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Sufficient Conditions for Integrability of Distortion Function $K_{f^{-1}}$

COSTANTINO CAPOZZOLI

Abstract. – *Assume that* Ω, Ω' *are planar domains and* $f : \Omega \xrightarrow{\text{onto}} \Omega'$ *is a homeomorphism belonging to Sobolev space* $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ *with finite distortion. We prove that if the distortion function* K_f *of* f *satisfies the condition* $\text{dist}_{EXP}(K_f, L^\infty) < 1$, *then the distortion function* $K_{f^{-1}}$ *of* f^{-1} *belongs to* $L^1_{\text{loc}}(\Omega')$. *We show that this result is sharp in sense that the conclusion fails if* $\text{dist}_{EXP}(K_f, L^\infty) = 1$. *Moreover, we prove that if the distortion function* K_f *satisfies the condition* $\text{dist}_{EXP}(K_f, L^\infty) = \lambda$ *for some* $\lambda > 0$, *then* $K_{f^{-1}}$ *belongs to* $L^p_{\text{loc}}(\Omega')$ *for every* $p \in \left(0, \frac{1}{2\lambda}\right)$. *As special case of this result we show that if the distortion function* K_f *satisfies the condition* $\text{dist}_{EXP}(K_f, L^\infty) = 0$, *then* $K_{f^{-1}}$ *belongs to intersection of* $L^p_{\text{loc}}(\Omega')$ *for all* $p \geq 1$.

1. – Introduction.

Recently there is a growing interest in studying properties of homeomorphisms, which can be proved also for the inverse maps (see [16], [13], [12], [10], [9], [8], [11]). For example let Ω and Ω' be planar domains and let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism, we have that if f belongs to Sobolev space $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and the differential Df vanishes almost everywhere on the zero set of Jacobian J_f of f, then also $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^2)$ and the differential Df^{-1} vanishes almost everywhere on the zero set of Jacobian $J_{f^{-1}}$ of f^{-1} (see [9]).

We are mainly concerned with homeomorphisms having finite distortion. Recall that a homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ has *finite distortion* if there is a measurable function $K(z) \geq 1$, finite almost everywhere, such that

$$|Df(z)|^2 \leq K(z) J_f(z) \quad \text{for a.e. } z \in \Omega. \tag{1}$$

Here $|Df(z)|$ stands for the operator norm of the differential matrix $Df(z) \in R^{2\times 2}$ defined by

$$|Df(z)| = \sup_{|h|=1} |Df(z)h|$$

and the Jacobian J_f belongs to Lebesgue space $L^1_{\text{loc}}(\Omega)$ and for every Borel set $B \subset \Omega$

$$\int_B J_f(z)dz \leq |f(B)|$$

(see [2], Corollary 3.3.6).

Inequality (1) is called *distortion inequality* for f. Observe that this inequality merely asks that the pointwise Jacobian $J_f(z) \geq 0$ for a.e. $z \in \Omega$ and that the differential $Df(z)$ vanishes at those points z where $J_f(z) = 0$. Geometrically, it means that at almost every point $z \in \Omega$ the differential $Df(z) : \mathbb{R}^2 \to \mathbb{R}^2$ deforms the unit disk onto an ellipse whose eccentricity is controlled by $K(z)$. Thus, in particular, the case $K = 1$ results in conformal deformations.

Given a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ having finite distortion, we define the *distortion function* of f, K_f, as

$$K_f(z) = \begin{cases} \dfrac{|Df(z)|^2}{J_f(z)} & \text{if } Df(z) \text{ exists and } J_f(z) > 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

Notice that K_f is the smallest function $K(z) \geq 1$ for which the distortion inequality (1) holds.

If $K_f \in L^\infty(\Omega)$, $K_f(z) \leq K$ for a.e. $z \in \Omega$, we say that f is *K-quasiconformal*. Clearly, in this case $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ and it is well known that also f^{-1} is K-quasiconformal i.e. $K_{f^{-1}} \in L^\infty(f(\Omega))$ and $K_{f^{-1}}(w) \leq K$ for a.e. $w \in f(\Omega)$ (see [2], Theorem 3.1.2).

Our results deal with the integrability of the distortion function $K_{f^{-1}}$ of f^{-1} in more general case.

Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism belonging to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ with finite distortion, we suppose that its distortion function K_f belongs to $L^1(\Omega)$ at least. In fact, in this case f^{-1} has finite distortion (see [9]) and therefore we can consider the distortion function $K_{f^{-1}}$ of f^{-1}. On the other hand the assumption K_f belonging to $L^1(\Omega)$ is also interesting because Hencl-Koskela obtain a better regularity for the inverse f^{-1} of f and precisely $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$. Moreover, they show that if K_f belongs to $L^{1-\delta}(\Omega)$, with $\delta \in (0, 1)$, then we may have that f^{-1} does not belong to $W^{1,1+\delta}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ (see example 1.4 in [9]). On the contrary in [13] the authors prove that if $f \in W^{1,p}(\Omega, \mathbb{R}^2)$, for some $p \in (1, 2]$, is a homeomorphism having finite distortion with distortion function K_f satisfying

$$M = \sup_{\delta \in (0,1)} \left(\delta \int_\Omega K_f(z)^{1-\delta}\, dz \right)^{\frac{1}{1-\delta}} < \infty,$$

then Df^{-1} belongs to grand Lebesgue space $L^{2)}(\Omega', \mathbb{R}^2)$, i.e

$$\|Df^{-1}\|_{L^{2)}(\Omega',\mathbb{R}^2)} = \sup_{\varepsilon\in(0,1)} \left(\varepsilon \int_{f(\Omega)} |Df^{-1}(w)|^{2-\varepsilon}\, dw \right)^{\frac{1}{2-\varepsilon}} < \infty.$$

Observe that K_f belonging to $L^1(\Omega)$ does not imply that $K_{f^{-1}}$ belongs to $L^1(\Omega')$, but even if K_f belongs to Orlicz space $EXP(\Omega)$, i.e. there exists $\lambda > 0$ for which $\int_\Omega e^{\frac{K_f(z)}{\lambda}} dz < \infty$, we may have that $K_{f^{-1}}$ does not belong to $L^1(\Omega')$ (see example in Section 3).

However denoting by $Hom(\Omega, \Omega')$ the set of all homeomorphisms between Ω and Ω' planar domains, we prove the following sufficient conditions for integrability of distortion function $K_{f^{-1}}$.

THEOREM 1. – *Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ having finite distortion. If the distortion function $K_f \in EXP(\Omega)$ satisfies the condition*

$$\text{dist}_{EXP}(K_f, L^\infty) < 1,$$

then

$$K_{f^{-1}} \in L^1_{\text{loc}}(\Omega').$$

This result is sharp in sense that the conclusion fails if $\text{dist}_{EXP}(K_f, L^\infty) = 1$.

THEOREM 2. – *Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ having finite distortion. If the distortion function $K_f \in EXP(\Omega)$ satisfies the condition*

$$\text{dist}_{EXP}(K_f, L^\infty) = \lambda \ \textit{for some} \ \lambda > 0,$$

then

$$K_{f^{-1}} \in L^p_{\text{loc}}(\Omega') \ \textit{for every} \ p \in \left(0, \frac{1}{2\lambda}\right).$$

COROLLARY 1. – *Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ having finite distortion. If the distortion function $K_f \in EXP(\Omega)$ satisfies the condition*

$$\text{dist}_{EXP}(K_f, L^\infty) = 0,$$

then

$$K_{f^{-1}} \in \bigcap_{p\geq 1} L^p_{\text{loc}}(\Omega').$$

The definition of $\text{dist}_{EXP}(\varphi, L^\infty)$ is given in Section 2 and we will prove Theorem 1, Theorem 2 and Corollary 1 in Section 3.

2. – Notations and preliminary results.

Let us first recall that given a square matrix A, the adjugate of A satisfies

$$A \,\mathrm{adj}A = I \,\det A \tag{3}$$

where $\det A$ denotes the determinant of A and I is the identity matrix.

An Orlicz function is a continuously increasing function

$$\mathcal{P} : [0, \infty) \to [0, \infty)$$

verifying

$$\mathcal{P}(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{P}(t) = \infty.$$

The Orlicz space $L^{\mathcal{P}}(\Omega)$ consist of those Lebesgue measurable functions φ defined in $\Omega \subset \mathbb{R}^2$ and valued in $\mathbb{R}$ such that

$$\int_\Omega \mathcal{P}\left(\frac{|\varphi(z)|}{\lambda}\right) dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$ (see [14]).

We denote by $EXP(\Omega)$ the Orlicz space corresponding to the Orlicz function $\mathcal{P}(t) = e^t - 1$. It consists of those measurable functions $\varphi : \Omega \to \mathbb{R}$ such that

$$\int_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$. $EXP(\Omega)$ is equipped with the Luxemburg norm

$$\|\varphi\|_{EXP(\Omega)} = \inf\left\{ \lambda > 0 : \fint_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz \leq 2 \right\}$$

where

$$\fint_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz = \frac{1}{|\Omega|} \int_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz.$$

Another Orlicz space of interest to us will be the Zygmund space $L^p \log^\beta L(\Omega)$ corresponding to the Orlicz function $\mathcal{P}(t) = t^p \log^\beta(e + t)$ with $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. It consists of those measurable functions $\varphi : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \left(\frac{|\varphi(z)|}{\lambda}\right)^p \log^\beta\left(e + \frac{|\varphi(z)|}{\lambda}\right) dz < \infty$$

for some $\lambda = \lambda(\varphi) > 0$. Also $L^p \log^\beta L(\Omega)$, with $\beta \geq 1 - p$, is equipped with the

Luxemburg norm

$$\|\varphi\|_{L^p \log^\beta L(\Omega)} = \inf\left\{ \lambda > 0 : \fint_\Omega \left(\frac{|\varphi(z)|}{\lambda}\right)^p \log^\beta\left(e + \frac{|\varphi(z)|}{\lambda}\right) dz \le 1 \right\}.$$

Notice that both are Banach spaces and that $EXP(\Omega)$ is the dual to the Zygmund space $L \log L(\Omega)$.

Let us recall that $L^\infty(\Omega)$ is not dense in $EXP(\Omega)$ and that in [3] (see also [6]) the authors established the following formula of the distance to $L^\infty(\Omega)$ for every function φ in $EXP(\Omega)$

$$\begin{aligned} \text{dist}_{EXP}(\varphi, L^\infty) &= \inf\{\psi \in L^\infty(\Omega) : \|\varphi - \psi\|_{EXP(\Omega)}\} \\ &= \inf\left\{ \lambda > 0 : \fint_\Omega e^{\frac{|\varphi(z)|}{\lambda}} dz < \infty \right\}. \end{aligned} \tag{4}$$

In particular we have that $\text{dist}_{EXP}(\varphi, L^\infty) = 0$, i.e. φ belongs to closure of $L^\infty(\Omega)$ in $EXP(\Omega)$, if and only if

$$e^{\frac{\varphi}{\lambda}} \in L^1(\Omega) \ \text{ for every } \ \lambda > 0.$$

Let $\Omega \subset \mathbb{R}^2$ be a domain. Every continuous open mapping defined on Ω having finite first partial derivatives almost everywhere in Ω, it is differentiable almost everywhere in Ω in the classical sense (see [7]). As every continuous mapping $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ is absolutely continuous on a.e. line parallel to the coordinate axes (see [17]) and therefore has finite first partial derivatives almost everywhere in Ω we have the following

LEMMA 1. – *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$. Then f is differentiable almost everywhere in Ω in the classical sense.*

DEFINITION 1. – *Let $f : \Omega \to \mathbb{R}^2$ be a measurable mapping. We say that f satisfies Lusin's condition $\mathcal{N}$ if for every measurable set $E \subset \Omega$*

$$|E| = 0 \qquad \Rightarrow \qquad |f(E)| = 0.$$

Recall that if $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$, then f satisfies Lusin's condition $\mathcal{N}$ (see [2], Theorem 3.3.7).

Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$, $B \subset \Omega$ a Borel set and let η a nonnegative Borel-measurable function on $\mathbb{R}^2$, we have

$$\int_B \eta(f(z))|J_f(z)|\, dz \le \int_{f(B)} \eta(w)\, dw.$$

This follows from [5, Theorem 3.1.8] together with the area formula for Lipschitz mapping. In particular, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ is an orientation preserving, i.e. $J_f(z) \geq 0$ for a.e. $z \in \Omega$, satisfying Lusin's condition $\mathcal{N}$ we have

$$\int_B \eta(f(z)) J_f(z)\, dz = \int_{f(B)} \eta(w)\, dw, \tag{5}$$

so

$$\int_B J_f(z) dz = |f(B)|$$

and

$$J_f(z) > 0 \ \text{ for a.e. } z \in \Omega.$$

Combining Theorems 1.3 and 6.1 of [9], Theorem 2.1 of [11] and a result due to Greco-Sbordone-C. Trombetti (see [8]) we can state the following result

THEOREM 3. – *If* $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$ *has finite distortion with*

$$K_f \in L^1(\Omega),$$

then

1. $J_f > 0$ *a.e. in* Ω;
2. $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ *has finite distortion and*

$$\int_{\Omega'} |Df^{-1}(w)|^2 dw = \int_{\Omega} K_f(z) dz;$$

3. $K_{f^{-1}}$ *has the form*

$$K_{f^{-1}}(w) = K_f(f^{-1}(w)) \ \textit{ for } \text{a.e. } w \in \Omega'. \tag{6}$$

Observe that, since $f \in \mathrm{Hom}(\Omega, \Omega')$, K_f and $K_{f^{-1}}$ defined at (2) and (6), are Borel-measurable functions. Moreover, if we assume only that the homeomorphism f belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, we may have that f^{-1} does not belong to $W^{1,1}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$. Indeed, consider the mapping $f : (0,2) \times (0,1) \to \mathbb{R}$ defined by

$$f(x, y) = (g^{-1}(x), y),$$

where g^{-1} is the inverse map of $g(t) = t + \varphi(t)$, with $\varphi : (0,1) \to (0,1)$ the Cantor ternary function. We have that f is a homeomorphism in $W^{1,\infty}_{\mathrm{loc}}$ whose inverse f^{-1} is of bounded variation, but it does not belong to $W^{1,1}_{\mathrm{loc}}$.

Recently, in [1] the authors obtained the following optimal regularity for Jacobian and for differential of a mapping with exponentially integrable distortion function.

THEOREM 4. – *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a mapping having finite distortion. Assume that the distortion function $K_f(z)$ satisfies the condition*

$$e^{\frac{K_f}{\lambda}} \in L^1_{\mathrm{loc}}(\Omega) \text{ for some } \lambda > 0.$$

Then we have

$$J_f \log^p(e + J_f) \in L^1_{\mathrm{loc}}(\Omega) \text{ for every } p \in \left(0, \frac{1}{\lambda}\right)$$

and

$$|Df|^2 \log^{p-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega) \text{ for every } p \in \left(0, \frac{1}{\lambda}\right).$$

Moreover this result is sharp in sense that the conclusion fails for $p = \frac{1}{\lambda}$ for every $\lambda > 0$.

As a special case of Theorem 4 we have

COROLLARY 2. – *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a mapping having finite distortion. Assume that the distortion function K_f satisfies the condition*

$$e^{\frac{K_f}{\lambda}} \in L^1_{\mathrm{loc}}(\Omega) \text{ for some } \lambda < 1.$$

Then

$$f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2).$$

Notice that under the same assumptions of Theorem 4, by an easy computation, we have

$$|Df|^2 \log^{-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega).$$

Indeed, we may use the elementary inequality

$$ab \le a \log(1 + a) + e^b - 1 \qquad (a, b \ge 0)$$

together with the fact that $t \mapsto \dfrac{t}{\log(e + t)}$ is an increasing function and that pointwise $|Df|^2 \le K_f J_f$ to find that

$$\frac{|Df|^2}{\log(e + |Df|^2)} \le \frac{K_f J_f}{\log(e + K_f J_f)} \le \lambda \frac{J_f}{\log(e + J_f)} \frac{K_f}{\lambda}$$

$$\le \lambda \left(J_f \frac{\log(1 + J_f / \log(e + J_f))}{\log(e + J_f)} + e^{\frac{K_f}{\lambda}} - 1 \right) \le \lambda (J_f + e^{\frac{K_f}{\lambda}} - 1)$$

for every $\lambda > 0$. We now integrate the previous estimate to obtain

$$\int_S \frac{|Df|^2}{\log(e+|Df|)} \leq 2\lambda\left(\int_S J_f + \int_S (e^{\frac{K_f}{\lambda}} - 1)\right)$$

for every $S \subset\subset \Omega$.

Finally, by Theorem 4 if $\lambda \in \left(0, \frac{1}{2}\right)$ and therefore

$$\mathrm{dist}_{EXP}(K_f, L^\infty) < \frac{1}{2}$$

we have

$$|Df|^2 \log(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega).$$

3. – Integrability of $K_{f^{-1}}$.

Let us start with following

PROOF OF THEOREM 1. – By hypothesis in particular K_f belongs to $L^1(\Omega)$, by Theorem 3 we have that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$, hence f^{-1} satisfies Lusin's condition $\mathcal{N}$. From (5) we then deduce that

$$J_{f^{-1}}(w) > 0 \quad \text{for a.e. } w \in \Omega'.$$

By Lemma 1 we know that f^{-1} is differentiable almost everywhere in Ω' in the classical sense. Moreover, we know that at each point of differentiability of f^{-1} such that $J_{f^{-1}}(w) > 0$ we have

$$Df(z) = \frac{1}{Df^{-1}(f(z))}. \tag{7}$$

By Theorem 3, f^{-1} has finite distortion. Let $T \subset\subset \Omega'$, using (3) we have

$$\int_T K_{f^{-1}}(w)\,dw = \int_T \frac{|Df^{-1}(w)|^2}{J_{f^{-1}}(w)}\,dw$$

$$= \int_T \frac{|\mathrm{adj}\,Df^{-1}(w)|^2}{J_{f^{-1}}(w)}\,dw = \int_T \frac{J_{f^{-1}}(w)}{|Df^{-1}(w)|^2}\,dw.$$

Applying (5) and (7) we obtain

$$\int_T K_{f^{-1}}(w)\,dw = \int_{f^{-1}(T)} \frac{1}{|Df^{-1}(f(z))|^2}\,dz = \int_{f^{-1}(T)} |Df(z)|^2\,dz. \tag{8}$$

Since $\mathrm{dist}_{EXP}(K_f, L^\infty) < 1$, then there exists $\lambda < 1$ such that $e^{\frac{K_f}{\lambda}} \in L^1_{\mathrm{loc}}(\Omega)$, by Corollary 2 $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ and therefore

$$K_{f^{-1}} \in L^1_{\mathrm{loc}}(\Omega').$$

□

To show that the conclusion of this theorem fails if $\mathrm{dist}_{EXP}(K_f, L^\infty) = 1$ we consider the following mapping (see [1]):

$$f(z) = \begin{cases} \dfrac{z}{|z|} \dfrac{1}{\sqrt{\log\left(e + \dfrac{1}{|z|}\right) \log\log\left(e + \dfrac{1}{|z|}\right)}} & \text{for } z \in \mathbb{D}(0,1) - \{0\} \\ 0 & \text{for } z = 0. \end{cases}$$

Indeed

$$f : \mathbb{D}(0,1) \to \mathbb{D}(0,R),$$

where $\mathbb{D}(0,1)$ denotes the disk of $\mathbb{R}^2$ centered at 0 with radius 1 and $\mathbb{D}(0,R)$ denotes the disk of $\mathbb{R}^2$ centered at 0 with radius

$$R = \frac{1}{\sqrt{\log(e+1)\log\log(e+1)}},$$

is a homeomorphism belonging to $W^{1,1}_{\mathrm{loc}}(\mathbb{D}(0,1), \mathbb{R}^2)$ with finite distortion such that the distortion function K_f satisfies

$$\mathrm{dist}_{EXP}(K_f, L^\infty) = 1,$$

indeed

$$e^{K_f} \in L^1(\mathbb{D}(0,1)) \ \text{ and } \ e^{\frac{K_f}{\lambda}} \notin L^1(\mathbb{D}(0,1)) \ \text{ for every } \ \lambda < 1,$$

while

$$K_{f^{-1}} \notin L^1(\mathbb{D}(0,R)).$$

Indeed, using the formulas in Chapter 11 of [14], we have

$$K_f(z) = \max\left\{ \frac{|z|\rho'(|z|)}{\rho(|z|)}, \frac{\rho(|z|)}{|z|\rho'(|z|)} \right\}$$

and

$$J_f(z) = \frac{\rho(|z|)\rho'(|z|)}{|z|}.$$

So

$$K_f(z) = \frac{2(1+e|z|)\log\left(e + \dfrac{1}{|z|}\right)\log\log\left(e + \dfrac{1}{|z|}\right)}{1 + \log\log\left(e + \dfrac{1}{|z|}\right)} \tag{9}$$

and

$$J_f(z) = \frac{1 + \log\log\left(e + \frac{1}{|z|}\right)}{2|z|^2(1+e|z|)\left(\log\left(e+\frac{1}{|z|}\right)\log\log\left(e+\frac{1}{|z|}\right)\right)^2}. \tag{10}$$

By (8), (9) and (10) we conclude that

$$\int_{\mathbb{D}(0,R)} K_{f^{-1}}(w)\,dw = \int_{\mathbb{D}(0,1)} |Df(z)|^2\,dz = \int_{\mathbb{D}(0,1)} K_f(z)J_f(z)\,dz$$

$$= \int_{\mathbb{D}(0,1)} \frac{1}{|z|^2\log\left(e+\frac{1}{|z|}\right)\log\log\left(e+\frac{1}{|z|}\right)}\,dz = \infty.$$

Our aim now is to prove the Theorem 2.

Proof of Theorem 2. – As in Theorem 1 we obtain that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ has finite distortion, f^{-1} satisfies Lusin's condition $\mathcal{N}$, f^{-1} is differentiable almost everywhere in Ω' in the classical sense and

$$J_{f^{-1}}(w) > 0 \quad \text{for a.e. } w \in \Omega'.$$

Moreover, we know that at each point of differentiability of f^{-1} such that $J_{f^{-1}}(w) > 0$ we have (7) and

$$J_f(z) = \frac{1}{J_{f^{-1}}(f(z))}. \tag{11}$$

Let $T \subset\subset \Omega'$ and let $p > 0$, using (3) we have

$$\int_T K_{f^{-1}}(w)^p\,dw = \int_T \frac{|Df^{-1}(w)|^{2p}}{J_{f^{-1}}(w)^p}\,dw$$

$$= \int_T \frac{|\mathrm{adj}\,Df^{-1}(w)|^{2p}}{J_{f^{-1}}(w)^p}\,dw = \int_T \frac{J_{f^{-1}}(w)^p}{|Df^{-1}(w)|^{2p}}\,dw$$

$$= \int_T \frac{J_{f^{-1}}(w)^{p-1}}{|Df^{-1}(w)|^{2p}}\,J_{f^{-1}}(w)\,dw.$$

Applying (5), (7) and (11) we obtain

$$\int_T K_{f^{-1}}(w)^p\, dw = \int_{f^{-1}(T)} \frac{J_{f^{-1}}(f(z))^{p-1}}{|Df^{-1}(f(z))|^{2p}}\, dz$$

$$= \int_{f^{-1}(T)} \frac{|Df(z)|^{2p}}{J_f(z)^p}\, J_f(z)\, dz = \int_{f^{-1}(T)} K_f(z)^p J_f(z)\, dz.$$

By inequality

$$K^p J \leq J \log^{2p}(e+J) + c(p,\lambda) e^{\frac{K}{\lambda}} \qquad (K, J, p, \lambda > 0)$$

(see [9], Lemma 5.1), we arrive at

$$\int_T K_{f^{-1}}(w)^p\, dw \leq \int_{f^{-1}(T)} \left(J_f(z) \log^{2p}(e + J_f(z)) + c(p,\lambda) e^{\frac{K_f(z)}{\lambda}} \right) dz. \tag{12}$$

By Theorem 4 we conclude

$$K_{f^{-1}} \in L^p_{\mathrm{loc}}(\Omega') \text{ for every } p \in \left(0, \frac{1}{2\lambda}\right).$$

□

Finally we prove the Corollary 1.

PROOF OF COROLLARY 1. – Since $\mathrm{dist}_{EXP}(K_f, L^\infty) = 0$, we have

$$e^{\frac{K_f}{\lambda}} \in L^1_{\mathrm{loc}}(\Omega) \text{ for every } \lambda > 0.$$

By (12) and by Theorem 4 we conclude

$$K_{f^{-1}} \in \bigcap_{p \geq 1} L^p_{\mathrm{loc}}(\Omega').$$

□

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