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Voltage-Current Characteristcs of Varistors and Thermistors

GIOVANNI CIMATTI

Abstract. – *The voltage-current characteristics of two classes of nonlinear resistors (varistors and thermistors) modelled as three-dimensional bodies is derived from the corresponding systems of nonlinear elliptic boundary value problems. Theorems of existence and uniqueness of solutions are presented, together with certain properties of monotonicity of the conductance.*

1. – Introduction.

Thermistors and varistors are highly nonlinear resistors [4] for which the electric conductivity cannot be assumed as constant, like in a ordinary resistors, but it strongly depends from the temperature (in thermistors) or from the electric field (in varistors). Their physical look does not differ from that of ordinary resistors (see figure 1).Thermistors and varistors are widely used as inrush current limiters, temperature sensors, for example in digital thermometer, self-resetting overcurrent protectors and self-regulating heating elements.

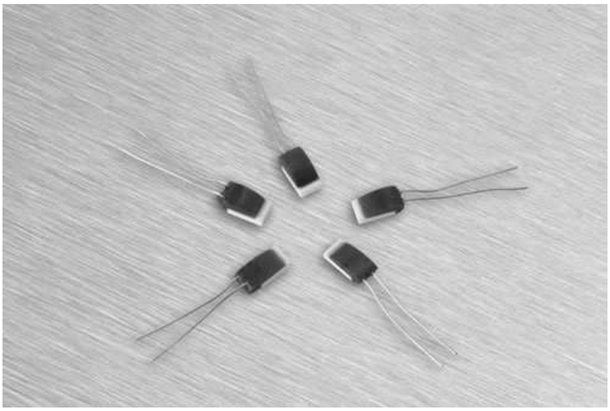

Fig. 1

In this paper we treat thermistors and varistors as three-dimensional bodies represented by open and bounded subsets Ω of $\boldsymbol{R}^3$, mainly with the goal of finding the voltage-current characteristic of these devices from general constitutive

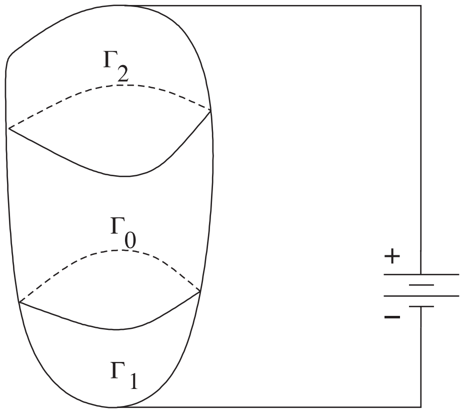

Fig. 2

equations. The regular boundary Γ of Ω consists of three parts, Γ_1, Γ_2 and Γ_0. Γ_1 and Γ_2 represent the electrodes to which a constant difference of potential V is applied. Γ_0 is the electrically insulated part of the body (see figure 2).

If $\varphi(X),\ X = (x_1, x_2, x_3)$ denotes the electric potential in Ω, then

$$\boldsymbol{J} = -\sigma\nabla\varphi \tag{1}$$

relates the electric field $-\nabla\varphi$ and the current density $\boldsymbol{J}$. From (1) and the equation $\nabla\cdot\boldsymbol{J} = 0$, expressing the conservation of charges, we obtain problem $P1$

$$\nabla\cdot(\sigma\nabla\varphi) = 0 \text{ in } \Omega,\ \ \varphi = 0 \text{ on } \Gamma_1,\ \ \varphi = V \text{ on } \Gamma_2, \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_0. \tag{2}$$

A second way of applying a difference of potential to a three-dimensional conductor is via an ordinary resistor R as in figure 3.

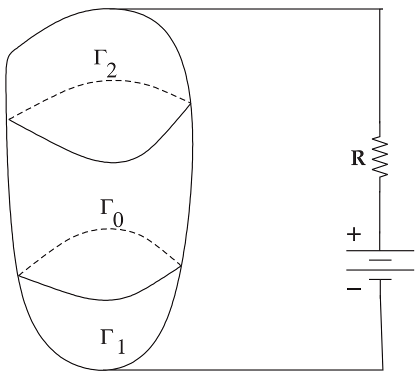

Fig. 3

This is a more realistic model, since the generator has, however small, an internal resistance. If ϕ is the unknown constant potential applied to Γ_2, we have the nonlocal elliptic problem $P2$: to find a function φ and a constant ϕ such that

$$\nabla \cdot (\sigma \nabla \varphi) = 0 \text{ in } \Omega,\ \varphi = 0 \text{ on } \Gamma_1,\ \varphi = \phi \text{ on } \Gamma_2, \tag{3}$$
$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_0,\ R\int_{\Gamma_2} \sigma \frac{\partial \varphi}{\partial n} d\Gamma = V - \phi.$$

The total current crossing the body is given in both cases by

$$I = \int_{\Gamma_2} \sigma \frac{\partial \varphi}{\partial n} d\Gamma. \tag{4}$$

When we are in the situation of figure 2, the functional relation (4) connecting V and I, crucial in the applications, is called the voltage-current characteristic of the device. When σ is a positive constant, problem $P1$ is simply the mixed problem for the laplacian. In this case the $V - I$ characteristic is the linear relation

$$I = \mathcal{K} V, \tag{5}$$

where $\mathcal{K}$ is the conductance of the body given by

$$\mathcal{K} = \sigma k. \tag{6}$$

It is easily seen that

$$k = \int_{\Gamma_2} \frac{\partial \psi}{\partial n} d\Gamma \tag{7}$$

where ψ is the solution of the problem

$$\Delta \psi = 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \Gamma_1,\ \psi = 1 \text{ on } \Gamma_2,\ \frac{\partial \psi}{\partial n} = 0 \text{ on } \Gamma_0. \tag{8}$$

Thus k depends only on Ω, Γ_1, Γ_2 and Γ_0. Equally easy is to verify that problem $P2$ has one and only one solution:

$$\varphi(X) = \frac{V}{R\mathcal{K} + 1} \psi(X), \quad \phi = \frac{V}{R\mathcal{K} + 1}$$

to which corresponds the total current

$$I = \frac{V\mathcal{K}}{R\mathcal{K} + 1}.$$

In varistors [4] the voltage-current characteristic is experimentally found to be of the form:

$$I = \mathcal{K} V^{\beta},\ \beta > 1. \tag{9}$$

The constant $\mathcal{K}$ is the conductance of the varistor. In this paper we propose a phenomenological model for three-dimensional varistors based on the nonlinear Ohm's law

$$\boldsymbol{J} = -\sigma(|\nabla\varphi|)\nabla\varphi \tag{10}$$

where the function $\sigma(t)$ is a property of the material. We assume, on physical grounds, $\sigma(t)$ continuous, defined on $[0,\infty)$ and such that

$$t \mapsto \sigma(t) \text{ is positive for } t > 0 \text{ and } \sigma(0) = 0. \tag{11}$$

In Section 2 we give a theorem of existence and uniqueness for problem $P2$, which is nonlinear if (10) holds, and prove that the model based on (10) gives precisely the empirical law (9) if $\sigma(t)$ is suitably chosen. Moreover, certain properties of dependence monotone of the conductance from the shape of the electrodes are presented. For thermistors, see [9], the Ohm's law takes the form

$$\boldsymbol{J} = -\sigma(u)\nabla\varphi \tag{12}$$

where $\sigma(u)$ is a given function of the temperature u. To have a closed system, we need to add to (2) the energy equation

$$-\nabla\cdot(\kappa(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 \tag{13}$$

and suitable boundary conditions for the temperature. The right hand side in (13) represents the Joule heating in the conductor. The system (2), (13) has been throughly investigated in recent years [7], [3], [1] under a variety of boundary conditions. A typical application of thermistor is as inrush current limiter. On "switch on" the thermistor limits the current due to its relatively high resistance at room temperature. As the electric current flows the thermistor heats reducing the value of its resistance and thus limits the damaging surge of currents to the load. In Section 6 we show, using a complex variable method, that in two-dimensional thermistors the total current is invariant for doubly connected domains with the same modulus.

2. – The Varistor.

If (10) holds Problem $P1$ becomes: $PV1$

$$\nabla\cdot(\sigma(|\nabla\varphi|)\nabla\varphi) = 0 \text{ in } \Omega,\ \varphi = 0 \text{ on } \Gamma_1,\ \varphi = V \text{ on } \Gamma_2,\ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_0. \tag{14}$$

Equation (14) is the Euler equation of the functional

$$J(\varphi) = \int_\Omega \mathcal{F}(\nabla\varphi)dx,\ \text{where } \mathcal{F} = F(|z|),\ z \in \boldsymbol{R}^3 \text{ and } F(\xi) = \int_0^\xi t\sigma(t)dt. \tag{15}$$

If we assume (11) and

$$\lim_{t\to\infty} \sigma(t)t^{2-p} = a > 0, \quad p \geq 2 \tag{16}$$

then [5] there exists $\varphi \in H$ such that

$$J(\varphi) = \inf\{J(w),\ w \in H\} \tag{17}$$

where

$$H = \{w \in H^{1,p}(\Omega),\ w = 0 \text{ on } \Gamma_1,\ w = V \text{ on } \Gamma_2\}. \tag{18}$$

Moreover, φ is a weak solution of $PV1$, i.e. satisfies

$$\varphi \in H,\ \int_\Omega \sigma(|\nabla\varphi|)\nabla\varphi \cdot \nabla v dX \text{ for all } v \in H^{1,p}(\Omega),\ v = 0 \text{ on } \Gamma_1,\ v = 0 \text{ on } \Gamma_2. \tag{19}$$

If $\sigma(t)$ is regular, φ gives a classical solution to problem $PV1$. Supposing, in addition to (11) and (16),

$$\sigma(t)t \text{ strictly increasing} \tag{20}$$

the solution of (19) is unique [11]. A variational formulation is also possible for the nonlocal problem $PV2$:

$$\nabla \cdot (\sigma(|\nabla\varphi|)\nabla\varphi) = 0 \text{ in } \Omega,\ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_0 \tag{21}$$

$$\varphi = 0 \text{ on } \Gamma_1,\ \varphi = \phi \text{ on } \Gamma_2, \tag{22}$$

$$R\int_{\Gamma_2} \sigma(|\nabla\varphi|)\frac{\partial\varphi}{\partial n} d\Gamma = V - \phi \tag{23}$$

where the unknowns are the function $\varphi(x)$ and the constant ϕ. (23) plays the role of a natural condition in the variational formulation. More precisely, let us define the functional

$$J(\varphi, \phi) = I_1(\varphi) + I_2(\phi) \text{ where} \tag{24}$$

$$I_1(\varphi) = \int_\Omega \mathcal{F}(\nabla\varphi)dX, \quad I_2(\phi) = \frac{1}{2R}(V - \phi)^2$$

which we assume, for the moment, as defined in the class

$$\mathcal{A} = \{\varphi(x) \in C^1(\bar{\Omega}) \cap C^2(\Omega),\ \varphi = 0 \text{ on } \Gamma_1,\ \varphi = \phi \text{ on } \Gamma_2\} \times \{\phi;\ \phi \in \boldsymbol{R}^1\}.$$

LEMMA 1. – *If* $(\bar{\varphi}(x), \bar{\phi})$ *minimizes* $J(\varphi, \phi)$ *in* $\mathcal{A}$, *then* $(\bar{\varphi}, \bar{\phi})$ *is a classical solution of problem* $PV2$.

PROOF. – Let $v(X) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, $v = 0$ on Γ_1, $v = \gamma$ (a constant) on Γ_2. Then, from the condition $\frac{d}{d\varepsilon} J(\bar{\varphi} + \varepsilon v, \bar{\phi} + \varepsilon\gamma)|_{\varepsilon=0} = 0$, we have

$$\int_\Omega \sigma(|\nabla\bar{\varphi}|)\nabla\bar{\varphi} \cdot \nabla v dX + \frac{1}{R}(\bar{\phi} - V)\gamma = 0 \tag{25}$$

for all $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, $v = 0$ on Γ_1, $v = \gamma$ on Γ_2 and for all $\gamma \in \boldsymbol{R}^1$. Choosing $\gamma = 0$ in (25) we obtain

$$\int_\Omega \sigma(|\nabla\bar{\varphi}|)\nabla\bar{\varphi} \cdot \nabla v dX = 0 \tag{26}$$

for all $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, $v = 0$ on $\Gamma_1 \cup \Gamma_2$. Integrating by parts in (26) we have (21) and (22). Again from (25) we get, taking into account (21),

$$\gamma\left[\int_{\Gamma_2} \sigma(|\nabla\bar{\varphi}|)\frac{\partial\bar{\varphi}}{\partial n} d\Gamma + \frac{1}{R}(\bar{\phi} - V)\right] = 0. \tag{27}$$

Since γ is arbitrary, (23) follows. □

If we assume (16) the lemma motivates the following weak formulation of problem $PV2$: To find (φ, ϕ), $\varphi \in H^{1,p}(\Omega)$, $p > 1$, $\varphi = 0$ on Γ_1, $\varphi = \phi$ on Γ_2, $\phi \in \boldsymbol{R}^1$ such that

$$\int_\Omega \sigma(|\nabla\varphi|)\nabla\varphi \cdot \nabla v dX + \frac{1}{R}(\phi - V)\gamma = 0 \tag{28}$$

for all $v \in H^{1,p}(\Omega)$, $v = 0$ on Γ_1, $v = \gamma$ on Γ_2 and for all $\gamma \in \boldsymbol{R}^1$. Under the sole assumption (11) neither uniqueness nor existence are, in general, to be expected for problem $PV2$. However, if we assume (20) in addition to (11) and (16) we have

THEOREM 1. – *There exists one and only one solution to problem $PV2$.*

PROOF. – Let us take the functional (24) in the class of admissibles

$$\mathcal{X} = \{\varphi \in H^{1,p}(\Omega),\ \varphi = 0 \text{ on } \Gamma_1,\ \varphi = \phi \text{ on } \Gamma_2\} \times \{\phi;\ \phi \in \boldsymbol{R}^1\}.$$

$\mathcal{X}$ is a reflexive Banach space with norm $\|\varphi\|_{W^{1,p}(\Omega)} + |\phi|$. By (16) and (20) $J(\varphi, \phi)$ is a convex functional, weakly lower semicontinuous and coercive in $\mathcal{X}$. Moreover, J is Gateaux differentiable. Therefore (see [5]) the minimum $(\bar{\varphi}, \bar{\phi})$ of $J(\varphi, \phi)$ exists in $\mathcal{X}$ and gives a solution to problem $PV2$. To prove uniqueness, let (φ_1, ϕ_1) and (φ_2, ϕ_2) be both solutions of $PV2$. Setting in (28) $\psi = \varphi_i$, $\gamma = \phi_i$ $i = 1, 2$ and subtracting the resulting equations, we obtain

$$B_1 + B_2 = 0, \text{ where}$$

$$B_1 = \int_\Omega [\sigma(|\nabla\varphi_1|)\nabla\varphi_1 - \sigma(|\nabla\varphi_2|)\nabla\varphi_2] \cdot (\nabla\varphi_1 - \nabla\varphi_2) dX, \quad B_2 = \frac{1}{R}(\phi_1 - \phi_2)^2.$$

By (20) we have (see [11] page 31)

$$[\sigma(|z_1|)z_1 - \sigma(|z_2|)z_2] \cdot (z_1 - z_2) > 0$$

whenever $z_1, z_2 \in \boldsymbol{R}^2$ and $z_1 \neq z_2$. Thus $B_1 = 0$ and $B_2 = 0$. This implies $\phi_1 = \phi_2$, $\nabla\varphi_1 = \nabla\varphi_2$ and, by (21), $\varphi_1 = \varphi_2$. Therefore $PV2$ has one and only one solution. □

REMARK 1. – Under the assumptions of Theorem 1 problem $PV2$ is strictly monotonic. This is the reason for the simplicity of the proof of existence and uniqueness. It would be interesting to study the structure of the set of solutions under more general hypotheses.

3. – The V-I characteristic of the varistor. Properties of monotonicity of the conductance.

Throughout this section we assume

$$\sigma(t) = \gamma t^{p-2}, \ p \geq 2, \ \gamma > 0. \tag{29}$$

Let $\psi(x)$ be the unique solution of the problem:

$$\nabla \cdot (|\nabla\psi|^{p-2}\nabla\psi) = 0 \text{ in } \Omega, \ \psi = 0 \text{ on } \Gamma_1, \ \psi = 1 \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma_0. \tag{30}$$

It is easy to verify that $\varphi = V\psi$ solves $PV1$ and that

$$I = \mathcal{K}V^{p-1}, \text{ where } \mathcal{K} = \gamma \int_{\Gamma_2} |\nabla\psi|^{p-2} \frac{\partial\psi}{\partial n} d\Gamma \tag{31}$$

is the corresponding V-I characteristic. Thus (29) predicts the empirical law (9). The conductance $\mathcal{K}$ of the varistor can also be defined via the minimum property

$$\mathcal{K} = \gamma \int_\Omega |\nabla\psi|^p dX = \gamma \inf\left\{ \int_\Omega |\nabla v|^p dX; \ v \in H \right\} \tag{32}$$

where $H = \{w \in H^{1,p}(\Omega), \ w = 0 \text{ on } \Gamma_1, \ w = 1 \text{ on } \Gamma_2\}$. We note that for a conductivity $\sigma(t)$ satisfying only (16), (20) and (11) the minimum property (32) does not apply. Moreover, the voltage-current characteristic is more complex than (31) and the notion of conductance becomes meaningless. The minimum property (32) has the following interesting consequences.

THEOREM 2. – *If in a varistor for which* (29) *holds the size of the electrodes is reduced i.e.*

$$\Gamma_2' \subseteq \Gamma_2, \ \Gamma_1' \subseteq \Gamma_1$$

leaving Ω unchanged, then the corresponding conductances satisfy

$$\mathcal{K}' \leq \mathcal{K}. \tag{33}$$

PROOF. – Let $H = \{v \in H^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_1,\ v = 1 \text{ on } \Gamma_2\}$ and $H' = \{w \in H^{1,p}(\Omega);\ w = 0 \text{ on } \Gamma_1',\ w = 1 \text{ on } \Gamma_2'\}$. We have

$$\mathcal{K} = \gamma \inf\left\{\int_\Omega |\nabla v|^p dX,\ v \in H\right\},\ \mathcal{K}' = \gamma \inf\left\{\int_\Omega |\nabla w|^p dX,\ w \in H'\right\}.$$

Since $H \subseteq H'$ we have (33). □

THEOREM 3. – *If the size of Ω is diminished, leaving the electrodes Γ_1 and Γ_2 unchanged the conductance diminishes.*

PROOF. – Let Ω and Ω' be the domains before and after the diminution. Define $H = \{v \in H^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma_1,\ v = 1 \text{ on } \Gamma_2\}$ and $H' = \{w \in H^{1,p}(\Omega');\ w = 0 \text{ on } \Gamma_1,\ w = 1 \text{ on } \Gamma_2\}$. If $v \in H$, then $w = v|_{\Omega'} \in H'$. By the minimum property of the conductance we have $\mathcal{K}' \leq \mathcal{K}$. □

These results can be used to estimate the conductance in cases of practical interest as in the following example. Let a cylinder of arbitrary cross-section G and length L be contained in Ω with the lower basis contained in Γ_1 and the upper basis in Γ_2 (figure 4).

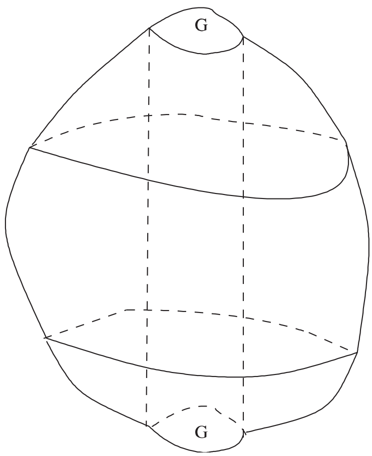

Fig. 4

The conductance of the cylinder is easily found to be $\mathcal{K}_c = \dfrac{\gamma|G|}{L^{p-1}}$, where $|G|$ is the two-dimensional measure of G. By Theorems 2 and 3, we have $\mathcal{K} \geq \mathcal{K}_c$.

4. – Non-homogeneous conductor.

In anisotropic conductors the conductivity is not a scalar but a symmetric tensor σ_{ij} which, in general, is also a function of space as e.g. when two pieces of different metals are soldered together. The Ohm's law reads in this case

$$\boldsymbol{J} = -\sum_{i,j=1}^{3} \sigma_{ij}(X)\frac{\partial\varphi}{\partial x_j}\boldsymbol{e}_j, \ \{\boldsymbol{e}_j\} \text{ a basis of} \{\boldsymbol{R}^3. \tag{34}$$

By Onsager's principle and the second principle of thermodynamics [8] σ_{ij} satisfies

$$\sigma_{ij}(X) = \sigma_{ji}(X) \tag{35}$$

and

$$\sum_{i,j=1}^{3} \sigma_{ij}(X)\xi_i\xi_j \geq \Lambda|\xi|^2 \text{ for all } \xi \in \boldsymbol{R}^3, \ \Lambda > 0. \tag{36}$$

The electric potential φ is now determined by the problem

$$\sum_{ij=1}^{3} \frac{\partial}{\partial x_i}\Big(\sigma_{ij}(X)\frac{\partial\varphi}{\partial x_j}) = 0 \text{ in } \Omega, \ \varphi = 0 \text{ on } \Gamma_1, \ \varphi = V \text{ on } \Gamma_2 \tag{37}$$

$$\sum_{i,j=1}^{3} \sigma_{ij}(X)\frac{\partial\varphi}{\partial x_j}n_i = 0, \ \boldsymbol{n} = \sum_{i=1}^{3} n_i\boldsymbol{e}_i \text{ unit vector normal to } \Gamma_2.$$

Moreover,

$$I = \sum_{i,j=1}^{3} \int_{\Gamma_2} \sigma_{ij}\frac{\partial\varphi}{\partial x_j}n_i d\Gamma \tag{38}$$

is the total current. The linear relation $I = \mathcal{K}V$ still holds with the conductance $\mathcal{K}$ given now by

$$\mathcal{K} = \sum_{i,j=1}^{3} \int_{\Gamma_2} \sigma_{ij}\frac{\partial\psi}{\partial x_j}n_i d\Gamma \tag{39}$$

where $\psi(X)$ solves

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\Big(\sigma_{ij}(X)\frac{\partial\psi}{\partial x_j}) = 0 \tag{40}$$

$$\psi = 0 \text{ on } \Gamma_1, \ \psi = 1 \text{ on } \Gamma_2, \ \sum_{i,j=1}^{3} \sigma_{i,j}(X)\frac{\partial\psi}{\partial x_j}n_i = 0.$$

In view of the possible discontinuities of $\sigma_{i,j}$, we write problem (40) in weak form as follows:

$$(41) \quad \psi \in H, \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(X) \frac{\partial \psi}{\partial x_i} \frac{\partial w}{\partial x_j} dX = 0,$$

$$\text{for all } w \in H^1(\Omega) \text{ such that } w = 0 \text{ on } \Gamma_1 \cup \Gamma_2,$$

where $H = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1,\ v = 1 \text{ on } \Gamma_2\}$, and the conductance as

$$(42) \qquad \mathcal{K} = \sum_{i,j=1}^{3} \int_\Omega \sigma_{ij}(X) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dX.$$

The two definitions (42) and (39) coincide when ψ and Γ are regular. $\mathcal{K}$ can also be characterized by the minimum property

$$(43) \qquad \mathcal{K} = \inf\left\{ \sum_{i,j=1}^{3} \int_\Omega \sigma_{ij}(X) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dX,\ v \in H \right\}.$$

Using (43) and reasoning as in Theorems 2 and 3, we can prove

THEOREM 4. – *(a) Let* σ_{ij}, σ'_{ij} *satisfy* (35) *and* (36) *and*

$$(44) \qquad \sum_{ij=1}^{3} \sigma'_{ij} \lambda_i \lambda_j \ge \sum_{ij=1}^{3} \sigma_{ij} \lambda_i \lambda_j, \text{ for all } \lambda \in \boldsymbol{R}^3.$$

Then

$$(45) \qquad \mathcal{K}' \ge \mathcal{K}.$$

(b) If in the same conductor Ω *we make the electrodes smaller i.e.*

$$\Gamma'_2 \subseteq \Gamma_2,\ \Gamma'_1 \subseteq \Gamma_1$$

then

$$\mathcal{K}' \le \mathcal{K}.$$

(c) If the size of Ω *is diminished leaving the electrodes unchanged, then*

$$\mathcal{K}' \le \mathcal{K}.$$

REMARK 2. – If Ω is diminished modifying the electrodes in the process, the conductance can either diminish or increase.

We give a simple application of Theorem 4. Let G be an open, bounded and connected subset of $\boldsymbol{R}^2$ and Ω the cylinder $\{(x, y, z);\ 0 < z < L,\ (x, y) \in G\}$ with the bases of the cylinder as electrodes. Assume $\sigma_M \geq \sigma(X) \geq \sigma_m > 0$. For the conductance $\mathcal{K}$ of Ω we have the bounds

$$\frac{\sigma_M |G|}{L} \geq \mathcal{K} \geq \frac{\sigma_m |G|}{L}$$

where $|G|$ is the two dimensional measure of G.

5. – The Reyleigh's method for estimating the conductance.

In his book "The Theory of Sound" J.W. Reyleigh proposed a trick for finding lower and upper bounds to the natural pitch of acoustical resonators. The method is based on an electrical analogy, interesting in itself, which permits to compute an upper bound to the conductance of a homogeneous conductor. Suppose the conductivity $\sigma = 1$ and divide Ω with a regular surface γ into two parts Ω' and Ω'' with boundary Γ' and Γ'' respectively. Define $\Gamma_j' = \Gamma' \cap \Gamma_j$, $\Gamma_j'' = \Gamma'' \cap \Gamma_j$, $j = 0, 1, 2$ where Γ_1 and Γ_2 represent, as usual, the electrodes of the conductor and Γ_0 the insulated part.

Let a be an arbitrary continuous function defined on γ and ψ be given by the problem

$$\Delta\psi = 0 \text{ in } \Gamma,\ \psi = 0 \text{ on } \Gamma_1,\ \psi = 1 \text{ on } \Gamma_2,\ \frac{\partial\psi}{\partial n} = 0,\ \text{on } \Gamma_0. \tag{46}$$

We have

$$\mathcal{K} = \int_\Omega |\nabla\psi|^2 dX. \tag{47}$$

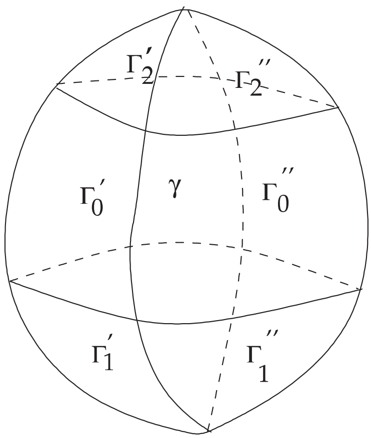

Fig. 5

Determine w' in Ω' and w'' in Ω'' with the problems

(48) $\Delta w' = 0$ in Ω', $w' = 0$ on Γ'_1, $w' = 1$ on Γ'_2, $\dfrac{\partial w'}{\partial n} = 0$ on Γ'_0, $w' = a$ on γ

(49) $\Delta w'' = 0$ in Ω'', $w'' = 0$ on Γ''_1, $w'' = 1$ on Γ''_2, $\dfrac{\partial w''}{\partial n} = 0$ on Γ''_0, $w'' = a$ on γ

and define in Ω, $w(X) = w'(X)$ in Ω', $w(x) = w''(X)$ in Ω''. w is continuous across γ and $\dfrac{\partial w}{\partial n}$ discontinuous, but if all the surfaces are regular $w \in H^1(\Omega)$. We claim that

$$\mathcal{K} \leq \int_\Omega |\nabla w|^2 dX. \tag{50}$$

Actually, if $u = w - \psi$ we have

(51) $\Delta u = 0$ in Ω' and in Ω'', $u = 0$ on Γ_1 and Γ_2, $\dfrac{\partial u}{\partial n} = 0$ on Γ_0.

Moreover, u is continuous across γ. Therefore

$$\int_\Omega |\nabla w|^2 dX = \int_\Omega |\nabla \psi|^2 dX + \int_\Omega |\nabla u|^2 dX + 2\int_\Omega \nabla\psi \cdot \nabla u dX.$$

Integrating by parts and recalling (46) and (51) we have

$$\int_\Omega \nabla\psi \cdot \nabla u dX = 0$$

thus (50) follows. The freedom in the choice of γ and a can be used to estimate $\int_\Omega |\nabla w|^2 dX$ and, by (50), the conductance $\mathcal{K}$ as in the following example. Let $\Omega = \{(x, y, z),\ 0<x<1,\ 0<y<1,\ 0<z<1\}$, $\Gamma_1 = \{(x, y, z),\ 0<x<1,\ 0<y<1,\ z = 0\}$, $\Gamma_2 = \{(x, y, z),\ 0<x<1/2,\ 0<y<1,\ z = 1\}$ and $\gamma = \{(x, y, z),\ x = 1/2,\ 0<y<1, 0<z<1\}$. Let $a = z$. We have $w' = z$. Moreover, the solution of (49) can be computed by separation of variables. In this way we obtain $\int_\Omega |\nabla w|^2 dX$ and therefore, by (50), an estimate of $\mathcal{K}$.

6. – The V-I characteristics for the thermistor.

In this section we study the voltage-current characteristics for a thin and homogeneous thermistor represented by a doubly connected domain Ω of the plane, bounded by two regular curves Γ_1 and Γ_2, ($\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma = \Gamma_1 \cup \Gamma_2$). Γ_1 and Γ_2 represent the electrodes to which a difference of potential V is applied. A constant temperature, assumed as zero in an empirical scale, is kept on the whole boundary Γ. The electrical and thermal conductivities σ and κ are supposed to be given functions of the temperature u. In many substances, e.g. semiconductors, this dependence is relevant. We assume σ and κ to be continuous and positive functions and to satisfy

$$\int_0^\infty \frac{\kappa(t)}{\sigma(t)}\,dt = \infty. \tag{52}$$

We note that (52) is verified by all materials satisfying the Wiedemann-Franz law. Under steady conditions, by the conservation of charge and energy, the potential $\varphi(x,y)$ and the temperature $u(x,y)$ are given by the boundary value problem (PT)

$$\nabla\cdot(\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega \tag{53}$$

$$-\nabla\cdot(\kappa(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 \text{ in } \Omega \tag{54}$$

$$\varphi = 0 \text{ on } \Gamma_1,\ \ \varphi = V \text{ on } \Gamma_2,\ \ u = 0 \text{ on } \Gamma. \tag{55}$$

If (52) holds, problem (PT) has one and only one solution [3]. The total current crossing the lamina is given by

$$I = \int_{\Gamma_2} \sigma \frac{\partial\varphi}{\partial n}\,ds. \tag{56}$$

We will use the fact that every doubly connected plane domain, as Ω, can be mapped conformally onto the annular domain

$$D = \{z; 1<|z|<r\},\ \ z = x+iy.$$

The number r is called the modulus of Ω, [10] and permits to divide the set of all plane doubly connected domains into equivalence classes $\{E_r\}$, each of which is characterized by r. The modulus can be computed by means of the Bergman function of the domain [2], [6], either exactly or approximately. Precise evaluations of the modulus exist e.g. for nonconcentric annuli, elliptic rings, cofocal elliptic rings, squares inside circles and squares inside squares. The theorem below permits to compute explicitly the total current if the modulus of Ω is known.

THEOREM 5. – *The total current is invariant in the equivalence class E_r and it is given by*

$$I = \frac{2\pi}{\ln r}\int_0^V \sigma\left(F^{-1}\left(\frac{V}{2}t - \frac{t^2}{2}\right)\right)dt \tag{57}$$

where

$$F(t) = \int_0^t \frac{\kappa(\xi)}{\sigma(\xi)} d\xi. \tag{58}$$

PROOF. – The equations (53) and (54) are conformally invariant (see [7]). Moreover, by direct calculations, it is possible to verify the crucial fact that also the total current (56) is invariant under conformal mapping. Thus we simply need to solve problem (PT) in the annular region D. Since there is only one solution if (52) holds, we can search directly for a solution depending uniquely on the radial variable ρ. To this end we use the transformation

$$\theta = \frac{\varphi^2}{2} + F(u). \tag{59}$$

It is easily seen that $\theta(\rho)$ solves the one-dimensional problem

$$(\rho\sigma(u)\theta')' = 0, \ \theta(1) = 0, \ \theta(r) = \frac{V^2}{2}. \tag{60}$$

We make the "ansatz" of the existence between θ and φ of a linear functional relation, which, in view of the boundary conditions, can only be of the form

$$\theta = \frac{V}{2}\varphi. \tag{61}$$

From (59) and (61) we have

$$u = F^{-1}\left(\frac{V}{2}\varphi - \frac{\varphi^2}{2}\right). \tag{62}$$

Therefore φ satisfies the problem

$$\left(\rho\sigma\left(F^{-1}\left(\frac{V\varphi}{2} - \frac{\varphi^2}{2}\right)\right)\varphi'\right)' = 0, \ \varphi(1) = 0, \ \varphi(r) = V$$

which can be solved setting

$$\psi = L(\varphi), \ \text{where } L(\varphi) = \int_0^{\varphi} \sigma\left(F^{-1}\left(\frac{Vt}{2} - \frac{t^2}{2}\right)\right) dt. \tag{63}$$

We find that $\psi(\rho)$ is given by

$$\psi(\rho) = \frac{a \ln \rho}{\ln r} \ \text{where } a = \int_0^V \sigma\left(F^{-1}\left(\frac{Vt}{2} - \frac{t^2}{2}\right)\right) dt.$$

On the other hand, from (63), we have

$$\sigma(0)\varphi'(r) = \frac{a}{r \ln r},$$

thus (57) follows. □

We may treat, with the same method, a second case in which Ω is a plane simply connected domain with boundary Γ. If we select four different points A, B, C, D on Γ, then Ω can be mapped conformally onto a rectangle R: $(0, a) \times (0, b)$ such that A, B, C, D go into $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$. The conformal modulus of Ω marked with the points A, B, C, D is a/b. We consider (53), (54) with the boundary conditions:

$$\varphi = 0 \text{ on } \Gamma_{AB},\ \varphi = V \text{ on } \Gamma_{CD},\ \frac{\partial \varphi}{\partial n} = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{DA} \cup \Gamma_{BC},\ u = 0 \text{ on } \Gamma,$$

$\Gamma_{DA} \cup \Gamma_{BC}$ is the part of the boundary electrically and thermally insulated. The problem in the rectangle can be solved again with the transformation (59). We find for the total current

$$I = \frac{a}{b} \int_0^V \sigma\left(F^{-1}\left(\frac{V}{2}t - \frac{t^2}{2}\right)\right) dt.$$

The geometry of the lamina enters, also in this case, only via its modulus a/b.

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