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Teresa Radice, Eero Saksman, Gabriella Zecca

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Isomorphisms of Royden Type Algebras Over $\mathbb{S}^1$

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Abstract. – *Let $\mathbb{S}^1$ and $\mathbb{D}$ be the unit circle and the unit disc in the plane and let us denote by $\mathcal{A}(\mathbb{S}^1)$ the algebra of the complex-valued continuous functions on $\mathbb{S}^1$ which are traces of functions in the Sobolev class $W^{1,2}(\mathbb{D})$. On $\mathcal{A}(\mathbb{S}^1)$ we define the following norm*

$$\|u\| = \|u\|_{L^\infty(\mathbb{S}^1)} + \left(\iint_{\mathbb{D}} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}}$$

where $\tilde{u}$ is the harmonic extension of u to $\mathbb{D}$.

We prove that every isomorphism of the functional algebra $\mathcal{A}(\mathbb{S}^1)$ is a quasi-symmetric change of variables on $\mathbb{S}^1$.

1. – Introduction.

Recent years have seen an intensive development of quasiconformal analysis and its relations to other areas in mathematics. We refer the reader to the monograph [2]. An interesting phenomenon to this direction is that for many function spaces (resp. function algebras) quasiconformal maps can be characterized as homeomorphisms such that the induced composition operator provides an isomorphism of the function space (resp. algebra).

One of the most interesting results in this direction was provided by H.M. Riemann in 1974. His result [9] shows that the BMO space of functions of bounded mean oscillation is of significance in connection with quasiconformal mappings of $\mathbb{R}^n$. More precisely, a K-quasiconformal homeomorphism of Ω onto Ω', $f : \Omega \to \Omega'$ induces a linear isomorphism

$$f^\sharp : BMO(\Omega') \to BMO(\Omega),$$

where $f^\sharp(u) = u \circ f$ for $u \in BMO(\Omega')$. The norm of $f^\sharp$ is bounded by a constant depending only on n and K,

$$\|u \circ f\|_{BMO(\Omega)} \leqslant C(K, n)\|u\|_{BMO(\Omega')}.$$

Conversely, under certain regularity assumptions on the homeomorphism $f : \Omega \to \Omega'$ the induced map $f^\sharp$ defines an isomorphism between $BMO(\Omega')$ and $BMO(\Omega)$ only if f is quasiconformal.

Similar type results remain true for other function spaces, we just mention here [1], [5], [10]. In this connection (and independently) it is natural to ask when two domains Ω and Ω' are quasiconformally equivalent; that is, if there exists a quasiconformal mapping $f : \Omega \to \Omega'$. In general, this problem is extremely difficult even in the plane. However, one implicit characterization of quasiconformally equivalent domains Ω and Ω' is that $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega')$, their respective Royden algebras, are algebraically isomorphic.

Given a domain Ω in $\mathbb{R}^n$, if $W^{1,n}(\Omega)$ denotes the Sobolev space of functions, the Royden algebra of Ω is defined as the algebra of functions $u \in C(\Omega) \cap W^{1,n}(\Omega)$ with the norm

$$\|u\| = \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^n(\Omega)}.$$

It is easily shown that a quasiconformal mapping $f : \Omega \to \Omega'$ always induces an algebra isomorphism between $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega)$. Conversely, an algebra isomorphism $T : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega')$ always induces a quasiconformal mapping of Ω onto Ω'. These theorems were proven for the two dimensional case by M. Nakai in [8], then generalized for n dimensions by L. G. Lewis in [7].

Let $\mathbb{S}^1$ and $\mathbb{D}$ be the unit circle and the unit disc in the plane and let us denote by $\mathcal{A}(\mathbb{S}^1)$ the function algebra of the (real-valued) continuous functions on $\mathbb{S}^1$ which are traces of functions in the Sobolev class $W^{1,2}(\mathbb{D})$, i.e.

$$\mathcal{A}(\mathbb{S}^1) = C(\mathbb{S}^1) \cap H^{\frac{1}{2}}(\mathbb{S}^1),$$

where $H^{\frac{1}{2}}$ stands for the standard Sobolev space of L^2-functions with half-derivative in L^2. We will equip $\mathcal{A}(\mathbb{S}^1)$ with the following norm

$$\|u\| = \|u\|_{L^\infty(\mathbb{S}^1)} + \left(\iint_{\mathbb{D}} |\nabla \tilde{u}|^2 \right)^{\frac{1}{2}} \tag{1.1}$$

where $\tilde{u}$ is the harmonic extension of u in $\mathbb{D}$. In the same spirit our aim here is to prove that every algebra isomorphism $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ induces a homeomorphism of $\mathbb{S}^1$ onto $\mathbb{S}^1$ (Theorem 1.3). The following

Proposition 1.1. – *Let $u, v \in \mathcal{A}(\mathbb{S}^1)$. Then*

$$\|uv\| \leqslant \|u\| \|v\| \tag{1.2}$$

(see Section 2 for the proof) implies that $\mathcal{A}(\mathbb{S}^1)$ is a normed algebra under pointwise multiplication, with the constant function 1 as identity. It is a real commutative Banach algebra with identity.

A classical theorem of Beurling and Ahlfors (see [3]) states that, given a homeomorphism f from the unit circle $\mathbb{S}^1$ onto itself, a necessary and sufficient condition for the existence of a quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ such that

$F|_{S^1} = f$ is that the homeomorphism f is *quasisymmetric*, i.e. there exists $D \geqslant 1$ such that for every $x \in \mathbb{R}$ and every $t \in (0, \pi)$

$$\frac{1}{D} \leqslant \frac{df(\Delta_{x+t,x})}{df(\Delta_{x,x-t})} \leqslant D \tag{1.3}$$

where df is the distributional derivative of f, $\Delta_{x',x''}$, $x' < x''$, denotes the (smaller) arc of the circle $\mathbb{S}^1$ with extremal points $e^{ix'}$ and $e^{ix''}$ respectively. The infimum of such constants D is named "*quasisymmetry constant*" of f. We have the following precise result.

THEOREM 1.2. – *Let $F : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal map and let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the quasisymmetric boundary-homeomorphism induced by F, i.e. $f(z) = \lim_{w \to z} F(z)$, $z \in \mathbb{S}^1$. Then, for any $v \in \mathcal{A}(\mathbb{S}^1)$ the double inequality*

$$\frac{1}{\sqrt{K}} \|v\| \leqslant \|v \circ f\| \leqslant \sqrt{K} \|v\| \tag{1.4}$$

holds.

PROOF. – Let $v \in \mathcal{A}(\mathbb{S}^1)$ and let $\tilde{u}$ be the harmonic extension of the composed function $u = v \circ f$ on $\mathbb{D}$. Obviously $\|u\|_\infty = \|v\|_\infty$. Moreover, let us observe that

$$\tilde{u}|_{\mathbb{S}^1} = (\tilde{v} \circ F)|_{\mathbb{S}^1} = u.$$

Since the harmonic extension $\tilde{u}$ minimizes the energy, by the chain rule formula

$$\iint_{\mathbb{D}} |\nabla \tilde{u}|^2 \leqslant \iint_{\mathbb{D}} |\nabla(\tilde{v} \circ F)|^2 dy = \iint_{\mathbb{D}} |\nabla \tilde{v}(F(y))({}^t DF)|^2 dy$$

$$\leqslant \iint_{\mathbb{D}} |\nabla \tilde{v}(F(y))|^2 |{}^t DF|^2 dy = \iint_{\mathbb{D}} |\nabla \tilde{v}(F(y))|^2 |DF|^2.$$

On the other hand, F is K-quasiconformal, hence for a.e. $y \in \mathbb{D}$ the distortion inequality

$$|DF(y)|^2 \leqslant K J_F(y)$$

holds, where we have denoted by J_F the Jacobian determinant of F. By a simple change of variables $x = F(y)$,

$$\iint_{\mathbb{D}} |\nabla \tilde{u}|^2 \leqslant K \iint_{\mathbb{D}} |\nabla \tilde{v}(F(y))|^2 J_F(y) dy$$

$$= K \iint_{\mathbb{D}} |\nabla \tilde{v}(x)|^2 dx < +\infty.$$

Hence, $u = v \circ f \in \mathcal{A}(\mathbb{S}^1)$ and

$$\|v \circ f\| \leqslant \|v\|_\infty + K^{\frac{1}{2}} \left(\iint_{\mathbb{D}} |\nabla \tilde{v}|^2 \right)^{\frac{1}{2}}.$$

It is well known that the inverse function F^{-1} of the K- quasiconformal map F is K- quasiconformal. Hence, the inverse function f^{-1} of f is the quasisymmetric homeomorphism induced by F^{-1}, i.e. $F^{-1}_{|_{\mathbb{S}^1}} = f^{-1}$. Then, by previous estimate, (1.4) is completely proved. □

In other words, under the assumptions of Theorem 1.2 we have proved that f defines a linear continuous operator $T : v \in \mathcal{A}(\mathbb{S}^1) \to v \circ f \in \mathcal{A}(\mathbb{S}^1)$, such that $T(v_1 v_2) = Tv_1 Tv_2$ for $v_1, v_2 \in \mathcal{A}(\mathbb{S}^1)$. Moreover, if for each $v(y) \in \mathcal{A}(\mathbb{S}^1)$ we set $T_f v(x) = v(f(x))$, then

$$T_f : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$$

is a bounded operator and $(T_f)^{-1} = T_{f^{-1}}$. We can conclude that T_f is a bicontinuous automorphism.

We shall henceforth call $T : A(\mathbb{S}^1) \to A(\mathbb{S}^1)$ an *isomorphism* if T is both topological and algebraic isomorphism. In other words, T is a bicontinuous bijection (i.e. an homeomorphism) that commutes with addition and multiplication. Our main result is the following

THEOREM 1.3. – *Let $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ be a algebra isomorphism. Then, there exists a quasisymmetric homeomorphism*

$$f : \mathbb{S}^1 \to \mathbb{S}^1$$

such that for each $u \in \mathcal{A}(\mathbb{S}^1)$ we have $(Tu)(x) = u(f(x))$ for any $x \in \mathbb{S}^1$, i.e.

$$Tu = u \circ f. \tag{1.5}$$

Let us observe that, in other words, Theorem 1.3 together with Theorem 1.2 says that the isomorphisms of the functional algebra $\mathcal{A}(\mathbb{S}^1)$ are in one to one correspondence with quasisymmetric changes of variables on $\mathbb{S}^1$. Also, the same results remain true for complex algebras $\mathcal{A}(\mathbb{S}^1)$, see Remark 3.4 at the end of Section 3.

2. – Preliminary results. Properties of an isomorphism $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$.

Let us first give the Proof (of Proposition 1.1).

PROOF. – Since the harmonic extension $w = \tilde{uv}$ minimizes the energy,

$$\|uv\| = \|uv\|_\infty + \left(\iint_{\mathbb{D}} |\nabla(\tilde{uv})|^2\right)^{\frac{1}{2}}$$

$$\leqslant \|uv\|_\infty + \left(\iint_{\mathbb{D}} |\nabla(\tilde{u}\tilde{v})|^2\right)^{\frac{1}{2}}.$$

Now, let us observe that

$$\nabla(\tilde{u}\tilde{v}) = (\nabla\tilde{u}) \cdot \tilde{v} + \tilde{u} \cdot (\nabla\tilde{v}),$$

and by the inequality $\|uv\|_\infty \leqslant \|u\|_\infty \|v\|_\infty$,

$$\|uv\| \leqslant \|u\|_\infty \|v\|_\infty + \left(\iint_{\mathbb{D}} |(\nabla\tilde{u}) \cdot \tilde{v} + \tilde{u} \cdot (\nabla\tilde{v})|^2\right)^{\frac{1}{2}}$$

$$\leqslant \|u\|_\infty \|v\|_\infty + \|(\nabla\tilde{u}) \cdot \tilde{v}\|_{L^2(\mathbb{D})} + \|\tilde{u} \cdot (\nabla\tilde{v})\|_{L^2(\mathbb{D})}.$$

By the maximal principle for the harmonic functions $\tilde{u}$ and $\tilde{v}$ we obtain

$$\|uv\| \leqslant \|u\|_\infty \|v\|_\infty + \|v\|_{L^\infty(\mathbb{S}^1)} \|\nabla\tilde{u}\|_{L^2(\mathbb{D})} + \|u\|_{L^\infty(\mathbb{S}^1)} \|\nabla\tilde{v}\|_{L^2(\mathbb{D})}$$

$$\leqslant \left(\|u\|_\infty + \|\nabla\tilde{u}\|_{L^2(\mathbb{D})}\right) \left(\|v\|_\infty + \|\nabla\tilde{v}\|_{L^2(\mathbb{D})}\right)$$

$$= \|u\|\|v\|.$$

This completes our proof. □

The next two lemmata list some properties of an isomorphism $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ that will be useful in the sequel. We will employ the following observation: an element $f \in \mathcal{A}(\mathbb{S}^1)$ is invertible (i.e. it has a multiplicative inverse) if and only if $f(w) \neq 0$ for every $w \in \mathbb{S}^1$. The necessity is obvious, and in order to deduce the sufficiency we employ the Douglas condition characterizing boundary functions whose harmonic extension have finite Dirichlet energy (see [4] and [2, p. 595]). In particular,

$$\|f\|^2 \sim \|f\|^2_{L^\infty} + \|f\|^2_{\dot{H}^{1/2}} < \infty,$$

where

$$\|f\|_{\dot{H}^{1/2}} := \left(\iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} |dx||dy|\right)^{1/2} \tag{2.1}$$

stands for the Douglas seminorm. If f does not vanish anywhere, the continuity of f implies that $|f| \geq c > 0$ on $\mathbb{S}^1$, and it follows that $1/f \in \mathcal{A}(\mathbb{S}^1)$ by observing that $|1/f(x) - 1/f(y)|^2 \leq c^{-4}|f(x) - f(y)|^2$.

LEMMA 2.1. – *Let* $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ *be an isomorphism. Then*

1. $T(c) = c$, *for any constant c. Especially,* $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ *is a continuous linear operator.*

2. $(Tf)(\mathbb{S}^1) = f(\mathbb{S}^1)$ *for any* $f \in \mathcal{A}(\mathbb{S}^1)$.

3. Let $c \in \mathbb{R}$; *then,* $\forall v \in \mathcal{A}(\mathbb{S}^1)$, $v \geqslant c \Rightarrow Tv \geqslant c$. *In particular,* Tv *preserves non-negativity (or non-positivity) of* v.

PROOF. – To prove *1.* let us observe that $T(1) = T(1^2) = (T(1))^2$. This means that

$$T(1) \in \{0, 1\}. \tag{2.2}$$

By continuity either $T(1)$ is identically zero or takes the constant value 1. The first case is ruled out since $T(1) = 0$ should imply for every $v \in \mathcal{A}(\mathbb{S}^1)$ that $T(v) = 0$. Hence $T(1) = 1$, and since by induction $T(nf) = nTf$ for any f and $n \in \mathbb{Z}$, we have $T(c) = c$ for all $c \in \mathbb{Q}$, which generalizes for all $c \in \mathbb{R}$ by continuity.

Equality *2.* is deduced by using *1.* as follows:

$$\lambda \in f(\mathbb{S}^1) \Longleftrightarrow f - \lambda \text{ is not invertible } \Longleftrightarrow T(f) - \lambda \text{ is not invertible} \Longleftrightarrow \lambda \in (Tf)(\mathbb{S}^1).$$

In turn, statement *3.* is an immediate consequence of part *2.* □

3. – Proof of Theorem 1.3.

Assume that $T : \mathcal{A}(\mathbb{S}^1) \to \mathcal{A}(\mathbb{S}^1)$ is an isomorphism. We start by *complexifying* both T and $\mathcal{A}(\mathbb{S}^1)$. The elements of the complex algebra $\mathcal{A}(\mathbb{S}^1)$ (with abuse we denote the complexification by the same symbol) are complex valued and complex functions on $\mathbb{S}^1$ such that the norm (1.1) is finite. Then the function $f : \mathbb{S}^1 \to \mathbb{C}$ belongs to the complex algebra if and only if both the real and imaginary part of f belong to (real) $\mathcal{A}(\mathbb{S}^1)$.

The isomorphism T is extended to the complexified algebra by setting for any real valued $u, v \in \mathcal{A}(\mathbb{S}^1)$

$$T(u + \imath v) = Tu + \imath Tv.$$

Let us remark that in this way we preserve the additivity and the multiplicativity of T. Indeed we have

$$T(u_1 + \imath v_1 + u_2 + \imath v_2) = T(u_1 + u_2) + \imath T(v_1 + v_2)$$

$$= T(u_1) + T(u_2) + \imath[T(v_1) + T(v_2)]$$

$$= T(u_1) + \imath T(v_1) + T(u_2) + \imath T(v_2)$$

$$= T(u_1 + \imath v_1) + T(u_2 + \imath v_2).$$

Moreover

$$T((u_1 + \imath v_1)(u_2 + \imath v_2)) = T(u_1u_2 - v_1v_2 + \imath(u_1v_2 + u_2v_1))$$

$$= T(u_1u_2 - v_1v_2) + \imath T(u_1v_2 + u_2v_1)$$

$$= Tu_1Tu_2 - Tv_1Tv_2 + \imath(Tu_1Tv_2 + Tu_2Tv_1)$$

$$= (Tu_1 + \imath Tv_1)(Tu_2 + \imath Tv_2)$$

$$= T(u_1 + \imath v_1)T(u_2 + \imath v_2).$$

It is obvious that the extended T yields an isomorphism of the (complex) algebra $\mathcal{A}(\mathbb{S}^1)$.

Now, let us denote by $e^{\imath\theta}$ the embedding of $\mathbb{S}^1$ in $\mathbb{C}$, i.e.

$$z = e^{\imath\theta} \in \mathbb{S}^1 \longrightarrow e^{\imath\theta} \in \mathbb{C}.$$

We have $e^{\imath\theta} \in \mathcal{A}(\mathbb{S}^1)$ so that $Te^{\imath\theta} \in \mathcal{A}(\mathbb{S}^1)$. Moreover, $T(e^{\imath\theta}) = (Tcos\theta) + \imath(T\sin\theta)$ where $(Tcos\theta)$ and $(T\sin\theta)$ are real functions in $\mathcal{A}(\mathbb{S}^1)$. Let us define f as the image by T of $e^{\imath\theta}$:

$$f(z) = \big(Te^{\imath\theta}\big)(z), \quad \forall z \in \mathbb{S}^1.$$

By definition $f \in \mathcal{A}(\mathbb{S}^1)$ and, obviously, it is the only candidate for the mapping satisfying (1.5).

We continue with several lemmata.

LEMMA 3.1. – *1. The complexified algebra and the isomorphism T satisfy*

$$(Tg)(\mathbb{S}^1) = g(\mathbb{S}^1) \quad \text{for any } g \in \mathcal{A}(\mathbb{S}^1).$$

2. The function f is unimodular: $|f(w)| = 1$ for all $w \in \mathbb{S}^1$. Moreover, if $g : \mathbb{S}^1 \to \mathbb{C}$ is a trigonometric polynomial, $g(u) = \sum_{n=-N}^{N} a_n u^n$, we have

$$(Tg)(w) = g(f(w)), \quad w \in \mathbb{S}^1.$$

PROOF. – The proof of part *1.* is identical with the proof in the real case (Lemma 2.1.*2.*). Since $e^{\imath\theta}$ takes values in $\mathbb{S}^1$, by part *1.* the same holds for $f = T(e^{\imath\theta})$. Especially, f is unimodular. Since $1 = e^{\imath\theta}e^{-\imath\theta}$ an application of T yields that $1 = f\overline{f}$, i.e. $\overline{f} = 1/f$. Part 2. follows immediately by the isomorphism property of g. □

We next observe that a suitable density argument allows us to replace the trigonometric polynomial g in part *2.* of the previous lemma by an arbitrary element from $\mathcal{A}(\mathbb{S}^1)$.

LEMMA 3.2. – *1. Let $g \in \mathcal{A}(\mathbb{S}^1)$ have the Fourier series $g(w) = \sum_{n=-\infty}^{\infty} \widehat{g}_n w^n$ for $w \in \mathbb{S}^1$. Then*

$$\|g\| \sim \|g\|_{L^\infty} + \left(\sum_{n=-\infty}^{\infty} n|\widehat{g}_n|^2\right)^{1/2}$$

2. Finite trigonometric polynomials are dense in $\mathcal{A}(\mathbb{S}^1)$.
3. For arbitrary $g \in \mathcal{A}(\mathbb{S}^1)$ it holds that $(Tg)(w) = g(f(w))$ for all $w \in \mathbb{S}^1$.

PROOF. – Part *1.* is well-known, but for readers convenience we sketch a proof. Let $g \in \mathcal{A}(\mathbb{S}^1)$ with the Fourier series

$$g(w) = \sum_{n=-\infty}^{\infty} \widehat{g}_n w^n. \tag{3.1}$$

Since g is continuous on $\mathbb{S}^1$, the harmonic extension of in $\mathbb{D}$ is

$$\tilde{g}(z) = \sum_{k=0}^{\infty} \widehat{g}_k z^k + \sum_{k=1}^{\infty} \widehat{g}_k \overline{z}^k.$$

We may then compute using ortogonality

$$\iint_{\mathbb{D}} |\nabla\tilde{u}|^2 = 2\iint_{\mathbb{D}} \left(\left|\frac{\partial\tilde{u}}{\partial z}\right|^2 + \left|\frac{\partial\tilde{u}}{\partial\overline{z}}\right|^2\right)$$

$$= 2\iint_{\mathbb{D}} \left(\left|\sum_{k=1}^{\infty} k\widehat{g}_k z^{k-1}\right|^2 + \left|\sum_{k=1}^{\infty} k\widehat{g}_{-k}\overline{z}^{k-1}\right|^2\right)$$

$$= 2\sum_{k=1}^{\infty} k^2|\widehat{g}_k|^2 \iint_{\mathbb{D}} |z|^{2k-2} + 2\sum_{k=1}^{\infty} k^2|\widehat{g}_{-k}|^2 \iint_{\mathbb{D}} |\overline{z}|^{2k-2}$$

$$= 2\pi \sum_{k=1}^{\infty} k\left(|\widehat{g}_k|^2 + |\widehat{g}_{-k}|^2\right),$$

where we observed that $\iint\limits_{\mathbb{D}} |z|^{2k-2} = 2\pi \int\limits_0^1 r^{2k-1} dr = \frac{4\pi}{2k}$. This clearly yields the claim.

In order to treat part *2.* assume again that given $g \in \mathcal{A}(\mathbb{S}^1)$ has the Fourier series (3.1) and consider then its N:th partial Fejer sum

$$g_N(w) := \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1}) u_n w^n$$

on $\mathbb{S}^1$. By part *1.* of the Lemma and dominated convergence theorem it is obvious that $\|g_N - g\|_{H^{1/2}(\mathbb{S}^1)} \to 0$ as $N \to \infty$. On the other hand, it is well-known that the Fejer partial sums also converge in $C(\mathbb{S}^1)$ (see e.g. [6, Thm 2.12]), whence $\|g_N - g\|_{C(\mathbb{S}^1)} \to 0$. Hence $\lim\limits_{N\to\infty} \|g_N - g\|_{\mathcal{A}(\mathbb{S}^1)} = 0$, which yields the stated density.

Finally, in order to treat claim *3* assume that $g \in \mathcal{A}(\mathbb{S}^1)$ is arbitrary and pick by part *2.* a sequence (g_N) of trigonometric polynomials with the property $\lim\limits_{N\to\infty} \|g_N - g\|_{\mathcal{A}(\mathbb{S}^1)} = 0$. By continuity of the isomorphism T we also have $\lim\limits_{N\to\infty} \|T(g_N) - T(g)\|_{\mathcal{A}(\mathbb{S}^1)} = 0$. By invoking part *2.* of Lemma 3.1 we especially see that

$$\|g_N \circ f - T(g)\|_{C(\mathbb{S}^1)} \to 0 \quad \text{as } N \to \infty. \tag{3.2}$$

The uniform convergence of g_N to g on $\mathbb{S}^1$ enables us to conclude immediately that $\lim\limits_{N\to\infty} \|g_N \circ f - g \circ f\|_{C(\mathbb{S}^1)} = 0$. As we combine this with (3.2) it follows that $T(g) = g \circ f$, which was to be shown. □

Observe that the last part of the previous Lemma shows that the isomorphism T actually is a composition operator with symbol f. Next we verify that f is one to one, i.e. T is induced by a homeomorphic change of variables.

LEMMA 3.3. – *The function $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism.*

PROOF. – Since the function $e^{i\theta}$ takes all the values on $\mathbb{S}^1$, also $f = T(e^{i\theta}) : \mathbb{S}^1 \to \mathbb{S}^1$ is surjective by part *1.* of Lemma 3.1. In turn, if f would not be injective, we could find distinct $w_1, w_2 \in \mathbb{S}^1$ such that $f(w_1) = f(w_2)$. According to part *3.* of the previous Lemma this yields that $T(g)(w_1) = g(f(w_1) = g(f(w_2)) = T(g)(w_2)$ for all $g \in \mathcal{A}(\mathbb{S}^1)$, which is impossible as T is surjective. Since $\mathbb{S}^1$ is compact, the bijective and continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism. □

We have now established Equality (1.5). Finally, what remains to verify is that f is quasisymmetric. Assume that the homeomorphism $f : T \to T$ is not quasisymmetric. We may assume that f is sense-preserving. Since f and f^{-1} are simultaneously quasisymmetric, there are adjacent intervals $I_k, I'_k \subset \mathbb{S}^1$ of the same length such that $|I_k| \to 0$ as $k \to \infty$ and such that $|f^{-1}(I_k)|/|f^{-1}(I'_k)|$ is not bounded. It is enough to construct a sequence of test functions $g_k \in \mathcal{A}(\mathbb{S}^1)$ that are supported on $2(I_k \cup I'_k)$ with $\|g_k\|_\infty = 1$ for all k and such that

$$\|g_k \circ f\|_{\dot{H}^{1/2}} / \|g_k\|_{\dot{H}^{1/2}} \to \infty \quad \text{as } k \to \infty.$$

Since the situation and our computations below are local, we may do the computations on the real axis. Moreover, by picking a subsequence and employing the scaling, reflection, and translation invariance of both (2.1) and quasisymmetry on the real line, we may arrive at the following situation: functions $f_1, f_2, \dots$ are dilated and translated (perhaps also suitably reflected) copies of a suitable restriction of f in such a way that

$$f_k([-1,0]) = [-1,0], \quad f_k([0,\varepsilon_k]) = [0,1] \quad \text{for each } k \geq 1,$$

where $\varepsilon_k \to 0$ as $k \to \infty$. In this situation it is enough to find a fixed compactly supported function $g \in H^{1/2}(\mathbb{R}) \cap C(\mathbb{R})$ that satisfies

$$\|g \circ f_k\|_{\dot{H}^{1/2}} / \|g\|_{\dot{H}^{1/2}} \to \infty \quad \text{as } k \to \infty.$$

We define our test function as follows:

$$g(x) = \begin{cases} 0 & \text{for} \quad x < -2 \\ x+2 & \text{for} \quad -2 \leqslant x < -1 \\ 1 & \text{for} \quad -1 \leqslant x < 0 \\ 1-x & \text{for} \quad 0 \leqslant x < 1 \\ 0 & \text{for} \quad x \geqslant 1 \end{cases} \tag{3.3}$$

Clearly $g \in H^{1/2}(\mathbb{R}) \cap C(\mathbb{R})$, and it suffices to show that the Douglas seminorm of $F_k := g \circ f_k$ blows up as $k \to \infty$. For that end one just needs to observe that $F_{k|[-1,0]} = 1$ and $F_{k|[\varepsilon_k,1]} = 0$. Namely, this leads to

$$\begin{aligned} \|F_k\|^2_{\mathcal{A}(\mathbb{S}^1)} &\geqslant \int_{-1}^{1}\int_{-1}^{1} \left(\frac{|F_k(x) - F_k(y)|}{|x-y|} \right)^2 dx dy \\ &\geqslant c \int_{-1}^{0} \left(\int_{\varepsilon_k}^{1} |x-y|^{-2} dx \right) dy \sim \log\left(\frac{1}{\varepsilon_k}\right), \end{aligned}$$

which proves the claim.

Remark 3.4. – Theorem 1.3 remains valid for any isomorphism of the complex algebra $\mathcal{A}(\mathbb{S}^1)$ as well. The proof remains unchanged. In particular, the counterparts of parts *1.* and *2.* of Lemma 2.1 remain valid. In a similar way one shows that $T(\imath) = \pm\, \imath$ (recall that we did not assume an isomorphism to be complex linear). Especially, from part *2.* it follows that an isomorphism keeps real things real, whence any (complex) isomorphism T is simply obtained as the complexification (or its conjugate) of the restriction of T on real functions.

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Teresa Radice, Dipartimento di Matematica e Applicazioni “R. Caccioppoli”.
Università degli Studi di Napoli “Federico II”
Via Cintia - Complesso Universitario Monte S. Angelo 80126 Napoli, Italy.
E-mail: teresa.radice@unina.it

Eero Saksman, Department of Mathematics and Statistics,
University of Helsinki, P.O. Box 68, 00014 Helsinki, Finland.
E-mail: eero.saksman@helsinki.fi

Gabriella Zecca, Dipartimento di Matematica e Applicazioni “R. Caccioppoli”.
Università degli Studi di Napoli “Federico II”
Via Cintia - Complesso Universitario Monte S. Angelo, 80126 Napoli, Italy.
E-mail: g.zecca@unina.it

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