$ in $\mathcal{L}(X)$ and $v(p)$ is a compact linear operator. This implies that $\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big)$ is a compact perturbation of the identity. Consequently, by the Fredholm alternative, $\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big)$ is invertible if and only if its kernel contains only the origin. Now, if x belongs to this kernel, then x is the initial value of a p-periodic mild solution to (5.1) by the last assertion of Theorem 2 and by Lemma 02. Then the uniqueness of the p-periodic mild solution to (5.1) implies that the kernel of $\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big)$ is reduced to the origin, and so $\frac{\partial}{\partial x} F\big(\lambda_0, u_0(0)\big)$ is invertible and the implicit function theorem applies. In view of Lemma 02, $F(\lambda, x) = 0$ means that x is the initial value of a p-periodic mild solution to $u' = Au + f(t, u, \lambda)$, and so we are done. □

EXAMPLE 3. – Consider the following periodic boundary value problem:

$$\begin{cases} u_t = \Delta u + g(t, x, u) + h(t) & \text{on }]0, +\infty[\times \mathbb{R}^k \\ u(t, \cdot)\ q\text{-periodic} & \text{for every } t \\ u(\cdot, x)\ p\text{-periodic} & \text{for every } x \in \Omega \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}^k$ is continuous and p-periodic, $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuously differentiable with $g(\cdot, x, u)$ p-periodic for every (x, u) and $g(t, \cdot, u)$ q-periodic for every (t, u). If there exist p-periodic functions u_0, h_0 such that $h \in C^1(\mathbb{R})$ and u_0 is a solution of the given problem when $h = h_0$ and the variational equation

$$\begin{cases} v_t = \Delta u + g_u\big(t, x, u_0(t)\big)\, v + h_0(t) & \text{on }]0, +\infty[\times \mathbb{R}^k \\ v(t, \cdot)\ q\text{-periodic} & \text{for every } t \\ v(\cdot, x)\ p\text{-periodic} & \text{for every } x \in \Omega \end{cases}$$

has only the trivial solution as p-periodic mild solution, then there exists $\delta > 0$ such that the given problem has a unique p-periodic mild solution whenever $\|h - h_0\|_{C^1} < \delta$.

This claim is a direct consequence of the previous theorem. In fact, consider the Banach space X [resp.: Y] of q-periodic [resp.: p-periodic] continuous fumctions $\mathbb{R} \to \mathbb{R}^k$ endowed with the L^∞-norm. Following the patterns of the example in § 6 of PAZY [5], we see that the given problem is equivalent to a periodic problem for an evolution equation

$$u' = Au + G(t, u) + h(t)$$

where A is the generator of a compact semigroup $T(t)$ on X, and $G(t, u) := g\big(t, \cdot, u(\,\cdot\,)\big)$. Then we have simply to apply Theorem 4 with $U := X$, $V := Y$, $\lambda := h, f(t, u, \lambda) := G(t, u) + h(t)$. □

THEOREM 5. – *Let X be a real Hilbert space and A the generator of a compact semigroup* $T(t)$ *on X. Assume that:*

- *A is symmetric and there is* $\lambda_0 > 0$ *such that*

$$(Ax|x) \leqslant -\lambda_0 \|x\|^2$$

for all $x \in \mathrm{dom}(A)$.

Let $f : \mathbb{R}^+ \times X \to X$ *be locally Lipschitz, bounded on bounded sets with* $f(\cdot, x)$ *p-periodic for every x and*

$$\limsup_{\|x\| \to \infty} \frac{\big(f(t, x)|x\big)}{\|x\|^2} < \lambda_0.$$

Then the evolution equation

$$u' = Au + f(t, u)$$

has p-periodic mild solutions and their initial values form a compact set in X.

PROOF. – We plan to use the Leray-Schauder topological degree. To this aim we start by considering a family of evolution equations. For every $0 \leqslant \tau \leqslant 1$ let $\varphi_\tau(t, x)$ be the value at t of the unique mild solution to

$$u' = Au + \tau f(t, u), \;\; u(0) = x.$$

By Lemma 02, the initial values of p-periodic mild solutions to the above evolution equation satisfy the identity

$$x = \varphi_\tau(p, x)$$

i.e. they are the fixed points of $\varphi_\tau(p, \cdot)$. As

$$\varphi_\tau(p, x) = T(p)x + \tau \int_0^p T(p - s) f\big(s, \varphi_\tau(s, x)\big)\, ds,$$

$\varphi_\tau(p,\cdot)$ is a completely continuous operator by Lemma 01. We claim that there is an a priori bound ρ for the fixed points of all $\varphi_\tau(p,\cdot)$. Assume the contrary, i.e. the existence of sequences $(x_n)_n$ and $(\tau_n)_n$ such that $\|x_n\| \to \infty$ and x_n is a fixed point of $\varphi_{\tau_n}(p,\cdot)$, and argue for a contradiction. To simplify notations, set $u_n(t) := \varphi_{\tau_n}(t, x_n)$. By the above, u_n is a p-periodic function. We shall reach the desired contradiction by showing that the u_n's are uniformly bounded. To this aim we fix n and constants $0<\lambda<\lambda_0$ and $R>0$ such that $x_n \neq 0$ and

$$\|x\| \geqslant R \quad \Rightarrow \quad \big(f(t,x)\big|x\big) \leqslant \lambda \|x\|^2$$

[as allowed by the definition of lim sup]. Consider the finite-dimensional Cauchy problems

$$u' = AP_k u + \tau_n P_k f(t, P_k u),\ u(a) = P_k\big(u_n(0)\big) \tag{5.3}$$

where P_k is the orthogonal projection of X onto the vector space spanned by the eigenspaces of A corresponding to its first k eigenvalues counting multiplicities. As A is the generator of a compact semigroup, its resolvent operators are compact (Theorem 1.1 of Pazy [5]). In view of the assumptions, $0 \in \rho(A)$ and consequently A^{-1} exists and is a compact operator. As A^{-1} is also symmetric, the union of the eigenspaces of A^{-1} is dense in X. As A and A^{-1} have the same eigenvectors, the $P_k(X)$'s are finite-dimensional. Moreover, f is locally Lipschitz. Thus we are in the position to apply Theorem 3: if $u_{n,k}$ is the unique solution to (5.3), then $\lim_k u_{n,k} = u_n$ uniformly on compacta. We claim that

($\star$) *For k sufficiently large, there is $t_k \in [0,p]$ such that $\|u_{n,k}(t_k)\| \leqslant R$.*

In fact, if $\|u_{n,k}(t)\| \geqslant R$ for all $t \in [0,p]$ and infinitely many k's, then for these k's we have

$$\begin{aligned}\frac{d}{dt}\|u_{n,k}(t)\|^2 &= 2\Big(P_k A u_{n,k}(t) + \tau_n P_k f\big(t, P_k u_{n,k}(t)\big)\Big| u_{n,k}(t)\Big)\\ &\qquad [\text{ as } A \text{ commutes with } P_k]\\ &= 2\Big(Au_{n,k}(t)\Big|u_{n,k}(t)\Big) + \tau_n\Big(f\big(t, P_k u_{n,k}(t)\big)\Big|P_k u_{n,k}(t)\Big)\\ &\leqslant 2(-\lambda_0+\lambda)\|u_{n,k}(t)\|^2.\end{aligned}$$

Calling $M := -\lambda_0 + \lambda$, from classical scalar differential inequalities in the finite-dimensional space $P_k(X)$ we get

$$\|u_{n,k}(t)\|^2 \leqslant v(t) \quad \text{whenever} \quad 0 \leqslant t \leqslant p$$

where v is the unique solution to the scalar Cauchy problem

$$\begin{cases} v' = 2Mv \\ v(0) = \|u_n(0)\|^2 \end{cases}.$$

Consequently $\|u_n(t)\|^2 \leqslant v(t)$ for $0 \leqslant t \leqslant p$ as $\lim_k u_{n,k} = u_n$. Since $M < 0$, we have

$$\|u_n(p)\|^2 \leqslant v(p) = \|u_n(0)\|^2 \, \mathrm{e}^{2Mp} < \|u_n(0)\|^2 = \|u_n(p)\|^2$$

which is a contradiction showing that ($\star$) holds true. Now set

$$N := \sup_{0 \leqslant t \leqslant p, \|x\| \leqslant R} \|f(t,x)\| \quad \text{and} \quad J_k := \{t \geqslant 0 : \|u_{n,k}(t)\| \leqslant R\}.$$

In view of ($\star$), $J_k \neq \emptyset$. In $P_k(X)$ we have

$$\begin{aligned} \frac{d}{dt}\|u_{n,k}(t)\|^2 &= 2\Big(P_k A u_{n,k}(t) + \tau_n P_k f\big(t, P_k u_{n,k}(t)\big)\big|u_{n,k}(t)\Big) \\ &\leqslant 2\begin{cases} -\lambda_0\|u_{n,k}(t)\|^2 + N\|u_{n,k}(t)\| & \text{if } t \in J_k, \\ (-\lambda_0 + \lambda)\|u_{n,k}(t)\|^2 & \text{otherwise} \end{cases} \\ &\leqslant 2M\|u_{n,k}(t)\|^2 + 2N\|u_{n,k}(t)\| \end{aligned}$$

where again $M := -\lambda_0 + \lambda$. Then from classical scalar differential inequalities we get

$$\|u_{n,k}(t)\| \leqslant w_k(t) \quad \text{whenever} \quad t_k \leqslant t \leqslant 2p$$

with w_k the unique solution to the scalar Cauchy problem

$$\begin{cases} w_k' = Mw_k + N \\ w_k(t_k) = R \end{cases}.$$

For, $z := w_k^2$ is a solution of $z' = 2Mz + 2N\sqrt{z}$ and $y := \|u_{n,k}(\,\cdot\,)\|^2$ satisfies the differential inequality $y' \leqslant 2My + 2N\sqrt{y}$. Passing to a subsequence if necessary, we assume $t_k \to t_\infty$ for a suitable $t_\infty \in [0,p]$. By the continuous dependence of solutions for scalar Cauchy problems, $w_k \to w_\infty$ uniformly on compacta, w_∞ being the unique solution to

$$\begin{cases} w_\infty' = Mw_\infty + N \\ w_\infty(t_\infty) = R \end{cases}.$$

This and $\lim_k u_{n,k} = u_n$ uniformly on compacta imply that

$$\|u_n(t)\| \leqslant w_\infty(t) + 1 \quad \text{whenever} \quad p \leqslant t \leqslant 2p$$

and consequently the u_n's are uniformly bounded in view of their p-periodicity. This is a contradiction, hence the existence of the a priori bound ρ is established. By the homotopy invariance of the topological degree we have

$$\deg\big(I - \varphi_1(p,\cdot), B, 0\big) = \deg\big(I - \varphi_0(p,\cdot), B, 0\big)$$

where B is the open ball centered at the origin with radius ρ. As φ_0 is the solution operator corresponding to $u' = Au$ whose only p-periodic mild solution is the trivial one (because $\|T(t)x\| \leqslant \text{const} \cdot e^{-\lambda_0 t}$),

$$\deg(I - \varphi_0(p, \cdot), B, 0) = \pm 1$$

by a well-known degree property related to compact linear operators. Consequently

$$\deg\big(I - \varphi_1(p, \cdot), B, 0\big) = \deg\big(I - T(p), B, 0\big) = \pm 1.$$

Then the solution property of the topological degree and Lemma 02 guarantee the existence of p-periodic mild solutions to the given evolution equation as φ_1 is its solution operator.

Finally, we note that the compactness of a bounded, closed set of fixed point of a completely continuous operator is well-known, hence we are done. □

COROLLARY. – *Let X and A be as in the statement of Theorem* 5. *Let $f : \mathbb{R}^+ \times X \to X$ be continuously differentiable, bounded on bounded sets and satisfy the following conditions:*

- *$f(\cdot, u)$ is p-periodic for every u;*
- *the partial derivative $f_u(t, x)$ is a symmetric linear operator for every t and x;*
- *there exists a constant $0 < \lambda < \lambda_0$ such that $(f_u(t, x) \cdot z | z) \leqslant \lambda \|z\|^2$ for every t, x and z.*

Then the evolution equation

$$u' = Au + f(t, u)$$

has exactly one p-periodic mild solution.

PROOF. – Let $\varphi(t, x)$ be the value at t of the unique mild solution to

$$u' = Au + f(t, u), \ u(0) = x.$$

In the proof of Theorem 5 we have seen that the initial values x of p-periodic mild solutions to the above evolution equation satisfy the identity

$$x - \varphi(p, x) = 0,$$

that $\varphi(p, \cdot)$ is a completely continuous operator with

$$\deg\big(I - \varphi(p, \cdot), B, 0\big) = \deg\big(I - T(p), B, 0\big) = \pm 1$$

where B is an open ball centered at the origin and containing the set F of all the fixed points of $\varphi(p, \cdot)$. The set F is non-empty and compact by Theorem 5. Fix $x \in F$. Following the patterns of the proof of Theorem 4, we see that $\frac{\partial}{\partial y} \varphi(p, x)$ is a compact linear operator on X and that $z = \frac{\partial}{\partial y} \varphi(p, x) \cdot z$ if and only if z is the initial value of a p-periodic mild solution to

$$v' = Av + f_u\big(t, \varphi(t,x)\big) \cdot v. \tag{5.4}$$

As

$$(f_u(t, \varphi(t,x)) \cdot y|y) \leqslant \lambda \|y\|^2, \tag{5.5}$$

applying Theorem 5 to (5.4) we see that the z's satisfying $z = \dfrac{\partial}{\partial y}\varphi(p,x) \cdot z$ form a compact set K. As K is also a vector space by the linearity of equation (5.4), it follows that $K = \{0\}$. Then $I - \dfrac{\partial}{\partial y}\varphi(p,x)$ is an invertible linear operator. This has two consequences:

(i) every point of F has a neighborhood where $I - \varphi(p,\cdot)$ is injective by virtue of the local inversion theorem. Consequently F is finite in view of its compactness. Call N the number of points in F;

(ii) a theorem of Leray-Schauder implies that

$$\deg\big(I - \varphi(p,\cdot), B, 0\big) = \deg\big(I - \frac{\partial}{\partial y}\varphi(p,x), B, 0\big)$$

where $B(x, \varepsilon_x)$ is the open ball centered at x with radius ε_x sufficiently small. In view of (5.5), we can repeat for the evolution equation (5.4) the argument in the proof of Theorem 5 and get

$$\deg\big(I - \frac{\partial}{\partial y}\varphi(p,x), B, 0\big) = \deg\big(I - T(p), B, 0\big).$$

Consequently

$$\deg\big(I - \varphi(p,\cdot), B(x,\varepsilon_x), 0\big) = \deg\big(I - T(p), B, 0\big).$$

Then (ii), (i) and the additivity property of the topological degree imply that

$$\begin{aligned}\deg\big(I - T(p), B, 0\big) &= \deg\big(I - \varphi(p,\cdot), B, 0\big) = \sum_{x\in F} \deg\big(I - \varphi(p,\cdot), B(x,\varepsilon_x), 0\big)\\ &= N \cdot \deg\big(I - T(p), B, 0\big).\end{aligned}$$

The only possibility for the validity of this identity is $N = 1$ because $\deg\big(I - T(p), B, 0\big) \neq 0$. $\square$

REFERENCES

[1] R. I. BECKER, *Periodic solutions of semilinear equations of evolution of compact type*, J. Math. Anal. Appl., **82** (1981), 33-48.

[2] A. CASTRO - A. C.LAZER, *Results on periodic solutions of parabolic equations suggested by elliptic theory*, Boll.U.M.I., **1**-B (1982), 1089-1104.

[3] T. Cazenave - A. Haraux, *An Introduction to Semilinear Evolution Equations*, Clarendon Press, Oxford, 1998.
[4] M. W. Hirsch - C. C. Pugh, *Stable manifolds for hyperbolic sets*, In: "Proc. Symp. Pure Math.", **vol. xiv**, Amer. Math. Society (Providence, 1970), 133-163.
[5] A. Pazy, *A class of semi-linear equations of evolution*, Israel J. Math., **20** (1975), 23-36.
[6] J. Sotomayor, *Smooth dependence of solutions of differential equations on initial data: a simple proof*, Bol. Soc. Brasil. Mat., **4** (1973), 55-59.
[7] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics* (Springer-Verlag, New York, 1988).
[8] G. Vidossich, *Continuous dependence for parabolic evolution equations*, In: "Recent Trends in Differential Equations", R. P. Agarwal (ed.), World Scientific (Singapore, 1992), 559-568.
[9] G. Vidossich, *A (semi-encyclopedic) first course in ODEs*, in a "never-ending" preparation since 1974.
[10] J. R. Ward Jr., *Boundary value problems for differential equations in Banach space*, J. Math. Anal. Appl., **70** (1979), 589-598.

SISSA, Via Beirut 2-4, 34014 Trieste, Italy
E-mail: vidossic@sissa.it

Received March 19, 2009 and in revised form July 8, 2009